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Topology at infinity of polynomial mappings and Thom regularity condition

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Abstract. We consider polynomial mappings which have atypical fibres due to the asymptotic behavior at infinity. Fixing some proper extension of the polynomial mapping, we study the localizability at infinity of the variation of topology of fibres and the possibility of interpreting local results at infinity into global results. We prove local and global Bertini–Sard–Lefschetz type statements for noncompact spaces and nonproper mappings and we deduce results on the homotopy type or the connectivity of the fibres of polynomial mappings.

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1. Introduction

Let $f: \mathbb{C}^n \to \mathbb{C}$ be a polynomial mapping. A value $a \in \mathbb{C}$ is called *atypical* for f if the mapping f is not topologically trivial over any disc centered at a. The set of atypical values is known to be finite and may be different from the set of *critical* values of f. We start giving a general setup for investigating the topology of complex polynomial mappings which have atypical values produced by so-called 'singularities at infinity'.

1.1. DEFINITION. Let $X \subset \mathbb{C}^n$ be an irreducible affine variety over $\mathbb{C}$ and let $f: X \to \mathbb{C}$ be algebraic. We say that $(\hat{f}, \mathbf{Y}, \mathcal{Z})$ is a *fibre-compactifying extension* (abbreviated *f.c.e.*) of f if $\hat{f}: \mathbf{Y} \to \mathbb{C}$ is an algebraic proper morphism which extends f, such that $\mathbf{Y}\backslash X$ is a Cartier divisor on the algebraic variety $\mathbf{Y}$ and $\mathcal{Z}$ is a smooth complex manifold containing $\mathbf{Y}$. We shall denote $\mathbf{Y}^\infty := \mathbf{Y}\backslash X$ and call it *the divisor at infinity*.

One may define similarly a f.c.e. for a mapping $g: X \to \mathbb{C}^p$ instead of a function f.

1.2. EXAMPLE. Let $G_f := \{(x,t) \in X \times \mathbb{C} \mid f(x) = t\}$ be the graph of f and let $\mathcal{C}$ be a compact smooth algebraic variety, $\mathcal{C} \supset X$. Let then $\mathbf{X} := \mathbf{X}(f, \mathcal{C})$ denote the Zariski closure of G_f in $\mathcal{C} \times \mathbb{C}$. If $\mathbf{X}\backslash X$ is a divisor, then $(t, \mathbf{X}, \mathcal{C} \times \mathbb{C})$ is a f.c.e. of f, where t denotes the restriction to $\mathbf{X}$ of the projection $\mathcal{C} \times \mathbb{C} \to \mathbb{C}$.

One may use the projective space $\mathbb{P}^n$ as the smooth compact $\mathcal{C}$. For $X = \mathbb{C}^n$, this was considered in several papers, e.g. [Br], [Di], [Pa-1], [ST] and [Pa-2]; notice that $\mathbf{X}(f, \mathbb{P}^n)$ is then a hypersurface in $\mathbb{P}^n \times \mathbb{C}$. Instead of $\mathbb{P}^n$, one might use a smooth toric compactification of $\mathbb{C}^n$ based on a subdivision into nonsingular cones of the Newton polyhedron at infinity of f, see [Ku], [Oka] and [Br].

The approach we propose relies on the study of the position of the levels of f relative to the nonzero levels of functions which locally define the divisor at infinity in some fibre-compactifying extension.

We shall introduce in this paper the *partial Thom stratification* (Definition 2.1) which is less demanding than the Whitney stratification, since it does not require Whitney (b) condition. Thom stratifications are really weaker than Whitney ones; we may send the reader to the famous example due to Briançon and Speder and to Bekka's remark in [Be, Introduction]. We shall see that ∂T-stratifications are also less restrictive than the (C)-regular stratifications introduced by Bekka [Be]. Nevertheless they are a good enough context to prove a basic local isotopy theorem for a nonproper mapping (Theorem 2.5) which may replace Thom's First Isotopy Lemma. As a byproduct, we give an entirely topological proof (Theorem 2.9) of a result due to Briançon, Maisonobe and Merle [BMM, Théorème 4.2.1] on the Thom regularity condition, whose original proof was based on $\mathcal{D}$-module techniques.

We show that ∂T-stratifications appear naturally at infinity and that there is a *canonical ∂T-stratification at* $\mathbf{Y}^\infty$, in case X is a smooth complex affine variety with isolated singularities (Theorem 3.6). Actually, this stratification is 'induced' by the *space of characteristic covectors at infinity* (Definition 3.4).

In Section 3 we define *localizability of the variation of topology of fibres* and prove that, for polynomial functions with *isolated $\mathcal{G}$-singularities* (Definition 4.2), where $\mathcal{G}$ is a ∂T-stratification at infinity, the variation of topology of the fibres is localizable (Theorem 4.3). The use of ∂T-stratifications allows us to prove, via Theorem 2.5, the existence of the *local monodromy* and *local variation mapping* of f at some isolated $\mathcal{G}$-singularity at infinity. These objects depend, of course, on the chosen f.c.e. of f.

We define the singular locus of a mapping with respect to a ∂T-stratification at infinity which satisfies Whitney (a) property, and use this in order to prove and apply Bertini–Sard–Lefschetz type results adapted to our affine situation. In case of a smooth complex affine X and $f: X \to \mathbb{C}$ with isolated $\mathcal{G}$-singularities we show that, up to the homotopy type, X can be built from a general fibre of f by attaching a certain number of cells, all of dimension $= \dim X$ (see Theorem 4.6 for a more general statement).

As another application of our approach, we prove that, for an *affine local complete intersection with isolated $\mathcal{G}$-singularities* $g: \mathbb{C}^{n+p} \to \mathbb{C}^p$ (Definition 6.1), the general fibre has the homotopy type of a bouquet of spheres (Theorem 6.2).

We also find a connectivity result (Theorem 5.5) on the fibres of any polynomial $f: \mathbb{C}^n \to \mathbb{C}$, in terms of its singularities with respect to a ∂T-stratification at $\mathbf{Y}^\infty$ with Whitney (a) property. We discuss in Section 3 an interesting example of a polynomial, from the point of view of its singularities and vanishing cyles at infinity.

Our point of view and results extend in particular the ones of [ST] – which were based on Whitney stratifications. Although the results in this paper concern complex mappings, some of them treat the real case too (e.g. Theorem 2.5). For some developments in the real case see [Ti-2].

2. ∂T-Stratifications and a local isotopy theorem

We introduce a stratification which is weaker than a Whitney stratification but will still allow us to prove a local isotopy theorem which will be used in the next section to investigate the topology at infinity. In the beginning, the setting is both real and complex analytic, but later we stick to the complex case.

Let $\mathcal{X}$ be a $\mathbb{K}$-analytic space and let $g: \mathcal{X} \to \mathbb{K}$ be a $\mathbb{K}$-analytic function, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Suppose that $\mathcal{X}$ is endowed with a complex (resp. real) stratification $\mathcal{G} = \{\mathcal{G}_\alpha\}_{\alpha\in\Lambda}$ such that $g^{-1}(0)$ is a union of strata. If $\mathcal{G}_\alpha \cap \overline{\mathcal{G}_\beta} \neq \emptyset$ then, by definition, $\mathcal{G}_\alpha \subset \overline{\mathcal{G}_\beta}$ and in this case we write $\mathcal{G}_\alpha < \mathcal{G}_\beta$.

2.1. DEFINITION. We say that $\mathcal{G}$ is a ∂T-stratification (*partial Thom stratification*) relative to g if the following condition is satisfied:

($\partial \mathbf{a}_g$) any two strata $\mathcal{G}_\alpha < \mathcal{G}_\beta$ with $\mathcal{G}_\alpha \subset g^{-1}(0)$ and $\mathcal{G}_\beta \subset \mathcal{X}\backslash g^{-1}(0)$ satisfy the Thom (a_g) regularity condition.

For the definition of the Thom (a_g) condition, one can consult for instance [GWPL, ch. I]. Local ∂T-stratifications exist since local Thom stratifications relative to a function exist, see e.g. [HL-1, Théorème 1.2.1], [Hi], [Be] (‘local’ means in some neighbourhood of a point x, for any $x \in \mathcal{X}$). But ∂T-stratifications are less demanding than Thom (or Thom-Whitney) stratifications. We give bellow an equivalent definition, in terms of the relative conormal.

2.2. Let $\mathcal{X} \subset \mathbb{K}^N$ be a $\mathbb{K}$-analytic variety. (In the real case, assume that $\mathcal{X}$ contains at least a regular point.) Let $U \subset \mathbb{K}^N$ be an open set and let $g: \mathcal{X} \cap U \to \mathbb{K}$ be $\mathbb{K}$-analytic. One defines the *relative conormal* $T^*_{g|\mathcal{X}\cap U} \subset T^*(\mathbb{K}^N)_{|\mathcal{X}\cap U}$, as follows, see [Te], [HMS]

$$T^*_{g|\mathcal{X}\cap U} := \text{closure}\{(y,\xi) \in T^*(\mathbb{K}^N) \mid y \in \mathcal{X}^0 \cap U, \xi(T_y(g^{-1}(g(y))) = 0\},$$

where $\mathcal{X}^0 \subset \mathcal{X}$ is the open dense subset of regular points of $\mathcal{X}$ where g is a submersion. The relative conormal is conical (i.e. $(y,\xi) \in T^*_{g|\mathcal{X}\cap U} \Rightarrow (y,\lambda\xi) \in T^*_{g|\mathcal{X}\cap U}, \forall\lambda \in \mathbb{K}^*$).

Let now $\mathcal{G}_\alpha < \mathcal{G}_\beta$ be two strata of a stratification $\mathcal{G}$ and let $(T^*_{g|\overline{\mathcal{G}_\beta}})_x := T^*_{g|\overline{\mathcal{G}_\beta}} \cap \pi^{-1}(x)$, where $\pi: T^*(\mathbb{K}^N) \to \mathbb{K}^N$ denotes the canonical projection. Then, locally at x, the condition $(\partial \mathrm{a}_g)$ for the strata $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ translates to

$$T^*_{\mathcal{G}_\alpha} \supset (T^*_{g|\overline{\mathcal{G}_\beta}})_x, \tag{1}$$

where $T^*_{\mathcal{G}_\alpha} \subset T^*(\mathbb{K}^N)$ denotes the conormal to the stratum $\mathcal{G}_\alpha$.

