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A CONVERGENCE PROBLEM FOR KERGIN INTERPOLATION

by JEAN PAUL CALVI

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Let E, F, G be three compact sets in $\mathbb{C}^n$. We say that (E,F,G) holds if for any choice of an interpolating array in F and of an analytic function f on G, the Kergin interpolation polynomial of f exists and converges to f on E. Given two of the three sets, we study how to construct the third in order that (E,F,G) holds.

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1. Formulating the problem

Let us first recall some basic facts for Kergin interpolation. Let Ω be a $\mathbb{C}$-convex domain in $\mathbb{C}^n$, i.e. for each complex line $l\subset\mathbb{C}^n$, $l\cap\Omega$ is empty or simply connected. Denote by $H(\Omega)$ the space of holomorphic functions on Ω and $P_d(\mathbb{C}^n)$ the space of polynomials whose degree does not exceed d.

Let $A=\{a_0,a_1,\dots,a_d\}$ be a subset of $d+1$ (nonnecessarily distinct) points in Ω, then there exists a unique continuous linear map:

$$K_A\colon H(\Omega)\to P_d(\mathbb{C}^n)$$

with the following properties.

(K1) For $i=0,1,\dots,d$ and $f\in H(\Omega)$, $K_A(f)(a_i)=f(a_i)$.

(K2) If $g\in H(\Omega)$ is of the form $g=f\circ u$ with u an affine map from $\mathbb{C}^n$ to $\mathbb{C}^m$ and $f\in H(u(\Omega))$ then

$$K_A(g)=K_{u(A)}(f)\circ u$$

where $u(A)=\{u(a_0),u(a_1),\dots,u(a_d)\}$. Thus if $m=1$

$$K_A(g)=L_{u(A)}(f)\circ u$$

where $L_{u(A)}(f)$ is the usual Lagrange Hermite interpolation polynomial of the one variable function f with respect to the points $u(a_0),\dots,u(a_d)$.

(K3) When all the points a_0, $a_1,\dots,a_d$ coincide, $K_A(f)$ is the Taylor expansion of f at the point $a(=a_0,a_1,\dots,a_d)$ and of degree d.

The polynomial $K_A(f)$ is called the Kergin interpolation polynomial of the function f with respect to the points $a_0, \dots, a_d$; we will also use the following alternative notation:

$$K_A(f) = K[a_0, a_1, \dots, a_d, f].$$

Constructive formulas and others algebraic properties are available in [**1**, **2**, **9**]. As well as in the classical one dimensional case, natural convergence problems arise for Kergin interpolation. Such problems are now quite well studied for entire functions, see [**4**, **5**, **1**], while very few is known for the general case: a topic to which is devoted the present note.

Definition 1. Given three compact sets E, F, G in $\mathbb{C}^n$, E and F being included in G, we say that the property (E, F, G) holds if for any triangular array of points $\{(a_d^i), d \in \mathbb{N}, 0 \leq i \leq d\}$ in F and any function f holomorphic in a neighbourhood of G, the Kergin polynomials $K[a_d^0, \dots, a_d^d, f]$, $d \in \mathbb{N}$ are well defined and converge uniformly to f on E as d tend to ∞.

We can now formulate the problem we wish to study.

Problem. Given two of the three compact sets E, F, G, construct the third and if possible optimality (in a sense to be made precise) in order that (E, F, G) holds.

The univariate version of this problem was first formulated and completely solved (in the general sense above) by Smirnov and Lebedev, see [**8**]. For the multidimensional case, some examples have already been studied by Bloom and Bos, see [**6**].

Since the $\mathbb{C}$-convex domains are the natural domains of existence of the Kergin operator, see [**2**, Prop. 3], we will not be surprised if a hypothesis of $\mathbb{C}$-convexity is needed for some of the compact sets E, F, G to expect that (E, F, G) holds.

Remark 1. (i) *If* (E, F, G) *holds and* $E' \subset E$, $F' \subset F$, $G \subset G'$ *then* (E', F', G') *also holds.*

(ii) *If* Φ *is an affine bijective map on* $\mathbb{C}^n$ *then* (E, F, G) *holds if and only if* $(\Phi(E), \Phi(F), \Phi(G))$ *holds.*

Proof. (i) is obvious and (ii) follows from the property (K2).

