A NOTE ON HOMOGENEOUS DENDRITES

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1. In graph-theoretic terms a homogeneous p-dendrite, $p \geq 2$, is defined as a finite singly-rooted tree in which the root has valency 1 while every other vertex has valency 1 or p. More descriptively, a homogeneous p-dendrite may be imagined to start from its root as the main, or 0th order, branch which proceeds to the first-order branch point where it gives rise to p first-order branches. Each of these either terminates at its other end (which is a second-order branch point) or it splits there again into p branches (which are of third order), and so on. The order of the dendrite is the highest order of a branch present in it. For completeness, a 0-th order dendrite is also allowed, this consists of the 0-th order branch alone.

Alternatively, if we consider a development in time rather than a structure in space, a homogeneous p-dendrite represents a history in which a single individual fissions into p identical individuals each of which either dies without descendants or else, fissions into p new indistinguishable individuals again.

We shall be interested here in the number $f_p(n)$ of (topologically) distinct n-th order p-dendrites. Our interest is motivated partly by biological and physical considerations relative to certain simple branching processes (number of various family-histories, number of distinct dendrites of a neuron, particle-showers, etc.) and partly by pure combinatorics.

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2. Here we determine the number $f_2(n) = f(n)$ of binary dendrites. To begin with, we have

(1) $$f(0) = 1, \qquad f(1) = 1.$$

Let $n \geq 0$ and consider an $(n+1)$-st order binary dendrite. There is here one first-order branch point, as shown in Figure 1, and this is followed by two structures one of which is an n-th order dendrite (position 1) and the other one an m-th order dendrite (position 2), with $m \leq n$.

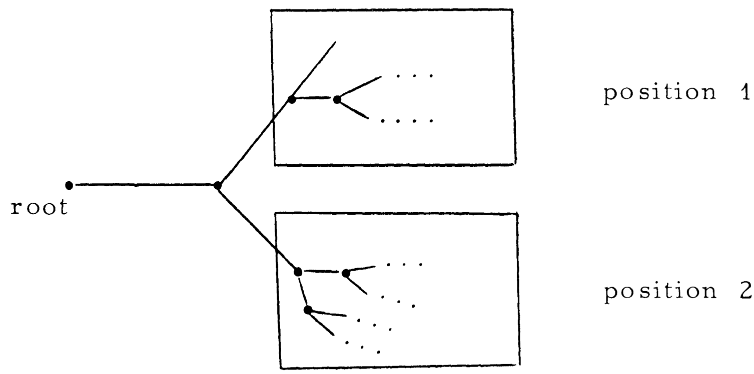

FIGURE 1

Suppose first that $m < n$. Then position 1 can be filled by any one of the $f(n)$ distinct n-th order dendrites and, independently, position 2 by any one of the $f(m)$ m-th order ones. Hence the total number of $(n+1)$-st order dendrites with $m < n$ is

(2) $$N_1 = f(n) \sum_{i=0}^{n-1} f(i)$$

When $m = n$, the n-th order dendrites in positions 1 and 2 can be identical, and this can occur in $f(n)$ ways, or they can be distinct, which can occur in

$$f(n)[f(n) - 1]/2$$

ways. Therefore the total number of $(n+1)$-st order dendrites with $m = n$ is

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$$N_2 = f(n)[f(n) + 1]/2.$$

Adding N_1 and N_2 we get the total number $f(n+1)$ of distinct $(n+1)$-st order dendrites:

$$(3) \qquad f(n+1) = f(n)\left[\sum_{i=0}^{n-1} f(i) + (1 + f(n))/2\right],$$

which may be written as

$$(4) \qquad \frac{f(n+1)}{f(n)} = \frac{1 + f(n)}{2} + \sum_{i=0}^{n-1} f(i).$$

Copying this equation with n replaced by $n-1$, subtracting from (4), and re-arranging, we get a nonlinear second-order recursion for f:

$$(5) \qquad f(n+1) = f(n)\left[\frac{f(n)}{f(n-1)} + \frac{f(n) + f(n-1)}{2}\right].$$

From this and (1) we compute successively

$f(2) = 2$, $f(3) = 7$, $f(4) = 56$, $f(5) = 2212$, $f(6) = 2595782$

and so on. It would be useful to have an explicit formula for $f(n)$ but this does not appear to be easy to get. Some rough bounds on $f(n)$ can be obtained as follows. By (5) we have

$$(6) \qquad f^2(n_o)/2 \le f(n_o+1) \le 2f^2(n_o);$$

therefore

$$[f^2(n_o)/2]^2/2 \le f(n_o+2) \le 2[2f^2(n_o)]^2$$

and generally for arbitrary n and any fixed n_o

$$(7) \qquad [f^{2^n}(n_o)]/2^{2^n-1} \le f(n+n_o) \le 2^{2^n-1}[f^{2^n}(n_o)].$$