2.3. DEFINITION. Let $\mathcal{X} \subset \mathbb{K}^N$ be a $\mathbb{K}$-analytic set endowed with a $\mathbb{K}$-analytic stratification $\mathcal{S} = \{\mathcal{S}_i\}_{i\in \mathrm{I}}$ which satisfies Whitney (a) property.

For a $\mathbb{K}$-analytic mapping $g = (g_1, \ldots, g_p): \mathcal{X} \to \mathbb{K}^p$, we define the *critical locus* of g with respect to $\mathcal{S}$ by

$$\mathrm{Sing}_{\mathcal{S}}\, g := \bigcup_{i\in \mathrm{I}} \mathrm{Sing}\, g_{|\mathcal{S}_i}.$$

Then $\mathrm{Sing}\, g$ is a closed subset of $\mathcal{X}$.

For $\mathbb{K}$-analytic functions $f, h: \mathcal{X} \to \mathbb{K}$ we say that the set

$$\Gamma_{\mathcal{S}}(f,h) := \mathrm{closure}\{\mathrm{Sing}_{\mathcal{S}}(f,h) \backslash (\mathrm{Sing}_{\mathcal{S}} f \cup \mathrm{Sing}_{\mathcal{S}} h)\}$$

is the *polar locus* with respect to $\mathcal{S}$.

Let $\check{\mathbb{P}}^{N-1}$ be the set of all hyperplanes of $\mathbb{P}^{N-1}$. A hyperplane $H \in \check{\mathbb{P}}^{N-1}$ is defined by a linear form $l_H: \mathbb{K}^N \to \mathbb{K}$ and we may occasionally identify these two objects. We have the following useful Bertini–Sard–Lefschetz type result, not easy to be found in the literature (we know of no reference in the real case).

2.4. LEMMA (Polar curve theorem). *Let $\mathcal{X} \subset \mathbb{K}^N$ be analytic or algebraic and let $f: \mathcal{X} \to \mathbb{K}$ be analytic or algebraic. Let $\mathcal{S} = \{\mathcal{S}_i\}_{i\in \mathrm{I}}$ be a finite family which stratifies $\mathcal{X}$ with Whitney* (a) *condition. Then there is an open $\Omega_f \subset \check{\mathbb{P}}^{N-1}$* (*Zariski-open in the complex case, resp. dense in the real analytic case*) *such that, for any $H \in \Omega_f$, $\Gamma_{\mathcal{S}}(l_H, f)$ is a curve or it is void.*

Proof. For any stratum $\mathcal{S}_i$ of dimension $\geqslant 1$ we consider the projectivised relative conormal $\mathbb{P}T^*_{f|\mathcal{S}_i} \subset \mathbb{P}T^*(\mathbb{K}^N)$. We identify $\mathbb{P}T^*(\mathbb{K}^N)$ with $\mathbb{K}^N \times \check{\mathbb{P}}^{N-1}$ and we denote by C_i the closure of $\mathbb{P}T^*_{f|\mathcal{S}_i}$ within $\mathbb{P}^N \times \check{\mathbb{P}}^{N-1}$. For any $i \in I$, let π be the projection on the first factor and γ the one on the second. Then the polar locus $\Gamma(l_H, h)$ is included in $\pi(\gamma^{-1}(H))$, for some $H \in \check{\mathbb{P}}^{N-1}$.

Since C_i is compact analytic, resp. algebraic, we may apply Verdier's [V, Théorème 3.3] to the proper morphism $\gamma: C_i \to \check{\mathbb{P}}^{N-1}$ to find an open $\Omega_i \subset \check{\mathbb{P}}^{N-1}$ with the properties claimed for Ω_f and such that $\forall H \in \Omega_i$, H is a regular value of the restriction $\gamma_{|(C_i)_{\mathrm{reg}}}$. Since $\forall i \in \mathrm{I}$, $\dim (C_i)_{\mathrm{reg}} = N$, it follows that either $\dim \gamma^{-1}(H) = 1$ or $\gamma^{-1}(H) = \emptyset$. We conclude by defining Ω_f as being the intersection $\cap_{i\in \mathrm{I}} \Omega_i$. □

The basic isotopy theorem we prove is the following.

2.5. THEOREM (open local isotopy). *Let* $(\mathcal{X}, x) \subset (\mathbb{K}^N, x)$ *be a* $\mathbb{K}$*-analytic irreducible space germ,* $\dim_x \mathcal{X} \geqslant 2$. *Denote by* B_ε *the open ball centered at* x*, of radius* ε *and by* $D_\eta \subset \mathbb{K}$ *the open disc* (*resp. interval*) *centered at* 0*, of radius* η*. Let* $h\colon (\mathcal{X}, x) \to (\mathbb{K}, 0)$ *be an analytic mapping transversal to a* ∂T*-stratification relative to* g*, denoted by* $\mathcal{G}$*. Suppose moreover that* $\mathcal{X}\backslash g^{-1}(0)$ *is smooth. Then for any* $\varepsilon > 0$ *small enough and* $0 < \delta \ll \varepsilon$*, the following restriction of* h

$$h_|\colon B_\varepsilon \cap h^{-1}(D_\delta) \cap (\mathcal{X}\backslash g^{-1}(0)) \to D_\delta$$

is a topologically trivial fibration.

Proof. $\mathcal{G}$ may fail to be a Whitney stratification, hence one cannot apply Thom's First Isotopy Lemma. Also $\mathcal{G}$ might not be (C)-regular in the sense of Bekka [Be, Sect. 2, Def. 1.1], nevertheless it is of a similar flavour.

The idea of proof is to lift by h the unit real (resp. complex) vector field $\partial/\partial t$ on D_δ to a vector field on $\mathcal{X}$ tangent to both $S_\varepsilon := \partial\overline{B_\varepsilon}$ and to all positive levels of g.

First choose $\varepsilon > 0$ such that for all $0 < \varepsilon' \leqslant \varepsilon$, the sphere $S_{\varepsilon'}$ is transversal to all strata of $\mathcal{G}$ and to all positive dimensional strata of the stratification induced by $\mathcal{G}$ on $h^{-1}(0)$ (by e.g. [HL-1, Th. 1.3.2] or [Loo, Lem. 2.2]). Our hypothesis on h implies that the levels of h are transversal to the levels of g different from 0, at any point of $B_\varepsilon\backslash g^{-1}(0)$, for ε small enough. We may therefore define a continuous vector field **v** on $B_\varepsilon \cap h^{-1}(D_\delta) \cap (\mathcal{X}\backslash g^{-1}(0))$, without zero, which is a pull-back of the unit vector field $\partial/\partial t$ on D_δ and is tangent to the levels of g. This follows for instance from the fact that $\mathcal{G}$ is so to say 'partially' (C)-regular, in the sense that $|g|^2$ is a control function for two strata $\mathcal{G}_\alpha < \mathcal{G}_\beta$ satisfying $\mathcal{G}_\alpha \subset g^{-1}(0)$ and $\mathcal{G}_\beta \subset \mathcal{X}\backslash g^{-1}(0)$. Therefore we may still construct a continuous vector field **v** like done by Bekka in [Be, p. 61, point 1].

But **v** might not be tangent to the sphere S_ε, so we have to modify it in the neighbourhood of $S_\varepsilon \cap h^{-1}(D_\delta)$. We need the following.

2.6. LEMMA. *With the same notations of Theorem* 2.5*, suppose that* $h\colon (\mathcal{X}, x) \to (\mathbb{K}, 0)$ *is transversal to* $\mathcal{G}$ *except possibly at* x*. Then for some* $0 < \alpha \ll \varepsilon$*,* $0 < \delta \ll \varepsilon$ *we have that* $h^{-1}(h(q))$ *is transversal to* $S_\varepsilon \cap g^{-1}(g(q))$*, for any* $q \in S_\varepsilon \cap h^{-1}(D_\delta) \cap g^{-1}(D^*_\alpha)$.

Proof. Our hypothesis implies that $\Gamma_\mathcal{G}(g, h) \cap g^{-1}(0) \subset \{x\}$. Indeed, if $y \in g^{-1}(0)$, $y \neq x$, then h is transversal to the level of g at any point within $\mathcal{N}_y \cap \mathcal{X}\backslash g^{-1}(0)$, where $\mathcal{N}_y$ is some small enough neighbourhood of y. It follows that $\Gamma_\mathcal{G}(g, h) \cap S_\varepsilon \cap g^{-1}(D_\alpha) = \emptyset$, for small enough ε and $0 < \alpha \ll \varepsilon$. We recall that ε was chosen small enough such that S_ε is transversal to all positive dimensional strata of the form $h^{-1}(0) \cap \mathcal{G}_\beta$, with $\mathcal{G}_\beta \in \mathcal{G}$.