2. Main results

Let N be any (complex) norm on $\mathbb{C}^n$, we let $B(a, r)$ denote the open N-ball with centre a and radius r. For any set X, let $\Delta_N(p, X)$ denote the N-distance from p to the boundary, ∂X, of X. All metric objects in this section refer to the norm N and so in the sequel we usually omit the subscript N. E, F, G always denote non empty compact sets in $\mathbb{C}^n$. If $z = (z_i)$ and $w = (w_i)$ then $\langle z, w \rangle = \sum_{i=1}^n z_i w_i$.

Definition 2. We define $F(E, G)$ to be the set of points $p \in \mathbb{C}^n$ such that there exists a ball with centre p which is included in G but contains E. Thus equivalently, $p\in F(E,G)$ if and only if

$$\max_{z\in E} N(z-p)-\Delta(p,G)\leqq 0.$$

Proposition 1. *$F(E,G)$ is a compact convex set.*

Proof. $F=F(E,G)$ is bounded since it is included in G and the fact that it is closed follows from the continuity of the function $p\to N(z-p)-\Delta(p,G)$ for each $z\in E$. Let us prove the convexity.

Let $p_1,p_2\in F$. We must show that the segment $[p_1,p_2]$ lies in F. By definition there exist two closed balls $B_1=\bar{B}(p_1,r_1)$ and $B_2=\bar{B}(p_2,r_2)$ which contain E and are included in G. We claim that for any point $p\in[p_1,p_2]$ there exists a closed ball with centre p containing E and included in $\bar{B}_1\cup\bar{B}_2$. This follows from the convexity of the function

$$\max_{x\in E} N(p-z)-\Delta(p,\partial(B_1\cup B_2))$$

which is equal to

$$\max_{z\in E,\, i=1,2}[N(p-z)+N(p-p_i)-r_i].$$

The proposition is proved. □

Proposition 2. *Let $E\subset G$. Suppose that G is regular $\mathbb{C}$-convex (see below) then $(E,F(E,G),G)$ holds.*

We say that a compact set G is $\mathbb{C}$-convex (respectively regular $\mathbb{C}$-convex) if G admits a basis of neighbourhoods composed of $\mathbb{C}$-convex domains (of $\mathbb{C}$-convex domains of the form $\Omega=\{\rho<0\}$ with $\rho\in C^2(\bar{\Omega})$, grad$\rho\neq 0$ on $\partial\Delta$). Any compact convex set is of this type.

Lemma 1. *Let Ω be a bounded $\mathbb{C}$-convex domain with smooth boundary (i.e. $\Omega=\{\rho<0\}$, $\rho\in C^2(\bar{\Omega})$). Let f be a function holomorphic on Ω and continuous on $\bar{\Omega}$. Finally, let $A=\{a_0,a_1,\dots,a_d\}$ be a subset of points in Ω, then for any $z\in\Omega$ the following Hermite type remainder formula holds:*

$$f(z)-K_A(f)(z)=$$

$$\frac{1}{(2i\pi)^n}\int_{\partial\Omega}\prod_{j=0}^{d}\frac{\langle\rho'(\xi),z-a_j\rangle}{\langle\rho'(\xi),\xi-a_j\rangle}\times\sum_{|\alpha|+\beta=n-1}\frac{f(\xi)\,\partial\rho(\xi)\wedge(\bar{\partial}\partial\rho(\xi))^{n-1}}{\langle\rho'(\xi),\xi-a\rangle^{\alpha}\langle\rho'(\xi),\xi-z\rangle^{\beta+1}}$$

where

$$\alpha=(\alpha_0,\alpha_1,\dots,\alpha_d),\ \langle\rho'(\xi),\xi-a\rangle^\alpha=\prod_{i=0}^{d}\langle\rho'(\xi),\xi-a_i\rangle^{\alpha_i},\ \rho'(\xi)$$

$$=\left(\frac{\partial\rho}{\partial z_i}(\xi)\right)_{1\leqq i\leqq n}\quad \text{and}\quad |\alpha|=\sum\alpha_i.$$

Proof. This formula is proved in [1], we refer the reader to that paper for the details but point out that the function

$$(\xi,z)\in\partial\Omega\times\Omega\rightarrow\langle\rho'(\xi),\xi-z\rangle$$

does not vanish. This is an important property of $\mathbb{C}$-convex domains which will be used in the sequel. □

Proof of Proposition 2. Let $\{(a_d^i),\ d\in\mathbb{N}, 0\leqq i\leqq d\}$ be a triangular array of points in $F=F(E,G)$ and f a function holomorphic in a neighbourhood of G. Because of the hypothesis of $\mathbb{C}$-convexity the Kergin polynomial

$$K[a_d^0,\dots,a_d^d] \tag{1}$$

is well defined for each d and thus we have only to prove that the polynomials in (1) converge uniformly to f on E as d tends to ∞.

