PRIMITIVE HOMOMORPHIC IMAGES
OF SEMIGROUPS

T. E. HALL

1. Introduction and summary

Consideration of the primitive homomorphic images of semigroups arises naturally from their matrix representation theory. (See, for example, papers [1] and [2].) This is basically because a semigroup of matrices, in which all the matrices have the same rank, is necessarily primitive. (The set of idempotents in the semigroup may be empty, however).

In his paper [2], Preston has shown (lemma 3) that a necessary condition for a regular semigroup to have a 0-restricted primitive homomorphic image is that the semigroup be categorical at zero, and (theorem 5) that for an inverse semigroup, this condition is also sufficient. (The terms "primitive", "categorical at zero", and "0-restricted" are defined below.) Furthermore, for an inverse semigroup $S = S^0$, the finest congruence giving a 0-restricted primitive homomorphic image of S may be written as

$$\pi = \{(x, y) \in S \times S : ax = ay \neq 0, \ xb = yb \neq 0$$
$$\text{for some } a, b \in S\} \cup \{(0, 0)\}.$$

(See [2], theorem 5, and the remark at the end of the proof.)

In this paper, we show, by a direct method, that if a semigroup $S = S^0$ is categorical at zero, then it has a maximal 0-restricted primitive homomorphic image; if S is also regular, then this image is S/π^*, where π^* is the congruence on S generated by the relation π above.

2. Semigroups which are categorical at zero

We repeat firstly some definitions from [2]. A semigroup $S = S^0$ is *primitive* if each of its non-zero idempotents is primitive. A semigroup $S = S^0$ is *categorical at zero* if $abc = 0$ implies either $ab = 0$ or $bc = 0$, for any elements a, b, c of S. A congruence ρ on a semigroup $S = S^0$ is *0-restricted* if $\{0\}$ is a ρ-class. A homomorphism ϕ of $S = S^0$ is *0-restricted* if the congruence $\phi \circ \phi^{-1}$ naturally induced on S by ϕ is 0-restricted.

Let $S = S^0$ be a semigroup which is categorical at zero.
Define

(1)
$$\rho_1 = \{(e, f) \in S \times S : e^2 = e, f^2 = f, ef = fe = f \neq 0\},$$
$$\rho_1' = \rho_1 \cup \rho_1^{-1} \cup \iota_S,$$

where ι_S is the identity relation on S,

$$\rho_{1,c} = \{(x, y) \in S \times S : x = uav, y = ubv,$$
$$\text{for some } (a, b) \in \rho_1', \text{ and for some } u, v \in S^1.\}$$

and

$$\rho_1^* = \rho_{1,c}^t = \rho_{1,c} \cup (\rho_{1,c} \circ \rho_{1,c}) \cup (\rho_{1,c} \circ \rho_{1,c} \circ \rho_{1,c}) \cup \cdots.$$

ρ_1^* is the congruence on S generated by the relation ρ_1. ([3], p. 18). We show
now that ρ_1^* is 0-restricted. We show first that

(i)   if $(e, f) \in \rho_1$, and $u, v \in S$, then $uev = 0$ if and only if $ufv = 0$.

Suppose $uev = 0$. Then since S is categorical at zero,

$$ue = 0 \quad \text{or} \quad ev = 0,$$

which implies

$$uf = uef = 0 \quad \text{or} \quad fv = fev = 0,$$

i.e.

$$ufv = 0.$$

Suppose $ufv = 0$. Then

$$uf = 0 \quad \text{or} \quad fv = 0,$$

i.e.

$$uef = 0 \quad \text{or} \quad fev = 0,$$

which means

$$ue = 0 \quad \text{or} \quad ev = 0$$

since $ef = fe = f \neq 0$ i.e.

$$uev = 0.$$

Setting $v = e$ we obtain

(ii) $ue = 0$ if and only if $uf = 0$, for any $(e, f) \in \rho_1, u \in S$,

and setting $u = e$ we obtain

(iii) $ev = 0$ if and only if $fv = 0$, for any $(e, f) \in \rho_1, v \in S$.

From (i), (ii) and (iii) we easily see that $uav = 0$ if and only if $ubv = 0$,
for any $(a, b) \in \rho_1', u, v \in S^1$. This means that $(0, 0)$ is the only pair in $\rho_{1,c}$
which contains 0, and this then is also true of ρ_1^*, i.e., ρ_1^* is 0-restricted.
S/ρ_1^* is thus also categorical at zero, but is not necessarily primitive.
Define now by induction ρ_{m+1}^*, the congruence generated by

(2) $\rho_{m+1} = \{(e, f) \in S \times S : e^2 \rho_m^* e, f^2 \rho_m^* f, ef \rho_m^* fe \rho_m^* f \neq 0\}.$