If the conclusion of this lemma is not true, then there would exist a sequence of points $p_i \in S_\varepsilon \cap g^{-1}(D^*_\alpha) \subset \mathcal{X} \backslash g^{-1}(0)$, with $p_i \to p \in S_\varepsilon \cap h^{-1}(0) \cap g^{-1}(0)$, such that the intersection of tangent spaces $T_{p_i} h^{-1}(h(p_i)) \cap T_{p_i} g^{-1}(g(p_i))$ is contained in $T_{p_i}(S_\varepsilon \cap \mathcal{X})$. Then, provided that the following limits exist (which one may assume without loss of generality), we would have

$$\lim T_{p_i} h^{-1}(h(p_i)) \cap \lim T_{p_i} g^{-1}(g(p_i)) \subset T_p S_\varepsilon. \tag{2}$$

On the other hand, let $\mathcal{G}_\phi \subset g^{-1}(0)$ be the stratum containing p. We have that $\lim T_{p_i} h^{-1}(h(p_i)) \supset T_p \mathcal{G}_\phi$, since $\mathcal{G}$ is a ∂T-stratification. Now since h is transversal to $\mathcal{G}_\phi$ and since $p \in h^{-1}(0)$, the set $h^{-1}(0) \cap \mathcal{G}_\phi$ must have positive dimension and we have assumed that S_ε is transversal to it. But this contradicts (2). □

We now complete the last part of the proof of Theorem 2.5. From Lemma 2.6, it follows that there exists a continuous vector field **w** without zero on some collar of $S_\varepsilon \cap h^{-1}(D_\delta) \cap g^{-1}(D^*_\alpha) \subset \overline{B_\varepsilon} \cap h^{-1}(D_\delta) \cap g^{-1}(D^*_\alpha)$, which is a lift of $\partial/\partial t$ by h and is tangent to $S_{\varepsilon''} \cap g^{-1}(p)$, for any $p \in D^*_\alpha$ and $\varepsilon' \leqslant \varepsilon'' \leqslant \varepsilon$, for some ε' close to ε.

Lastly, there exists another continuous vector field **u** without zero on $B_\varepsilon \cap h^{-1}(D_\delta) \backslash g^{-1}(D_\alpha)$ which is again a lift of $\partial/\partial t$ and is tangent to the sphere S_ε (by the choice of ε).

We then glue those three **u**, **v**, **w** by a partition of unity and get a vector field which trivializes the fibration $h_|$. □

We stick to the complex case during the rest of this section. We prove the following Lefschetz type result, which extends (from the point of view of the stratification involved in the statement) the result of Hamm and Lê [HL-2, Cor. 4.2.2].

2.7. COROLLARY. *Let $(\mathcal{X}, x)$ be a complex germ and let $h, g\colon (\mathcal{X}, x) \to (\mathbb{C}, 0)$ be complex functions such that h is transversal to a ∂T-stratification with respect to g, except possibly at x. Assume that $\mathcal{X} \backslash g^{-1}(0)$ is smooth of dimension n. Then for any ε small enough, any $0 < \delta \ll \varepsilon$ and any $\eta \in D^*_\delta$, the pair*

$$(B_\varepsilon \cap h^{-1}(D_\delta) \cap \mathcal{X} \backslash g^{-1}(0), B_\varepsilon \cap h^{-1}(\eta) \cap \mathcal{X} \backslash g^{-1}(0))$$

is $(n-1)$-connected.

Proof. Under the stated conditions, the following pair

$$(B_\varepsilon \cap h^{-1}(D_\delta) \cap \mathcal{X} \backslash g^{-1}(0), (S_\varepsilon \cap h^{-1}(D_\delta)) \cup (B_\varepsilon \cap h^{-1}(\eta) \cap \mathcal{X} \backslash g^{-1}(0)))$$

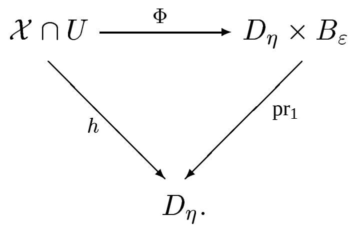
is $(n-1)$-connected, by [HL-2, Th. 4.2.1]. See also the proof of [*loc. cit.*], since we have given an equivalent (slightly different) form of their result.

Now by our Lemma 2.6, the restriction of h to $S_\varepsilon \cap h^{-1}(D_\delta) \cap \mathcal{X} \backslash g^{-1}(0)$ induces a topologically trivial fibration over D_δ. The conclusion follows. □

We end this section by a result concerning ∂T-stratifications, which is a kind of a converse of Theorem 2.5. Let us first define the local stratified triviality property, following [BMM, 4.1].

So let $\mathcal{X} \subset \mathbb{C}^N$ be a complex analytic set endowed with a stratification $\mathcal{G}$ having Whitney (a) condition. Let $\mathcal{G}_\alpha$ be a stratum, $\dim \mathcal{G}_\alpha > 0$, let $x \in \mathcal{G}_\alpha$ and let $h: (\mathbb{C}^N, x) \to (\mathbb{C}^p, 0)$ be a submersion transversal to $\mathcal{G}_\alpha$. Let D_η denote the open ball centered at $0 \in \mathbb{C}^p$, of radius η and let B_ε be the ball within $h^{-1}(0)$ centered at x, of radius ε. For $\varepsilon > 0$ small enough, $\mathcal{G}$ induces a Whitney (a) stratification on $h^{-1}(0)$; therefore B_ε is stratified by the intersections of $h^{-1}(0)$ with the strata of $\mathcal{G}$. Then $D_\eta \times B_\varepsilon$ is endowed with the product stratification.

2.8. DEFINITION. The stratification $\mathcal{G}$ of $\mathcal{X}$ satisfies the *local stratified triviality* property (abbreviated LST) if and only if for any point $x \in \mathcal{X}$ and any submersion $h: (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$ transversal to $\mathcal{G}_\alpha$ at x, where $x \in \mathcal{G}_\alpha$ and $0 < p \leqslant \dim \mathcal{G}_\alpha$, there are $\eta > 0$, $\varepsilon > 0$, a neighbourhood U of x within $\mathbb{C}^N$ and a stratified homeomorphism Φ such that the following diagram commutes



One notices that the LST property is preserved when slicing by hyperplanes transversal to the strata. Moreover, the LST property is preserved when restricting to $\overline{\mathcal{G}_\beta}$, where $\mathcal{G}_\beta$ is some stratum (since Φ is a stratified homeomorphism). It is verified for instance by Whitney stratifications (by Thom–Mather first isotopy lemma).

2.9. THEOREM. *Let $\mathcal{X} \subset \mathbb{C}^N$ be endowed with a complex stratification $\mathcal{G}$ which satisfies Whitney* (a) *condition, such that $g^{-1}(0)$ is a union of strata and that $\mathcal{G}$ verifies the* local stratified triviality *property* (*abbreviated* LST). *Then $\mathcal{G}$ is a ∂T-stratification* (*and hence a Thom* (a_g)*-stratification*).

This is a remarkable result due to Briançon, Maisonobe and Merle [BMM, Th. 4.2.1]. The original proof uses some important $\mathcal{D}$-modules technical results. We give bellow a self-contained proof using a polar curve argument.

Proof. By absurd, let $y \in g^{-1}(0)$ be a point where the $(\partial \mathrm{a}_g)$ condition for $\mathcal{G}$ fails. We then consider the locus $Y \subset g^{-1}(0)$ of all points within some neighbourhood of p where $(\partial \mathrm{a}_g)$ fails. Let $\mathcal{T}$ be some complex Thom (a_g)-stratification of a neighbourhood of y_0 in $\mathcal{X}$, which locally refines $\mathcal{G}$ and such that new strata occur only within Y (which is possible, by the definition of Y).

Let $\mathcal{T}_0$ be a stratum of $\mathcal{T}$ which is of maximal dimension among the strata which are included in Y and which contain y_0 in their closure. Let now $\mathcal{G}_0$ be a stratum of $\mathcal{G}$ of maximal dimension among those which intersect $\mathcal{T}_0$ and take some point $y_0 \in \mathcal{G}_0 \cap \mathcal{T}_0$. Notice that $\dim \mathcal{T}_0 < \dim \mathcal{G}_0$, by the choices we have made and the definition of Y. Denoting by p the dimension of $\mathcal{T}_0$, let V be a linear affine space in $\mathbb{C}^N$ such that $\dim V = N - p$, $V \ni y_0$, $V \pitchfork_{y_0} \mathcal{G}_0$ and that, within a small enough neighbourhood of y_0, $V \cap \mathcal{T}_0 = \{y_0\}$.

Let us fix some $\xi_0 \in (T^*_{g|\mathcal{X}})_{y_0}$ such that $\xi_0 \notin T^*_{\mathcal{G}_0}(\mathbb{C}^N)$. Then there is a Zariski-open dense subset of linear $(N-p)$-planes V through y_0 which verify the conditions imposed above and moreover have the property

$$(y, \xi_{0|V}) \notin T^*_{\mathcal{G}_0 \cap V}(V). \tag{3}$$

We shall fix such a V for the rest and we shall identify it with $\mathbb{C}^{N-p}$. Notice that the chosen V has the property that

$$(T^*_{g|\mathcal{X}\cap V})_x \subset T^*_{\mathcal{G}_x \cap V}(\mathbb{C}^{N-p}), \tag{4}$$

$\forall x \in g^{-1}(0) \cap V$ within some neighbourhood of y_0, $x \neq y_0$, where $\mathcal{G}_x$ denotes the stratum to which x belongs.

The stratification $\mathcal{G}$ induces on $\mathcal{X} \cap V$ a stratification $\mathcal{K}$ which also has Whitney (a) property. By (3), there is some stratum $\mathcal{K}_1$ of $\mathcal{K}$ such that

$$(T^*_{g|\overline{\mathcal{K}_1}})_{y_0} \not\subset T^*_{\mathcal{K}_0}(\mathbb{C}^{N-p}), \tag{5}$$

where $\mathcal{K}_0 = \mathcal{G}_0 \cap V$.

We identify ξ_0 with a linear form on $\mathbb{C}^N$ and denote by l its restriction to $\mathcal{K}_1$. We consider the germ at y_0 of the polar locus $\Gamma_{\mathcal{K}}(l, g_{|\overline{\mathcal{K}_1}})$. By the properties (4) and (5), we get $\Gamma_{\mathcal{K}}(l, g_{|\overline{\mathcal{K}_1}}) \cap g^{-1}(0) = \{y_0\}$. Since non void, this polar locus must be of positive dimension. This is so because in our case $\Gamma_{\mathcal{K}}(l, g_{|\overline{\mathcal{K}_1}}) = \text{closure}\{\pi_0(\gamma^{-1}(l))\}$, where $\pi_0\colon (\mathbb{P}T^*_{g|\overline{\mathcal{K}_1}})_{|\mathcal{K}_1} \to \mathcal{K}_1$ and $\gamma\colon \mathbb{P}T^*_{g|\overline{\mathcal{K}_1}} \to \check{\mathbb{P}}^{N-p-1}$ are the canonical projections and moreover the complex variety $\mathbb{P}T^*_{g|\overline{\mathcal{K}_1}}$ has dimension $N-p$ (see also the proof of Lemma 2.4). Then $\Gamma_{\mathcal{K}}(l, g_{|\overline{\mathcal{K}_1}})$ is precisely of dimension 1: if it were of dimension $d > 1$, then it would intersect the hypersurface $g^{-1}(0)$ along a $(d-1)$-dimensional set, which would contradict the fact that this intersection is just the point y_0.