3. Throughout this section we assume that $p = 3$ and we get a formula analogous to (5) for the case of ternary splitting. Let $f_3(n) = f(n)$, put $n \geq 2$ and consider an $(n+1)$-st order ternary dendrite. In analogy to Figure 1 we have now three positions to be filled by ternary dendrites of orders n, m, and k, with $0 \leq k \leq m \leq n$. Suppose first that $k < m < n$, then any k-th order, m-th order, and n-th order dendrites can fill, independently, their respective positions and so the number N_1 of ternary dendrites of $(n+1)$-st order, with $k < m < n$, is

$$(8) \qquad N_1 = f(n) \sum_{m=1}^{n-1} \sum_{k=0}^{m-1} f(k)f(m).$$

When $k < m = n$ the corresponding number is

$$(9) \qquad N_2 = f(n) \left[\frac{f(n) + 1}{2} \right] \sum_{k=0}^{n-1} f(k).$$

When $k = m = n$, there are three cases to consider because among the n-th order dendrites filling the three positions there may be one, two or three distinct ones. The total number N_3 is here

$$(10) \qquad N_3 = f(n) + f(n)[f(n) - 1] + f(n)[f(n) - 1][f(n) - 2]/6$$

$$= f(n)[f^2(n) + 3f(n) + 2]/6.$$

Finally, when $k = m < n$, the contribution to the total is

$$(11) \qquad N_4 = \frac{f(n)}{2} \sum_{k=0}^{n-1} f(k)[f(k) + 1].$$

Adding the numbers N_1, N_2, N_3, N_4 from the equations (8), (9), (10), (11) we get

(12)

$$f(n+1) = f(n) \sum_{m=1}^{n-1} \sum_{k=0}^{m-1} f(m)f(k) + \frac{f(n)[f(n)+1]}{2} \sum_{k=0}^{n-1} f(k)$$

$$+ \frac{f(n)}{2} \sum_{k=0}^{n-1} f(k)[f(k)+1] + \frac{f(n)}{6} [f^2(n) + 3f(n) + 2].$$

Therefore

(13)

$$\frac{f(n+1)}{f(n)} - \frac{f^2(n) + 3f(n) + 2}{6} = \sum_{m=1}^{n-1} \sum_{k=0}^{m-1} f(m)f(k) + \frac{f(n)+1}{2} \sum_{k=0}^{n-1} f(k)$$

$$+ \frac{1}{2} \sum_{k=0}^{n-1} f(k)[f(k)+1].$$

Denote the left-hand side of (13) by $F(n)$; taking first differences, we get

$$F(n) - F(n-1) = f(n-1) \sum_{k=0}^{n-2} f(k) + \frac{f(n)+1}{2} f(n-1) + \frac{f(n) - f(n-1)}{2} \sum_{k=0}^{n-2} f(k)$$

$$+ \frac{1}{2} f(n-1)[f(n-1) + 1]$$

so that

(14)

$$F(n) - F(n-1) - \frac{f(n-1)}{2} [f(n) + f(n-1)] - f(n-1) = \frac{f(n) + f(n-1)}{2} \sum_{k=0}^{n-2} f(k).$$

Denote the left-hand side of (14) by $G(n)$ and put

$$H(n) = \frac{2G(n)}{f(n) + f(n-1)}$$

so that (14) is now simply

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$$H(n) = \sum_{k=0}^{n-2} f(k);$$

taking first differences again, we eliminate all the sums and get

$$H(n) - H(n-1) = f(n-2).$$

Substituting successively for H, G, F, we get after some tedious algebra

(15)

$$f(n+1) = f(n)[f(n) + f(n-1)][f(n) + f(n-1) + f(n-2)]/6$$
$$+ f(n) \frac{f(n) + f(n-1)}{f(n-1) + f(n-2)} \left[\frac{f(n)}{f(n-1)} - \frac{f(n-1)}{f(n-2)} \right] + f^2(n)/f(n-1).$$

Direct inspection shows that $f(0) = 1$, $f(1) = 1$, $f(2) = 3$, now the recursion formula (15) yields

$$f(3) = 31, \quad f(4) = 8401, \quad f(5) = 100\,130\,704\,103 \quad \text{etc.}$$

4. It is possible to obtain in the same way successive formulas, analogous to (5) and (15), for $f_4(n)$, $f_5(n)$, etc. However their complexity grows very rapidly, and a recursion formula valid for a general $f_p(n)$ appears to be difficult to get. We shall obtain instead the general analogue of (3) and (12).