Let us take D a $\mathbb{C}$-convex domain of the form $D=\{\rho<0\}$, $\rho\in C^2(\bar{D})$, containing G and such that f is holomorphic on a neighbourhood of $\bar{D}$. The function $(\xi,t)\rightarrow|\langle\rho'(\xi),\xi-t\rangle|$ is continuous and does not vanish on the compact set $\partial D\times G$ and hence its infimum c on this compact set is not negative. Thus, since E and F are included in G we have, with the previous notation:

$$\sum_{|\alpha|+\beta=n-1}1/(|\langle\rho'(\xi),\xi-a_d^i\rangle^\alpha\langle\rho'(\xi),\xi-z\rangle^{\beta+1}|)\leqq c^{-n}\binom{n+d+1}{n-1}. \tag{2}$$

On the other hand

$$\frac{1}{(2i\pi)}\partial\rho(\xi)\wedge(\bar{\partial}\,\partial\rho(\xi))^{n-1}$$

is a bounded measure on ∂D.

Hence applying the remainder formula of Lemma 1 (this is possible) we see that the Kergin interpolation polynomials (1) will converge uniformly to f on E if we prove that there exists a positive real number $\delta<1$ such that for any $z\in E$, $\xi\in\partial D$ and $p\in F$

$$\left|\frac{\langle \rho'(\xi), z-p\rangle}{\langle \rho'(\xi), \xi-p\rangle}\right| \leqq \delta. \tag{3}$$

Since in this case, by the above estimates, we would have

$$\max_{z\in E} |f(z)-K[a_d^0, a_d^1, \ldots, a_d^d, f](z)| \leqq C\binom{d+n+1}{n-1}\delta^{d+1} \tag{4}$$

where C is a constant independent of d and then the left term in (4) tends to 0 as d tends to ∞.

Let $p\in F$. Then by Definition 2, there exists a closed ball $\bar{B}(p,r)$ such that $E\subset \bar{B}(p,r)\subset G$ and so for any $\xi\in\partial D$ we have

$$\langle \rho'(\xi), E\rangle \subset \langle \rho'(\xi), \bar{B}(p,r)\rangle \subset \langle \rho'(\xi), G\rangle \subset \langle \rho'(\xi), \bar{D}\rangle. \tag{5}$$

We remark that the last inclusion is strict. The point $\langle \rho'(\xi), \xi\rangle$ is a boundary point of the last set and since N is a complex norm, the second is disc with centre $\langle \rho'(\xi), p\rangle$ in the complex plane. We therefore have (make a drawing!)

$$|\langle \rho'(\xi), \xi-p\rangle| > |\langle \rho'(\xi), z-p\rangle|. \tag{6}$$

We note that the conclusion is false if the second set is not a disc.

Now, by (6), the left hand side of (3) is a continuous function strictly bounded by 1 on the compact set $E\times F\times\partial D$. The existence of $\delta<1$ follows and the proposition is proved. □

Definition 3. Let $F\subset G$. The set $E=E(F,G)$ is defined by $E=\bigcap_{p\in F}\bar{B}_p$, where $\bar{B}_p$ is the closed ball which has the maximal radius among all those with centre p and included in G. This is obviously a compact convex set.

Definition 4. The set $G=C(E,F)$ is defined by $G=\bigcup_{p\in F}\bar{B}^p$ where $\bar{B}^p$ is the closed ball with minimal radius among those with centre p and containing E. This is a compact set starshaped with respect to any point in E.

Corollary 1. *Let $F\subset G$. Suppose that G is regular $\mathbb{C}$-convex then $(E(F,G),F,G)$ holds.*

Proof. Let $\tilde{F}=F(E(F,G),G)$ then by Proposition 2, $(E(F,G),\tilde{F},G)$ holds and since $F\subset\tilde{F}$, (E,F,G) also holds (see the Remark 1). □

Corollary 2. *Suppose that G is regular $\mathbb{C}$-convex and $G(E,F)\subset G$ then (E,F,G) holds.*

Proof. Let $\tilde{E}=E(F,G)$ then by Corollary 1, $(\tilde{E},F,G)$ holds. Since $\tilde{E}\supset E$, (E,F,G) also holds. □

In some cases a refinement of this last corollary is possible.