The LST property inherited by $\mathcal{X} \cap V$ and then by $\overline{\mathcal{K}_1}$ implies that the complex links $\text{CL}(\overline{\mathcal{K}_1}, y_0)$ and $\text{CL}(\overline{\mathcal{K}_1} \cap g^{-1}(0), y_0)$ are contractible (see [GM] for the definition and properties of the complex link). We consider the germ at y_0 of the mapping $(l, g_|)\colon \overline{\mathcal{K}_1} \to \mathbb{C}^2$ (where $\dim \mathcal{K}_1 \geqslant 2$) and take a small enough polydisc neighbourhood $\mathcal{P}$ of y_0 (see [Lê-1] and also [Ti-1] for the

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definition). The slice $l^{-1}(l(y_0)+\varepsilon)\cap\mathcal{P}$, for $\varepsilon > 0$ small enough, is nothing else but the complex link $\mathrm{CL}(\overline{\mathcal{K}_1}, y_0)$. This slice is obtained, up to homotopy type, from $l^{-1}(l(y_0)+\varepsilon)\cap g^{-1}(0)\cap\mathcal{P}$ (which itself is just the complex link $\mathrm{CL}(\overline{\mathcal{K}_1}\cap g^{-1}(0), y_0)$), by attaching a finite number of cells of dimension $= \dim_{\mathbb{C}}\mathcal{K}_1 - 1$ (compare to [ST, proof of Prop. 4.5]). These cells come from the singularities of the function g on $l^{-1}(l(y_0)+\varepsilon)\cap\mathcal{P}$, which are exactly the points where the polar curve $\Gamma_{\mathcal{K}}(l, g_{|\overline{\mathcal{K}_1}})$ intersects $l^{-1}(l(y_0)+\varepsilon)\cap\mathcal{P}$. Since $\Gamma_{\mathcal{K}}(l, g_{|\overline{\mathcal{K}_1}})$ is a nonvoid curve, the number of these intersection points is greater than 0, hence the number of cells is also greater than zero (to one singular point there corresponds at least one cell). But this contradicts the contractibility of $\mathrm{CL}(\overline{\mathcal{K}_1}, y_0)$. □

3. ∂T-Stratifications and singularities at infinity

We shall now focus on an algebraic morphism $f\colon X\to\mathbb{C}$, where $X\subset\mathbb{C}^n$ is a complex affine variety.

3.1. DEFINITION. (∂T-stratifications at infinity). Let $(\hat{f}, \mathbf{Y}, \mathcal{Z})$ be a f.c.e. of f. If a stratification $\mathcal{S}$ of some neighbourhood of $\mathbf{Y}^\infty$ within $\mathbf{Y}$, with $\mathbf{Y}^\infty$ a union of strata, has the property that, locally at any $y\in\mathbf{Y}^\infty$, it is a ∂T-stratification with respect to g, where $g=0$ is some local equation of $\mathbf{Y}^\infty$ at y, then we call $\mathcal{S}$ a ∂T-stratification (at infinity) at $\mathbf{Y}^\infty$.

Partial Thom stratifications at infinity do exist. We may construct one as follows. Take a Whitney stratification of the algebraic space $\mathbf{Y}$, having $\mathbf{Y}^\infty$ as union of strata (which is possible, see e.g. [GWPL, 2.7]). Then Theorem 2.9 shows that this is also a ∂T-stratification at $\mathbf{Y}^\infty$.

We would like to construct some weaker stratification, not to involve Whitney conditions, but arising somehow naturally from the relative conormal. We first need some preliminary stuff.

3.2. PROPOSITION. *Let $(\hat{f}, \mathbf{Y}, \mathcal{Z})$ be a f.c.e. of f. Let $g=0$ be a reduced local equation of the divisor $\mathbf{Y}^\infty$ at y. Then $(T^*_{g|\mathbf{Y}\cap U})_y$ does not depend on the choice of g, where U is some open subset of $\mathcal{Z}$ and $T^*_{g|\mathbf{Y}\cap U}\subset T^*(U)$.*

The proof will clearly follow from the next lemma, which we prove in both real and complex cases, for later use (e.g. Theorem 4.3 or [Ti-2]).

3.3. LEMMA. *Let $(\mathcal{X}, x)\subset(\mathbb{K}^N, x)$ be a germ of a $\mathbb{K}$-analytic space and let $g\colon(\mathcal{X}, x)\to(\mathbb{K}, 0)$ be a $\mathbb{K}$-analytic function. Let $\gamma\colon\mathcal{X}\to\mathbb{K}$ be analytic such that $\gamma(x)\neq 0$ and denote by W some neighbourhood of x in $\mathbb{K}^N$. Then $(T^*_{g|\mathcal{X}\cap W})_x = (T^*_{\gamma g|\mathcal{X}\cap W})_x$.*

Proof. Let $g^{-1}(0)$ be denoted $\mathcal{Y}$. Suppose first that $(\mathcal{X}, x)$ is smooth. We have $\operatorname{grad} \gamma g = \gamma \operatorname{grad} g + g \operatorname{grad} \gamma$, hence

$$\frac{\operatorname{grad} \gamma g}{\|\operatorname{grad} g\|} = \gamma \frac{\operatorname{grad} g}{\|\operatorname{grad} g\|} + \operatorname{grad} \gamma \frac{g}{\|\operatorname{grad} g\|}.$$

Since γ is analytic, $\|\operatorname{grad} \gamma\|$ and γ are bounded within some neighbourhood of x. By the following Łojasiewicz inequality [Lo], valid in a neighbourhood of x

$$\|\operatorname{grad} g\| \geqslant |g|^{\theta}, \quad \text{for some } 1 > \theta > 0,$$

we get that $g/\|\operatorname{grad} g\|$ tends to zero as the point tends to x. Therefore, along any sequence of points tending to x, we have $\lim(\operatorname{grad} \gamma g/\|\operatorname{grad} g\|) = \gamma(x) \lim(\operatorname{grad} g/\|\operatorname{grad} g\|)$, hence the limits of the directions $\operatorname{grad} \gamma g$ and $\operatorname{grad} g$ are the same.

In the general case we resolve $\mathcal{X}$ within an embedded resolution $p\colon \tilde{\mathcal{X}} \to \mathcal{X}$, to a smooth variety $\tilde{\mathcal{X}}$. This is an isomorphism over $\mathcal{X}_{\text{reg}} \backslash \mathcal{Y}$. Now apply the result in the smooth case for the functions $g \circ p$ and $(\gamma \circ p)(g \circ p)$, then pull down to the conormal of $\mathcal{X}$, by the following diagram

$$\begin{array}{ccccc}
(T^*\mathbb{K}^N)_{|\tilde{\mathcal{X}}\cap p^{-1}(W)} & \hookleftarrow & p^*(T^*\mathbb{K}^N)_{|\mathcal{X}\cap W} & \longrightarrow & (T^*\mathbb{K}^N)_{|\mathcal{X}\cap W} \\
 & \searrow & \downarrow & & \downarrow{\scriptstyle \pi} \\
 & & \tilde{\mathcal{X}} \cap p^{-1}(W) & \xrightarrow{\;p\;} & \mathcal{X} \cap W.
\end{array}$$

3.4. DEFINITION. Let $y \in \mathbf{Y}^{\infty}$ and let $g = 0$ define $\mathbf{Y}^{\infty}$ within some affine open $U \subset \mathbf{Y}$ containing y. Let $\pi\colon T^*_{g|\mathbf{Y}\cap U} \to \mathbf{Y} \cap U$ be the canonical projection. We denote by $\mathcal{C}^{\infty}(U \cap \mathbf{Y}^{\infty})$ the subspace $\pi^{-1}(\mathbf{Y}^{\infty} \cap U)$ of $T^*_{g|\mathbf{Y}\cap U}$ and we call it the *space of characteristic covectors at infinity*. We also denote its fibre $\pi^{-1}(y)$ by $(\mathcal{C}^{\infty})_y$.

The notation $(\mathcal{C}^{\infty})_y$ is not ambiguous, since this fibre does not depend on the local equation $g = 0$, as shown by Proposition 3.2.

3.5. *Construction of the canonical ∂T-stratification at* $\mathbf{Y}^{\infty}$. Let $X \subset \mathbb{C}^n$ be a complex affine variety with isolated singularities and let $(\hat{f}, \mathbf{Y}, \mathcal{Z})$ be a f.c.e. of $f\colon X \to \mathbb{C}$. Let $\mathcal{G}$ be a stratification of $\mathbf{Y}$ in some neighbourhood of $\mathbf{Y}^{\infty}$, such that $\mathbf{Y}^{\infty}$ is a union of strata. It follows from (1) that $\mathcal{G}$ is a ∂T-stratification at $\mathbf{Y}^{\infty}$ if and only if

$$\mathcal{C}^{\infty}(U \cap \mathbf{Y}^{\infty}) \subset \bigcup_{\mathcal{G}_\alpha \in \mathcal{G} \cap \mathbf{Y}^{\infty}} T^*_{\mathcal{G}_\alpha \cap U}, \tag{6}$$

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for any open $U \subset \mathbf{Y}$.