Then by using the method above, and considering the images of e, f, u, v in S/ρ_m^*, where $(e, f) \in \rho_{m+1}$, $u, v \in S$, we can easily show that ρ_{m+1}^* is 0-restricted, if we assume ρ_m^* to be 0-restricted. By induction we have that ρ_n^* is 0-restricted for all natural numbers n. Define now

$$(3) \qquad\qquad \rho = \bigcup_{n=1}^{\infty} \rho_n^*.$$

Clearly ρ is 0-restricted and, since $\rho_n^* \subseteq \rho_{n+1}^*$ for each n, is also a congruence. It is an easy matter to show that S/ρ is primitive; for let us consider any idempotents E, $F \in S/\rho$ such that $EF = FE = F \neq 0$. Choose any elements $e \in E$, $f \in F$. Then $e\rho e^2$, $f\rho f^2$, and $ef\rho fe\rho f \neq 0$. Since $\rho = \bigcup_{n=1}^{\infty} \rho_n^*$, there exist numbers n_1, n_2, n_3, n_4, such that $e\,\rho_{n_1}^*\,e^2$, $f\,\rho_{n_2}^*\,f^2$, $ef\,\rho_{n_3}^*\,fe$ and $fe\,\rho_{n_4}^*\,f$. Choosing $i = \max\{n_1, n_2, n_3, n_4\}$, and using $\rho_n^* \subseteq \rho_{n+1}^*$, we obtain $e\,\rho_i^*\,e^2$, $f\,\rho_i^*\,f^2$, $ef\,\rho_i^*\,fe\,\rho_i^*\,f \neq 0$, whence $(e, f) \in \rho_{i+1} \subseteq \rho$, i.e. $E = F$, and so S/ρ is primitive.

If β is a 0-restricted primitive congruence on S, it must contain ρ_1^*, and if it contains ρ_m^*, it must contain ρ_{m+1}^*, and so by induction it contains ρ. We have thus proved the following result.

THEOREM 1. *Let $S = S^0$ be a semigroup, categorical at zero. Then the relation ρ, defined by the equations* (1), (2) *and* (3), *is the finest 0-restricted primitive congruence on S.*

3. Semigroups which are regular and categorical at zero

Let $S = S^0$ be a semigroup which is regular and categorical at zero. Define on S the congruences ρ_1^* and ρ as before. Also define

$$(4) \qquad \pi = \{(x, y) \in S \times S : ax = ay \neq 0, xb = yb \neq 0, \text{ for some } a, b \in S\}$$

and denote by π^* the congruence on S generated by π.

THEOREM 2. *Let $S = S^0$ be a regular semigroup, categorical at zero. Then $\rho = \rho_1^* = \pi^*$, as defined by equations* (1), (2), (3) *and* (4); *whence each is the finest 0-restricted primitive congruence on S.*

PROOF. We show first that $\rho = \rho_1^*$. For this it will suffice to show that S/ρ_1^* is primitive. Take any elements E, F of S/ρ_1^* such that $E^2 = E$, $F^2 = F$ and $EF = FE = F \neq 0$. Since $E^2 = E$, $F^2 = F$, from Lallement's result, lemma 2.2, paper [4], there exist idempotents $e, f \in S$, such that $e \in E$, $f \in F$. We thus have $e^2 = e$, $f^2 = f$, and $ef\,\rho_1^*\,fe\,\rho_1^*\,f \neq 0$. Let $(ef)'$ be an inverse of ef. Then $ef(ef)'ef = ef \neq 0$ and so $ef(ef)'e \neq 0$. Now

$$[ef(ef)'e][ef(ef)'e] = ef(ef)'e$$

and

$$e[ef(ef)'e] = [ef(ef)'e]e = [ef(ef)'e] \neq 0.$$

Thus

$$e\, \rho_1^*\, ef(ef)'e,$$

whence

$$fe\, \rho_1^*\, fef(ef)'e\, \rho_1^*\, f(ef)'e\, \rho_1^*\, ef(ef)'e\, \rho_1^*\, e.$$

But $fe\, \rho_1^*\, f$, so $f\, \rho_1^*\, e$, i.e. $E = F$, and S/ρ_1^* is primitive, and so we have

$$\rho = \rho_1^*.$$

Now choose any pair $(x, y) \in \pi$. Then there exist elements $a, b \in S$ such that $ax = ay \neq 0$, and $xb = yb \neq 0$. Also there exist elements A, B, X, Y in S/ρ_1^* such that $a \in A$, $b \in B$, $x \in X$, $y \in Y$, and then $AX = AY \neq 0$ and $XB = YB \neq 0$ since ρ_1^* is 0-restricted. Since S/ρ_1^* is primitive and regular, it is the 0-direct union of a set of completely 0-simple subsemigroups, from Preston's paper [2], theorem 1. Clearly A, B, X, Y are in the same completely 0-simple subsemigroup, and this means that $X = Y$, i.e.

$$(x, y) \in \rho_1^*.$$

Therefore

$$\pi \subseteqq \rho_1^*,$$

whence

$$\pi^* \subseteqq \rho_1^*.$$

But clearly $\rho_1 \subseteqq \pi$, whence $\rho_1^* \subseteqq \pi^*$. Therefore

$$\pi^* = \rho_1^*.$$

This completes the proof of the theorem.

By a method similar to that used above, it can also be shown that for periodic semigroups which are categorical at zero,

$$\rho = \rho_1^*.$$

Acknowledgement

Note added in proof. Since this paper was submitted, the paper 'Matrix representations of semigroups' by D. B. McAlister (*Glasgow Math. J.* 8 (1967), 1—13), has appeared. It contains strongly related results.

References

[1] W. D. Munn, Matrix representations of inverse semigroups, *Proc. London Math. Soc.* 14 (1964), 165—181.

[2] G. B. Preston, Matrix representations of inverse semigroups, *J. Aust. Math. Soc.* (to appear).

[3] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups* (Math. Surveys, number 7, Amer. Math. Soc. 1961).

[4] G. Lallement, Congruences et équivalences de Green sur un demi-groupe régulier, *C. R. Acad. Sc. Paris*, Série A, **262** (1966), 613—616.

Department of Mathematics
Monash University, Clayton