Proposition 3. *Let $F \subset E$. Then $(E, F, G(E,F))$ holds.*

Note that no hypothesis of $\mathbb{C}$-convexity is formulated in the proposition.

Proof. The remainder formula of Lemma 1 is of no interest here and we have to find another one.

First we prove that there exists a domain D with smooth boundary, containing E and such that f is holomorphic on D and continuous on $\bar{D}$. Let us choose Ω a domain containing G such that f is holomorphic in a neighbourhood of $\bar{\Omega}$ and for any $p \in F$ an open ball $B(p, r_p)$ with $E \subset B(p, r_p) \subset \Omega$. It follows that

$$G \subset \bigcup_{p \in F} B(p, r_p) \subset \Omega.$$

We may cover the compact set G by a finite number of open balls, say

$$G \subset \bigcup_{i=1}^{q} B(p_i, r_{p_i}) := O.$$

Hence f is holomorphic in a neighbourhood of $\bar{O}$ which admits a basis of neighbourhoods of bounded smooth domains. This can be seen by smoothly approximating the continuous function $\rho(z) = \inf_{i=1,\dots,q}(N(z-p_i) - r_{p_i})$. The existence of D is thus proved.

Let v be any point in E. By hypothesis there exists a closed N-ball $\bar{B}(v,r)$ containing E and included in G. We can find $\tilde{N}$ a norm smooth (C^2) away from the origin and close enough to N, i.e. $(1-\varepsilon)N \leqq \tilde{N} \leqq N$ with ε small, such that the open $\tilde{N}$-ball $B_{\tilde{N}}(v,r) := \tilde{B}$ contains E and is included in D.

Now, for $\xi \in \mathbb{C}^n$, $\xi \neq v$, we define $s(\xi) = \tilde{N}'(T(\xi,\xi))$ where $T(\xi, w) = v + r(w-v)/(\tilde{N}(\xi - v))$. Then s is a C^2 function in a neighbourhood of ∂D and for any $z \in \tilde{B}$, $\langle s(\xi), \xi - z \rangle \neq 0$. Indeed the complex hyperplane $\langle s'(\xi), \xi - z \rangle = 0$ is the image of the complex tangent hyperplane to $\tilde{B}$ at the point $T(\xi,\xi)$ by the affine map $w \to T(\xi, w)$. Hence, by the general Koppelman Cauchy's formula, see [3, p. 28] or [**10**, Theorem 16.5.4], we have for any $z \in \tilde{B}$:

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_{\partial D} \frac{f(\xi)}{\langle s(\xi), z - \xi \rangle^n} \sum_{k=1}^{n} (-1)^{k-1} s_k(\xi)\, ds_{[k]} \wedge d\xi \tag{7}$$

where $ds_{[k]} = ds_1 \wedge \cdots \wedge ds_{k-1} \wedge ds_{k+1} \wedge \cdots \wedge ds_n$. This last formula leads to a convenient remainder formula.

Let $A = \{a_0, a_1, \dots, a_d\}$ be points in F. The Kergin interpolation polynomial is well defined since f is holomorphic on the convex set $\tilde{B}$ and for any $z \in \tilde{B}$ we have

$$f(z)-K_A(f)(z)=$$

$$\frac{(n-1)!}{(2i\pi)^n}\int_{\partial D}\prod_{j=0}^{d}\frac{\langle s(\xi),z-a_i\rangle}{\langle s(\xi),\xi-a_j\rangle}\times\sum_{|\alpha|+\beta=n-1}f(\xi)\sum_{k=1}^{n}\frac{(-1)^{k-1}s_k(\xi)\,ds_{[k]}\wedge d\xi}{\langle s(\xi),\xi-a\rangle^{\alpha}\langle s(\xi),\xi-z\rangle^{\beta}}. \tag{8}$$

This formula can be proved by interpolating the holomorphic kernel in (7), as is done in [2] or [5], taking into account that E and F are included in $\tilde{B}$ and that $f\to K_A(f)$ is a continuous linear map on $H(\tilde{B})$.

Finally to prove that $(E,F,G(E,F))$ holds, we can proceed as in the proof of Proposition 2, by using the remainder formula (8) and remarking that the function $(\xi,t)\to\langle s(\xi),\xi-t\rangle$ is continuous and does not vanish on the compact set $\partial D\times B(v,r)$. The proposition is proved. □

Remark 2. In the one dimensional case there is only a complex norm (except for multiplication by a positive scalar), no hypothesis on the compact sets are needed and the set $F(E,G)$ (respectively $E(F,G),G(E,F)$) is optimum that is, cannot be enlarged (respectively enlarged, diminished) without losing the property (E,F,G), see [**8**, 1.3.5]. In multidimensional case, optimality can be proved only for very particular compact sets; see the examples below.