Let U be small enough such that $\mathbf{Y}^\infty \cap U$ is defined by one equation $g = 0$. Then the space of characteristic covectors at infinity $\mathcal{C}^\infty(U \cap \mathbf{Y}^\infty)$ is a Lagrangean conic space, see [HMS], and therefore has the following property

$$\mathcal{C}^\infty(U \cap \mathbf{Y}^\infty) = \bigcup_{j \in \Delta} T^*_{Z_j}, \tag{7}$$

where Z_j are certain irreducible complex subspaces of $\mathbf{Y}^\infty \cap U$, with $\cup_{j\in\Delta} Z_j = \mathbf{Y}^\infty \cap U$ and Δ is a finite set (by [HMS] or [BMM, Sect. 2]).

Now there are stratifications of $\mathbf{Y}^\infty \cap U$ with connected strata, such that $(Z_i)_{\mathrm{reg}} \setminus \cup_{j \neq i} Z_j$ is a stratum, $\forall i \in \Delta$. We take the coarsest of all such stratifications. Together with the stratum $(\mathbf{Y} \cap U) \setminus (\mathbf{Y}^\infty \cup \mathrm{Sing}(X))$, this yields a stratification of $\mathbf{Y} \cap U$. We shall call it the *local canonical stratification*. One may globalize it on $\mathbf{Y}$, as the following result shows.

3.6. THEOREM. *Let $X \subset \mathbb{C}^n$ be a complex affine variety with isolated singularities and let $(\hat{f}, \mathbf{Y}, \mathcal{Z})$ be a f.c.e. of $f: X \to \mathbb{C}$. Then there is a unique ∂T-stratification $\mathcal{T}$ at $\mathbf{Y}^\infty$ such that, at any point $y \in \mathbf{Y}^\infty$, $\mathcal{T}$ coincides with the local canonical stratification.*

Proof. Let $\{(U_i, g_i)\}_{i \in \mathrm{I}}$ be a family such that $\{U_i\}_{i\in\mathrm{I}}$ is a family of open subsets of $\mathbf{Y}$ which covers $\mathbf{Y}^\infty$ and that $\mathbf{Y}^\infty \cap U_i$ is defined by $g_i = 0$, $\forall i \in \mathrm{I}$. Notice that the local canonical stratification defined from $\mathcal{C}^\infty(U_i \cap \mathbf{Y}^\infty)$ as in 3.5 is a ∂T-stratification, since it verifies the condition (6).

On any intersection $U_i \cap U_j$ the spaces of characteristic covectors at infinity $\mathcal{C}^\infty(U_i \cap \mathbf{Y}^\infty)$ and $\mathcal{C}^\infty(U_j \cap \mathbf{Y}^\infty)$ coincide, by Proposition 3.2. We have shown in the above construction 3.5 that the local canonical stratification depends only on the space of characteristic covectors at infinity. Therefore one may patch together the local canonical stratifications to a global one. □

3.7. DEFINITION. We call the stratification $\mathcal{T}$ given by Theorem 3.6 the *canonical ∂T-stratification at $\mathbf{Y}^\infty$*.

4. Localizing the variation of topology at the singularities at infinity

In case of a polynomial mapping $f: \mathbb{C}^n \to \mathbb{C}$, a still unsolved problem is how to detect the change in topology from a nearby fibre to a smooth atypical fibre of f. Results in several variables [Pa-1], [ST], [Pa-2] were recently obtained in case f has in some sense *isolated singularities at infinity*. The aim was essentially to localize the global change of topology at a finite number of points on a given compactification of the atypical fibre. One reason for this restriction is that the isolated case is rather well understood – for instance we dispose of a local result in the complex case, the stratified Bouquet Theorem [Ti-1], which describes the homotopy type of the Milnor fibre at an isolated critical point – whereas there is quite much lack of knowledge about local nonisolated singularities. Let us first give a precise meaning to 'localizable'. We still keep the previous notations.

4.1. DEFINITION. Let $(\hat{f}, \mathbf{Y}, \mathcal{Z})$ be a f.c.e. of an algebraic morphism $f: X \to \mathbb{C}$ and let $f^{-1}(t_0)$ be a fibre with at most isolated singularities. We say that the variation of topology of the fibres of f at t_0 is *localizable* if there exists a finite set $\{a_1, \ldots, a_k\} \in \hat{f}^{-1}(t_0) \subset \mathbf{Y}$, for each $i \in \{1, \ldots, k\}$ a complete system $\mathcal{N}_i = \{N_i^j\}_{j \in \mathbb{N}}$ of neighbourhoods of the point a_i and for any $J = (j_1, \ldots, j_k) \in \mathbb{N}^k$ there is $\delta_J > 0$ such that the restriction $f_|: (X \setminus \cup_{i=1,k} N_i^{j_i}) \cap f^{-1}(D_\delta) \to D_\delta$ is a trivial fibration, for any radius $\delta \leqslant \delta_J$.

In order to study the localizability problem we shall use ∂T-stratifications at infinity.

4.2. DEFINITION. Let X have at most isolated singularities. Let $(\hat{f}, \mathbf{Y}, \mathcal{Z})$ be a f.c.e. of $f: X \to \mathbb{C}$ and let $\mathcal{G}$ be a stratification of $\mathbf{Y}$ which is a ∂T-stratification at $\mathbf{Y}^\infty$. We say that f has *isolated $\mathcal{G}$-singularities* (with respect to $(\hat{f}, \mathbf{Y}, \mathcal{Z})$) if f has at most isolated singularities and $\hat{f}$ is transversal to $\mathcal{G}$ at all but a finite number of points of $\mathbf{Y}^\infty$.

This extends the notion of isolated *$\mathcal{W}$-singularity* of [ST], defined with respect to a certain canonical Whitney stratification $\mathcal{W}$ of $\mathbf{Y}$, in case $X = \mathbb{C}^n$ and $\mathbf{Y} = \mathbf{X}(f, \mathbb{P}^n)$.

4.3. THEOREM. *Let X have at most isolated singularities and let f have isolated $\mathcal{G}$-singularities. Then the variation of topology of the fibres of f at t_0 is localizable, $\forall t_0 \in \mathbb{C}$.*

Proof. Let $\{a_1, \ldots, a_k\} \in \hat{f}^{-1}(t_0) \cap \mathbf{Y}^\infty$ be the points where $\hat{f}$ is not transversal to $\mathcal{G}$, for some fixed $t_0 \in \mathbb{C}$. Let B_i, B_i' be small enough open balls within $\mathcal{Z}$, centered at a_i such that $B_i \subset \overline{B_i} \subset B_i'$. For some $p \in \hat{f}(t_0) \cap \mathbf{Y}^\infty \setminus \{a_1, \ldots, a_k\}$, let V_p denote a small enough open neighbourhood of p within $\mathcal{Z}$. We cover the compact $\hat{f}(t_0) \cap \mathbf{Y}^\infty \setminus \cup_{i=1}^k B_i'$ by a finite number of such neighbourhoods and denote by W their union and by $\mathcal{V} = \{V_p\}_p$ the covering. One may assume that $W \cap B_i = \emptyset$, $\forall i \in \{1, \ldots, k\}$. Let $\phi: W \to \mathbb{R}_{\geqslant 0}$ be the C^∞-function defined by $\phi = \Sigma_{V \in \mathcal{V}} \alpha_V |g_V|^2$, where $\{\alpha_V\}_{V \in \mathcal{V}}$ is a partition of unity relative to $\mathcal{V}$ and $g_V = 0$ is the equation of $\mathbf{Y}^\infty$ within V. We now show that f is transversal to all levels of ϕ within $W' \cap X$, for some open $W' \subset W$ which also covers the compact $\hat{f}(t_0) \cap \mathbf{Y}^\infty \setminus \cup_{i=1}^k B_i'$. Any $p \in \hat{f}(t_0) \cap \mathbf{Y}^\infty \setminus \cup_{i=1}^k B_i'$ is in the overlap of several $V \in \mathcal{V}$. On some small enough neighbourhood of p, the local equations are related by $g_{V_j} = \lambda_{js} g_{V_s}$, with λ_{js} an analytic function and $\lambda_{js}(p) \neq 0$. Now $(T_\phi^*)_p = (T_{|g_V|^2}^*)_p$, by Lemma 3.3. Let $\mathcal{G}_0$ be the stratum of $\mathcal{G}$ which contains p (thus of dimension $\geqslant 1$, since $p \notin \{a_1, \ldots, a_k\}$). But $\hat{f}$ is transversal to $\mathcal{G}_0$ at p, hence f is transversal to the positive levels of ϕ within some small enough neighbourhood of

p, since $(T^*_{g_{V_j}})_p \subset (T^*_{\mathcal{G}_0})_p$. We may therefore construct a continuous lift $\mathbf{v}$ of $\partial/\partial t$ by $\hat{f}$ on $W' \cap X$, for some open W' as above, such that it is tangent to the positive levels of ϕ. On the other hand we may define a vector field $\mathbf{v}_i$ on $B'_i \cap X$, as in the proof of Theorem 2.5, which is a lift of $\partial/\partial t$ by f and is tangent to $\partial\overline{B_i}\backslash\mathbf{Y}^\infty$. We finally glue the vector fields $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_k$ by a partition of unity to get a vector field $\mathbf{w}$ which is again a lift of $\partial/\partial t$ by f and is tangent to $\partial\overline{B_i}\backslash\mathbf{Y}^\infty$, $\forall i \in \{1, \ldots, k\}$. Thus, by integrating $\mathbf{w}$, we would get a trivialisation of $f_{|W'\backslash(\mathbf{Y}^\infty \cup_{i=1}^k B_i)}$ over a sufficiently small disc D centered at $t_0 \in \mathbb{C}$.

We also have to deal with the eventual singular points $\{b_1, \ldots, b_r\}$ of the fibre $f^{-1}(t_0)$. We take a sufficiently small ball β_j at each such point. Then the sphere $\partial\overline{\beta_j}$ cuts transversally any fibre of f over a sufficiently small disc D centered at t_0.

It is standard how to lift $\partial/\partial t$ by f to a vector field on $X \cap f^{-1}(D) \cap \beta\backslash \cup_{j=1}^r \beta_j$ which is tangent to the spheres $\partial\overline{\beta_j}$, where β is a large enough ball in $\mathbb{C}^n$.