Consider an (n+1)-st order p-dendrite. Referring to Figure 1, we have here p positions to fill instead of two, and we suppose that the j-th position contains a p-dendrite of order k_j. To meet the enumerative conditions we must have

(16) $$0 \leq k_1 \leq k_2 \leq \ldots \leq k_p = n.$$

It is important to know where the strict inequality occurs between k_i and k_{i+1}, $i = 1, \ldots, p-1$. There are 2^{p-1} sequences of $p-1$ signs each of which is "<" or "="; any such sequence will be denoted by r and called an ordering, and the set of all 2^{p-1} orderings will be denoted by R. Once an ordering $r \in R$

is given, the monotonicity properties of (16) are known; further, irrespective of the values of the indices k_i two different orderings will lead to different (n+1)-st order dendrites.

Let $r \in R$, if in r we find a sequence such as

$$k_1 = k_2 = \ldots = k_m < \ldots, \ldots < k_i = k_{i+1} = \ldots = k_{i+m-1} < \ldots,$$

$$\text{or } \ldots < k_{p-m+1} = k_{p-m+2} = \ldots = k_p$$

we call it a step of length m. In particular,

$$k_1 < \ldots, \ldots < k_i < \ldots, \ldots < k_p$$

are steps of length 1. Let $g = g(r)$ be the total number of steps in r and let $m_j = m_j(r)$ be the length of the j-th consecutive one $(j = 1, 2, \ldots, g(r))$ so that

$$1 \leq g(r) \leq p, \quad \sum_{j=1}^{g(r)} m_j(r) = p.$$

Consider now the j-th step, of length $m_j(r)$; this corresponds to filling $m_j(r)$ positions with p-dendrites of the same order, say s_j. By the enumeration conditions of the problem we deal here with combinations in which repetitions are allowed, and there are $f_p(s_j)$ possibilities of filling each position. Therefore the $m_j(r)$ positions can be filled in

$$\binom{f_p(s_j) + m_j(r) - 1}{m_j(r)}$$

ways. Hence the number of ways in which all the positions can be filled, once the ordering r as well as the values of the indices s_j are fixed, is

$$\prod_{j=1}^{g(r)} \binom{f_p(s_j) + m_j(r) - 1}{m_j(r)}$$

Allowing for suitable variation of indices s_j corresponding to the same ordering r, we find that the total number of ways of filling all the positions for a fixed ordering r is

$$\sum_{s_{g(r)-1}=g(r)-2}^{n-1} \cdots \sum_{s_3=2}^{s_4-1} \sum_{s_2=1}^{s_3-1} \sum_{s_1=0}^{s_2-1} \prod_{j=1}^{g(r)} \binom{f_p(s_j) + m_j(r) - 1}{m_j(r)} .$$

Summing over all the 2^{p-1} orderings to get the grand total number of ways of filling all the positions we get finally

(17)

$$f_p(n+1) = \sum_{r \in R} \left[\sum_{s_{g(r)-1}=g(r)-2}^{n-1} \cdots \sum_{s_3=2}^{s_4-1} \sum_{s_2=1}^{s_3-1} \sum_{s_1=0}^{s_2-1} \prod_{j=1}^{g(r)} \binom{f_p(s_j) + m_j(r) - 1}{m_j(r)} \right]$$

which generalizes (3) and (12) to arbitrary p.

Of the 2^{p-1} terms in the square brackets there is exactly one, namely

$$\binom{f_p(n) + p - 1}{p}$$

containing no summation; this corresponds to having all p positions filled with maximal (n-th order) dendrites. Therefore

(18)

$$f_p(n+1) - \binom{f_p(n) + p - 1}{p} = \sum_{s_{p-1}=p-2}^{n-1} \cdots \sum_{s_3=2}^{s_4-1} \sum_{s_2=1}^{s_3-1} \sum_{s_1=0}^{s_2-1} \prod_{j=1}^{g(r)} f_p(s_j) + Q_{p-2}$$

where the first term on the right corresponds to all p positions having dendrites of different orders (assuming that n is large enough) and has $p-1$ summations, while Q_{p-2} is the sum of all the other terms, each of which has $\leq p-2$ summations. Denote the left-hand side of (18) by $F(n)$; taking first differences one finds that $F(n) - F(n-1)$ is of the form

$$(19) \qquad f_p(n-1)S_1 + T_{p-3} + T_{p-4} + \ldots + T_1 + T_0$$

where S_1 is a single term with $p-2$ summations, and T_i is a sum of terms with i summations. One repeats now the same number-of-summations reduction procedure by taking the first difference of

$$G(n) = [F(n) - F(n-1) - T_0]/f_p(n-1)$$

to get an expression similar to (19):

$$G(n) - G(n-1) = \varphi[f_p(n), f_p(n-1)]S_1' + T_{p-4}' + \ldots + T_0'$$

where φ is a rational function. The whole process is carried out $p-1$ times and one ends up with a nonlinear p-step recurrence relation

$$f_p(n+1) = R[f_p(n), f_p(n-1), \ldots, f_p(n-p+1)]$$

where R is a rational function with integer coefficients.

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