3. Examples

We just give two examples for which some optimality is achieved.

Proposition 4. *Let N be a complex norm. Let r, s, t be three positive numbers such that $2s+r=t$ and define $E=\bar{B}_N(0,r)$, $F=\bar{B}_N(0,s)$ and $G=\bar{B}_N(0,t)$ then (E,F,G) holds optimally, i.e., if $E'\supset E, E'\neq E$ (respectively $F'\supset F, F'\neq F$; $G'\subset G$, $G\neq G'$) then (E',F,G) (respectively (E,F',G); (E,F,G')) no longer holds.*

Proof. That (E,F,G) holds follows from an application of Proposition 2. Let us prove for example that given r and t, if $N(p)>(t-r)/2$ then $(E,F\cup\{p\},G)$ does not hold. Let l be the complex line passing through p and 0. Then $l\cap E$, $l\cap F$, $l\cap G$ are three discs with centre 0 and radius respectively r, s, t in the one dimensional complex space l normed by the restriction of N. Let us denote these discs respectively by $D(r)$, $D(s)$, $D(t)$. Next, let us choose a one variable function f holomorphic in a disc $D(r')$ with $t<r'<N(p)+r$ but not in any larger domain.

In view of Cartan's theorem, see [7], the function f can be extended to a function still denoted by f, holomorphic in a neighbourhood of G. The Taylor expansion of f at the point p cannot converge to f uniformly on E otherwise the one dimensional Taylor expansion at the point p of f restricted to l would converge on a disc containing $D(r)$ hence also somewhere outside $D(r')$ which would be a contradiction. Since Taylor polynomials are Kergin interpolation polynomials, see (K3), the claim is proved. □

Proposition 5. *Let* $I=[-1,+1]\subset\mathbb{R}\subset\mathbb{C}$ *and* $a_i\in[0,1]$, $i=0,\ldots,n$. *Define* $E=I^n\subset\mathbb{R}^n\subset\mathbb{C}^n$, $F=\times_{i=1,\ldots,n}[-a_i,a_i]$ *and*

$$G=\{z\in\mathbb{C}^n/\exists p\in E/|z_i-p_i|\leqq 1+a_i,\quad i=1,\ldots,n\}$$

then (i) (E,F,G) *holds and* (ii) G *is the smallest convex set with this property.*

Proof. To prove that (E,F,G) holds we may apply Corollary 2 by using the norm $N(z)=\max_{i=1,\ldots,n}|z_i|/(1+a_i)$. Next, an inspection of the functions $1/(\pm a_i+z_i)$ and their Taylor expansion at points $(0,\ldots,0,\ \pm a_i,0,\ldots,0)$ whose coordinates are only 0 except at the ith place, shows that G must contains the product of the sets

$$D_i=\left\{t\in\mathbb{C}\Big/\left|\frac{t-p}{u-p}\right|,u,p\in I\right\}$$

and since G is the convex hull of $D_1\times D_2\ldots\times D_n$ we are done. □

We note that, by making use of the Remark 1 (ii), we obtain similar results when $[-1,1]$ and $[-a_i,a_i]$ are replaced by any concentric intervals.

Remark 3. When N is the Euclidean norm the result in Proposition 4 has been first given by Bloom and Bos in [6]. They also proved (i) in Proposition 5 in a different way.

Remark 4. Let E be symmetric with respect to 0. Let R be the N-diameter of E. Then by arguing as in the proof of Proposition 4, we can prove that $(E,E,B(0,R))$ holds and R is the smallest radius with this property.

Remark 5. Suppose that E and F (for example) lie in a complex subspace Π of $\mathbb{C}^n$ of dimension m and let N be any norm on Π then if we construct $G(E,F)\subset\Pi$, $(E,F,G(E,F))$ holds in $\mathbb{C}^n$. In particular the one dimensional solution of the problem leads to optimal solution in $\mathbb{C}^n$ for compact sets lying on a complex line.

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Laboratoire D'analyse Complexe
Universite Paul Sabatier
118, Route de Narbonne
31062 Toulouse Cedex
France
E-mail: Calvi @ cix.cict. fr