Glueing this vector field to the one before, we finally get, by integration, a global topologically trivial fibration:

$$f_|\colon (X\backslash(\cup_{i=1}^k B_i \cup \cup_{j=1}^r \beta_j)) \cap f^{-1}(D) \to D,$$

for a sufficiently small disc (resp. interval) D centered at $t_0 \in \mathbb{C}$. □

4.4. *Local variation mapping at infinity*. Suppose we are under the conditions of Theorem 4.2. Let $p \in \mathbf{Y}^\infty \cap \hat{f}^{-1}(t_0)$ be an isolated $\mathcal{G}$-singularity of f. We may associate to such a point certain topological invariants, as follows. Let B_ε be an open ball at p within $\mathcal{Z}$, for some small enough $\varepsilon > 0$. Then by Theorem 2.5 and Lemma 2.6, the following restriction of f

$$f_|\colon X \cap \overline{B_\varepsilon} \cap f^{-1}(D_\delta\backslash\{t_0\}) \to D_\delta\backslash\{t_0\} \tag{8}$$

is a topologically trivial fibration, where D_δ is a disc of radius δ, with $0 < \delta \ll \varepsilon$, centered at $t_0 \in \mathbb{C}$. Moreover, the construction used in 2.5 and 2.6 gives that the restriction of f to $X \cap \partial\overline{B_\varepsilon} \cap f^{-1}(D_\delta)$ is topologically trivial over D_δ.

Let us denote by F the fibre $X \cap \overline{B_\varepsilon} \cap f^{-1}(\gamma)$ of the fibration (8), for some $\gamma \in D_\delta\backslash\{t_0\}$ and let $\partial F := X \cap \partial\overline{B_\varepsilon} \cap f^{-1}(\gamma)$. By the above remarks, there exists a geometric monodromy of F which is the identity on ∂F. This proves the following.

4.5. PROPOSITION. *Let $f\colon X \to \mathbb{C}$ have an isolated $\mathcal{G}$-singularity at $p \in \mathbf{Y}^\infty \cap \hat{f}^{-1}(t_0)$. Then there exists a* local variation mapping

$$\mathrm{var}\colon H_\bullet(F, \partial F) \to H_\bullet(F),$$

such that $h - \mathrm{Id} = \mathrm{var} \circ j$, where $h\colon H_\bullet(F) \to H_\bullet(F)$ denotes the monodromy, Id *is the identity and $j\colon H_\bullet(F) \to H_\bullet(F, \partial F)$ is induced by the inclusion $F \hookrightarrow$*

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$(F, \partial F)$. □

The following result compares the space X to the general fibre of f, up to homotopy.

4.6. THEOREM. *Let $X \subset \mathbb{C}^n$ be an affine complex variety of dimension d with at most isolated singularities and such that, at each such singularity, X is a local complete intersection. Let f have isolated $\mathcal{G}$-singularities with respect to some f.c.e. $(\hat{f}, \mathbf{Y}, \mathcal{Z})$ and to some stratification $\mathcal{G}$ of $\mathbf{Y}$ which is a ∂T-stratification at infinity.*

Then, up to homotopy type, the space X is obtained from a general fibre of f to which one attaches a number, say λ_f, of cells of real dimension d.

Proof. By Theorem 4.3, the variation of topology of the fibres of f is localizable at a finite number of points on X or on $\mathbf{Y}^\infty$. For those points on X, the local situation is managed by the classical result of Milnor [Mi] (in the smooth case) and Lê [Lê-2, Th. 5.1] (in the singular case): if $f\colon (X, b_j) \to (\mathbb{C}, 0)$ is an analytic function with isolated singularities on a local complete intersection X of dimension d, then the pair $(B \cap X \cap f^{-1}(D), B \cap X \cap f^{-1}(\eta))$ is $(d-1)$-connected, where B is a small enough ball centered at b_j and D a small enough disc at $0 \in \mathbb{C}$, $\eta \in D^*$. For the points 'at infinity', we may apply Corollary 2.7 to get that the pair $(B \cap (\mathbf{Y} \backslash \mathbf{Y}^\infty) \cap f^{-1}(D), B \cap (\mathbf{Y} \backslash \mathbf{Y}^\infty) \cap f^{-1}(\eta))$ is also $(d-1)$-connected. We may conclude that the space X is built up from a fibre $f^{-1}(\eta)$ which moves within a fibration, encountering a finite number of isolated singularities $\{a_1, \ldots, a_k, b_1, \ldots, b_r\}$. By invoking Switzer's result [Sw, Prop. 6.13], at each such point one has to attach a number (= the local 'Milnor number') of d-cells. The total number of cells is the sum of all these local Milnor numbers.

The Stein space X is a CW-complex of dimension $\leqslant d$, by a well known result due to Hamm [H-1]. As a nice particular case of Theorem 4.6, we then get the following bouquet result which improves a previous one due to Siersma and the author [ST, Th. 3.1].

4.7. COROLLARY. *Under the assumptions of Theorem* 4.6, *suppose in addition that X is $(d-1)$-connected. Then the general fibre of f has the homotopy type of a bouquet of spheres $\vee S^{d-1}$.* □

We end by an example of a simple polynomial with an interesting local behavior at infinity.

4.8. EXAMPLE. Let f be the following complex polynomial mapping (which was undoubtly known to Newton, as polynomial in two variables) $x + x^2y : \mathbb{C}^3 \to \mathbb{C}$ in 3 variables x, y, z and consider its f.c.e. $(t, \mathbf{X}, \mathbb{P}^3 \times \mathbb{C})$ (see Example 1.2). We first recall from [ST], in an equivalent formulation, the definition of a t-singularity at infinity of f: we say that f is *t-regular* at $p \in \mathbf{X}^\infty$ if and only if $(p, \mathrm{d}\, t) \notin (\mathcal{C}^\infty)_p$. Otherwise we say that p is a *t-singularity of f at infinity*. Another interpretation of p being a t-singularity at infinity of f is that the function $t - t(p)$ has vanishing cycles at p, that is $p \in \mathrm{supp}(\Phi_{t-t(p)}(\mathbb{C}_\mathbf{X}))$. This fact was noticed by Parusinski [Pa-2, Prop. 1.1, Cor. 1.5] and follows from Ginsburg's Theorem [Gi], [BMM, Théorème 3.4.2] and Sabbah's intersection index formula [Sa, 4.6], after applying a slicing argument that reduces the problem to the isolated singularities case.

We show that f is not t-regular at a whole line $L := \{x_0 = x = t = 0\}$ within $\mathbf{X}^\infty \cap t^{-1}(0)$, hence has a nonisolated t-singularity. This also implies that f has nonisolated $\mathcal{G}$-singularities at infinity, where $\mathcal{G}$ is the canonical ∂T-stratification at $\mathbf{X}^\infty$ (see Definition 3.7). The proof goes like this.

Let $F := xx_0^2 + x^2y - tx_0^3$. The t-regularity in the chart $y \neq 0$ is equivalent to $|y| \cdot \|\partial f/\partial x\| \not\to 0$, as $\|x, y\| \to \infty$, cf. [ST, p. 780]. But in our example $|y| \cdot \|1 + 2xy\|$ tends to 0, for instance if $y \to \infty$, $x = 1/y^3 - 1/2y$, $z = ay$. The limit points in $\mathbf{X}^\infty$ are the 1-dimensional set L. The only atypical value is 0. Moreover, for any algebraic change of coordinates $\phi \in \mathrm{Aut}(\mathbb{C}^3)$, the polynomial $f \circ \phi$ has a single atypical value, with nonisolated t-singularity at infinity. Indeed, the singularities at infinity cannot be isolated since then the general fibre would be homotopically a bouquet $\bigvee S^2$; but in our example the general fibre is a circle.

On the other hand, one can easily verify that, for a general linear form $l = ax+by+cz$ with $c \neq 0$, the polar curve $\Gamma(l, f)$ is void. This implies that, by a generic linear change of coordinates, one has $\Gamma(x', f') = \Gamma(y', f') = \Gamma(z', f') = \emptyset$, where f' is the polynomial f in the new coordinates. One can take for instance $x' = x+z$, $y' = y+z$, $z' = z$ and then $f' = x'-z'+(x'-z')^2(y'-z')$. Let then $(t', \mathbf{X}', \mathbb{P}^3 \times \mathbb{C})$ be the f.c.e. of f', let $p \in \mathbf{X}'^\infty$ and let $y_0 = 0$ be some local equation of $\mathbf{X}'^\infty$ at p. The voidness of the polar loci above implies, by an easy computation, that the local polar locus $\Gamma_p(t', y_0)$ at p is also void. It then follows that one can lift the vector field $\partial/\partial t$ by t' to a complex vector field $\mathbf{v}$ tangent to the nonzero levels of y_0 within $N_p \cap \mathbf{X}' \backslash \mathbf{X}'^\infty$, where N_p is some small enough open neighbourhood of p in $\mathbb{P}^3 \times \mathbb{C}$. By integrating $\mathbf{v}$, one concludes that, for some small enough disc $D \ni f'(p)$, the restriction $f'_|: N_p \cap (f')^{-1}(D) \to D$ is a topologically trivial fibration. One may interpret this by saying that f' is locally trivial at infinity at any $p \in \mathbf{X}'^\infty$ (which however *does not imply* that f has no atypical value!).

We have seen before that f is not t-regular at infinity, hence (by the linear change of coordinates), f' is not t'-regular at infinity at the whole line $L' = \{x' - z' = t' = 0\}$. In particular, for all $p \in L' \subset \mathbf{X}'^\infty$, t' is not locally trivial at p since t' *has vanishing cycles at* p.

One may therefore conclude that the condition 'no vanishing cycles at $p \in \mathbf{X}^\infty$' is not necessary for f' to be locally trivial at p.

5. Connectivity of fibres

When the mapping $f: X \to \mathbb{C}$ has non-isolated $\mathcal{G}$-singularities then, in principle, we cannot localize the variation of topology of fibres at infinity. Nevertheless, we shall prove some connectivity results for the fibres of f, using a Lefschetz type method.

We denote by H_s the affine hyperplane $\{l_H = s\} \subset \mathbb{C}^n$, where $H \in \check{\mathbb{P}}^{n-1}$ is a hyperplane defined by a linear form $l_H: \mathbb{C}^n \to \mathbb{C}$.

5.1. GENERAL SLICE. Let $X \subset \mathbb{C}^n$ be an affine algebraic variety and let $f: X \to \mathbb{C}$ be algebraic. Consider the f.c.e. $(t, \mathbf{X}, \mathbb{P}^n \times \mathbb{C})$ of f (cf. Example 1.2) and let $\mathcal{W} = \{\mathcal{W}_i\}_{i \in \mathrm{I}}$ be a finite complex stratification of $\mathbf{X}$ satisfying Whitney (a) property and with $\mathbf{X}^\infty$ as union of strata. By a Bertini type argument similar to the one in the proof of Lemma 2.4, there is an open dense $\Omega_X \subset \check{\mathbb{P}}^{n-1}$ and a finite set $A_X \subset \mathbb{C}$ such that $\forall H \in \Omega_X, \forall s \in \mathbb{C} \backslash A_X$, the affine hyperplane H_s is transversal to $\mathcal{W}_i$, for all $i \in \mathrm{I}$ such that $\mathcal{W}_i \subset X$.

For strata $\mathcal{W}_i \subset \mathbf{X}^\infty \subset \mathbb{P}^{n-1} \times \mathbb{C}$ we use again a Bertini type argument for the projection $T^*_{t|\mathcal{W}_i} \to \check{\mathbb{P}}^{n-1}$ to deduce the following: there is an open dense $\Omega_{\mathbf{X}^\infty} \subset \check{\mathbb{P}}^{n-1}$ such that $\forall H \in \Omega_{\mathbf{X}^\infty}$, the hyperplane $H \times \mathbb{C}$ is transversal to $\mathcal{W}_i$ within $\mathbb{P}^{n-1} \times \mathbb{C}$, except possibly at a finite number of points, for all $i \in \mathrm{I}$ such that $\mathcal{W}_i \subset \mathbf{X}^\infty$. Let then denote $\Omega_{\mathbf{X}} := \Omega_X \cap \Omega_{\mathbf{X}^\infty}$.

Consider now the restriction $f_{|H_s}: X \cap H_s \to \mathbb{C}$. We identify the hyperplane $\overline{H_s} \subset \mathbb{P}^n$ to $\mathbb{P}^{n-1}$ and consider the space $\mathbf{X}(f_{|H_s}, \mathbb{P}^{n-1})$. Let $H \in \Omega_{\mathbf{X}}$ and $s \in \mathbb{C} \backslash A_X$. Then $\mathbf{X}^\infty(f_{|H_s}, \mathbb{P}^{n-1}) = \mathbf{X}^\infty(f, \mathbb{P}^n) \cap (H \times \mathbb{C})$. We may therefore define the following induced stratification.

5.2. DEFINITION. For $H \in \Omega_{\mathbf{X}}$ and $s \in \mathbb{C} \backslash A_X$, we denote by $\mathcal{W}_{H_s}$ the stratification of $\mathbf{X}(f_{|H_s}, \mathbb{P}^{n-1})$ of which all strata of dimension $\geqslant 1$ have the following form

$$\begin{cases} \mathcal{W}_i \cap H_s, & \text{if } \mathcal{W}_i \subset X \text{ or} \\ (\mathcal{W}_i \cap (H \times \mathbb{C})) \backslash \{p \in \mathcal{W}_i \mid \mathcal{W}_i \not\pitchfork H \times \mathbb{C}\}, & \text{if } \mathcal{W}_i \subset \mathbf{X}^\infty. \end{cases}$$

5.3. DEFINITION. Keeping the above notations, we define the *critical locus at infinity* of f with respect to the stratification $\mathcal{W}$ by

$$\mathrm{Sing}^\infty_{\mathcal{W}} f := \bigcup_{\mathcal{W}_j \subset \mathbf{Y}^\infty} \mathrm{Sing}\, t_{|\mathcal{W}_j}.$$

$\mathrm{Sing}^\infty f$ is a closed subset of $\mathbf{Y}^\infty$, since $\mathcal{W}$ has Whitney (a) property.

With these definitions, we have the following.

5.4. LEMMA. *There exists a Zariski-open set* $\Omega_t \subset \check{\mathbb{P}}^{n-1}$ *and a finite set* $A \subset \mathbb{C}$ *such*

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that, if $H \in \Omega_t \subset \check{\mathbb{P}}^{n-1}$ *and* $s \in \mathbb{C}\backslash A$, *then* $\dim \mathrm{Sing}_{\mathcal{W}_{H_s}} f_{|H_s} \leqslant \dim \mathrm{Sing}_{\mathcal{W}} f - 1$ *and* $\dim \mathrm{Sing}^{\infty}_{\mathcal{W}_{H_s}} f_{|H_s} \leqslant \dim \mathrm{Sing}^{\infty}_{\mathcal{W}} f - 1$, *in case* $\dim \mathrm{Sing}_{\mathcal{W}} f \geqslant 1$ *and* $\dim \mathrm{Sing}^{\infty}_{\mathcal{W}} f \geqslant 1$.

Proof. Let $H \in \Omega_f \cap \Omega_{\mathbf{X}}$ and $s \in \mathbb{C}\backslash A_X$. Then, by Lemma 2.4, dim $\mathrm{Sing}_{\mathcal{W}_{H_s}} f_{|H_s} = \dim(\Gamma(l_H, f) \cap H_s) \cup (\mathrm{Sing}_{\mathcal{W}} f \cap H_s)$. By Bertini–Sard, $\dim(\mathrm{Sing}_{\mathcal{W}} f \cap H_s) \leqslant \dim \mathrm{Sing}_{\mathcal{W}} f - 1$. There is a finite set $A \subset \mathbb{C}$ such that $A \supset A_X$ and that $\dim(\Gamma(l_H, f) \cap H_s) \leqslant 0$ whenever $H \in \Omega_f \cap \Omega_{\mathbf{X}}$ and $s \in \mathbb{C}\backslash A$. Hence the first inequality in our statement holds for $H \in \Omega_f \cap \Omega_{\mathbf{X}}$ and $s \in \mathbb{C}\backslash A$.

To prove the latter inequality in our statement we proceed as follows. Take a stratum $\mathcal{W}_i \subset \mathbf{X}^{\infty} \subset \mathbb{P}^{n-1} \times \mathbb{C}$. From 5.1 it follows that, for $H \in \Omega_{\mathbf{X}^{\infty}}$, the restriction $t_{|}\colon \mathcal{W}_i \cap (H \times \mathbb{C})\backslash \mathrm{Sing}\, t_{|\mathcal{W}_i} \to \mathbb{C}$ has a finite number of critical points. Notice that the set $\mathrm{Sing}\, t_{|\mathcal{W}_i}$ is contained in a finite number of fibres of $t_{|\mathcal{W}_i}$, let those be $(t_{|\mathcal{W}_i})^{-1}(t_j)$, for $j \in K_i$.

Then, for $H \in \Omega_{\mathbf{X}}$ and $s \in \mathbb{C}\backslash A$, we have $\dim \mathrm{Sing}^{\infty}_{\mathcal{W}_{H_s}} f_{|H_s} \leqslant \dim \cup_{i\in \mathrm{I}, j \in K_i}$ $(\mathrm{Sing}\, t_{|\mathcal{W}_i} \cap H \times \{t_j\})$. Again by Bertini–Sard, there is another open dense $\hat{\Omega} \subset \check{\mathbb{P}}^{n-1}$ such that, if $H \in \hat{\Omega}$, then $\dim(\mathrm{Sing}\, t_{|\mathcal{W}_i} \cap H \times \{t_j\}) \leqslant \dim \mathrm{Sing}\, t_{|\mathcal{W}_i} - 1$, $\forall i \in \mathrm{I}$, $\forall j \in K_i$.

Finally, taking $\Omega_t = \Omega_f \cap \Omega_{\mathbf{X}} \cap \hat{\Omega}$ and the finite set A as in the first part of this proof, we get the both claimed inequalities. □

We may then strenghten the connectivity result of Siersma and the author [ST, Cor. 3.6] as follows.

5.5. THEOREM. *Let* $f\colon \mathbb{C}^n \to \mathbb{C}$ *be any polynomial mapping. Consider the f.c.e.* $(t, \mathbf{X}, \mathbb{P}^n \times \mathbb{C})$ *of* f *and let* $\mathcal{W} = \{\mathcal{W}_i\}_{i\in \mathrm{I}}$ *be a finite complex stratification of* $\mathbf{X}$ *such that it has Whitney* (a) *property, it is a* ∂T*-stratification at* $\mathbf{X}^{\infty}$ *and* $\mathbb{C}^n$ *is one of its strata. Then the general fibre of* f *is at least* $(n - 2 - \dim(\mathrm{Sing}\, f \cup \mathrm{Sing}^{\infty}_{\mathcal{W}} f))$*-connected. Moreover, a special fibre is at least* $(n - 3 - \dim(\mathrm{Sing}\, f \cup \mathrm{Sing}^{\infty}_{\mathcal{W}} f))$*-connected.*

Proof. Remark first that if $H \in \Omega_{\mathbf{X}}$ an d $s \in \mathbb{C}\backslash A$ then $\mathcal{W}_{H_s}$ is a Whitney (a) stratification of $\mathbf{X}(f_{|H_s}, \mathbb{P}^{n-1})$, it is a ∂T-stratification at $\mathbf{X}^{\infty}(f_{|H_s}, \mathbb{P}^{n-1})$ and H_s is one of its strata.

We use a Lefschetz type argument as follows. Slicing a fibre $F_{\varepsilon} := f^{-1}(\varepsilon)$ by a general hyperplane $H_s \subset \mathbb{C}^n$, we get a fibre $F_{\varepsilon} \cap H_s$ of $f_{|H_s}$, which is a polynomial mapping in one variable less. Then the pair $(F_{\varepsilon}, F_{\varepsilon} \cap H_s)$ is $(\dim F_{\varepsilon} - 1)$-connected, by a result of Hamm [H-2]. To prove our statement inductively, we also need $\dim \mathrm{Sing}\, f_{|H_s} \leqslant \dim \mathrm{Sing}\, f - 1$ and $\dim \mathrm{Sing}^{\infty}_{\mathcal{W}_{H_s}} f_{|H} \leqslant \dim \mathrm{Sing}^{\infty}_{\mathcal{W}} f - 1$. This is done by Lemma 5.4. By repeated slicing and succesively cutting down dimensions, we arrive at a polynomial h which is the restriction of f to a linear affine space, with $\dim \mathrm{Sing}\, h \leqslant 0$ and $\dim \mathrm{Sing}^{\infty}_{\mathcal{T}} h \leqslant 0$, where $\mathcal{T}$ is an induced Whitney (a), ∂T-stratification at infinity, obtained by repeated general slicing. This implies that h has isolated t-singularities with respect to $\mathcal{T}$.

If F_ε is a general fibre then its general slice is also a general fibre of the restriction $f_{|H_s}$. We then conclude the induction by applying Corollary 4.7 to the polynomial h. In case we start with an atypical fibre F_a, we may get at the end an atypical fibre $\mathcal{F}_a$ of h. Let then $\mathcal{F}_\eta$ be a nearby general fibre of h. Then $\mathcal{F}_\eta$ is homotopy equivalent to a bouquet $\vee S^q$ of spheres of dimension $q = \dim \mathcal{F}_\eta$. To prove the last part of our theorem, we only need to show that $\pi_i(\mathcal{F}_a) = \pi_i(\mathcal{F}_\eta)$, for $i \leqslant q-2$.

We shall make use of Theorem 4.3, its proof and notations. By this theorem, the variation of topology of the fibres of f over a small enough disc D centered at $a \in \mathbb{C}$ is localizable at a finite set of points $\{a_1, \dots, a_k\} \in \mathcal{F}_a$ and $\{b_1, \dots, b_r\} \in \overline{\mathcal{F}_a} \cap \mathbf{X}^\infty$. Consider a large enough ball $\beta \subset \mathbb{C}^n$ such that $\mathcal{F}_a \cap \beta$ is diffeomorphic to $\mathcal{F}_a$ and that $\partial\overline{\beta} \pitchfork \mathcal{F}_\gamma$, $\forall \gamma \in D$. Then, for $\eta \in D$, $\mathcal{F}_a \cap \beta$ is obtained from $\mathcal{F}_\eta \cap \beta$ by attaching a finite number of $(q+1)$-cells corresponding to the isolated singularities a_i.

For some singularity at infinity b_j, let $g_j = 0$ be a local equation of $\mathbf{X}^\infty$ at b_j. Then we consider the germ of the mapping $(t, g_j)\colon \mathbf{X} \to \mathbb{C}^2$ at b_j and a corresponding small enough polydisc neighbourhood $\mathcal{P}_j$ at b_j. As in the proof of Theorem 4.3, there is a nonvoid polar curve $\Gamma_{b_j}(t, g_j)$ at b_j, which cuts $\mathcal{F}_\eta$ at a finite number of points within $\mathcal{P}_j$. This shows that $\mathcal{F}_\eta$ is built from $\mathcal{F}_\eta \cap \beta$ by attaching a certain number of q-cells for each such intersection point, and for all $j \in \{1, \dots, r\}$. In conclusion, we have shown the following homotopy equivalences

$$\mathcal{F}_a \overset{\text{ht}}{\simeq} \mathcal{F}_a \cap \beta \overset{\text{ht}}{\simeq} (\mathcal{F}_\eta \cap \beta) \cup \mathbf{e}^{q+1}\text{-cells}, \quad \mathcal{F}_\eta \overset{\text{ht}}{\simeq} (\mathcal{F}_\eta \cap \beta) \cup \mathbf{e}^{q}\text{-cells} \overset{\text{ht}}{\simeq} \vee S^q,$$

which imply the desired equality of homotopy groups. Now the proof is complete. □

6. Application to affine complete intersections

Let $g = (g_1, \dots, g_p)\colon \mathbb{C}^{n+p} \to \mathbb{C}^p$, $n > 0$, be a polynomial mapping. If at some point, g is a local complete intersection with isolated singularity, then its local Milnor fibre at this point is a bouquet of n-spheres, by Hamm's result, see [H-1]. The context we have developed allows us to prove a result of this type for *global mappings*.

Let us introduce some notations. In analogy to Definition 1.1 we may define a fibre-compactifying extension $(\hat{g}, \mathbf{Y}, \mathcal{Z})$ of the mapping g. Following 1.2, an example of a f.c.e. is the following. Let $\mathbf{X}$ be the algebraic closure of the graph of g, within $\mathbb{P}^{n+p} \times \mathbb{C}^p$. The intersection $\mathbf{X} \cap (H^\infty \times \mathbb{C}^p)$ will be denoted by $\mathbf{X}^\infty$, where H^∞ is the hyperplane at infinity $\mathbb{P}^{n+p} \backslash \mathbb{C}^{n+p}$. The data $(t, \mathbf{X}, \mathbb{P}^{n+p} \times \mathbb{C}^p)$ define then a f.c.e. of the mapping g, where $t\colon \mathbf{X} \to \mathbb{C}^p$ is the restriction to $\mathbf{X}$ of the projection $\mathbb{P}^{n+p} \times \mathbb{C}^p \to \mathbb{C}^p$.

6.1. DEFINITION. Let $(\hat{g}, \mathbf{Y}, \mathcal{Z})$ be a f.c.e. of g and let $\mathcal{W} = \{\mathcal{W}_i\}_{i \in \mathrm{I}}$ be a finite complex stratification of $\mathbf{Y}$ with Whitney (a) property, which is a ∂T-stratification at $\mathbf{Y}^\infty$ and has $\mathbb{C}^{n+p}$ as one of its strata. Let us denote $\mathrm{Sing}_{\mathcal{W}}\hat{g} := \mathrm{Sing}_{\mathcal{W}}g \cup \mathrm{Sing}_{\mathcal{W}}^\infty g$ and $\Delta(\hat{g}) := \hat{g}(\mathrm{Sing}_{\mathcal{W}}\hat{g})$.

We say that $g: \mathbb{C}^{n+p} \to \mathbb{C}^p$ is an *affine local complete intersection* (*a.l.c.i.*) *with isolated $\mathcal{W}$-singularities* iff the restriction $\hat{g}_|: \mathrm{Sing}_{\mathcal{W}}\hat{g} \to \Delta(\hat{g})$ is a finite mapping. Such a g is in particular a local complete intersection with isolated singularity at any point of $\mathbb{C}^{n+p}$.

6.2. THEOREM. *Let $g: \mathbb{C}^{n+p} \to \mathbb{C}^p$ be a polynomial mapping and let $(\hat{g}, \mathbf{Y}, \mathcal{Z})$ be a f.c.e. of g. Let $\mathcal{W}$ be a finite complex stratification of* $\mathbf{Y}$ *with Whitney* (a) *property, which is a ∂T-stratification at $\mathbf{Y}^\infty$ and has $\mathbb{C}^{n+p}$ as one of its strata. If g is an a.l.c.i. with isolated $\mathcal{W}$-singularities then the general fibre of g has the homotopy type of a bouquet $\vee S^n$.*

Proof. $\mathrm{Sing}_{\mathcal{W}}\hat{g}$ is closed analytic (since $\mathcal{W}$ has Whitney (a) property). Since $\hat{g}_{|\mathrm{Sing}\,\hat{g}}$ is finite, the discriminant $\Delta(\hat{g})$ is also closed analytic, of dimension $< p$. On the other hand, for any $y \in \Delta(\hat{g})$, the set $\hat{g}^{-1}(y) \cap \mathrm{Sing}_{\mathcal{W}}\hat{g}$ is finite. Notice that for $p = 1$ this means that the polynomial g has isolated $\mathcal{W}$-singularities, hence we are in the conditions of Corollary 4.7 and the conclusion follows.

For $p > 1$, we use induction on p. There is a Zariski-open subset Ω_Δ in the set of linear projections from $\mathbb{C}^p$ to $\mathbb{C}^{p-1}$ such that, if $l \in \Omega_\Delta$, then the restriction of l to the analytic subset $\Delta(\hat{g}) \subset \mathbb{C}^p$ is a finite mapping. We may assume, by a linear coordinate change, that l is the projection along the last coordinate of $\mathbb{C}^p$. Let's consider the polynomial mapping $g' = (g_1, \ldots, g_{p-1}) : \mathbb{C}^{n+p} \to \mathbb{C}^{p-1}$ and let us define $\hat{g'}: \mathbf{Y} \to \mathbb{C}^{p-1}$, $\hat{g'} := l \circ \hat{g}$. Then the data $(\hat{g'}, \mathbf{Y}, \mathcal{Z})$ represent a f.c.e. of g'. Moreover g' is an a. l.c.i. with isolated singularities with respect to the same stratification $\mathcal{W}$. Therefore we may assume by induction that the general fibre of g', say $X := (g')^{-1}(\varepsilon)$, is homotopically a bouquet $\vee S^{n+1}$. Moreover, the algebraic mapping $g_p: X \to \mathbb{C}$ has isolated singularities with respect to the f.c.e. $\hat{g}_p: \overline{X} = \hat{g}^{-1}(\{\varepsilon\} \times \mathbb{C}) \to \mathbb{C}$ and the stratification on $\overline{X}$ induced by $\mathcal{W}$. Now the general fibre of $g_p: X \to \mathbb{C}$ is just the general fibre of the initial polynomial mapping g. We apply Corollary 4.7 to conclude. □

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