COLLINEATION GROUPS CONTAINING PERSPECTIVITIES

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To H. S. M. Coxeter, on his sixtieth birthday

1. Introduction. Let **P** be a projective plane of finite order n and Γ a group of collineations of **P**. Gleason **(6)** and Wagner **(10)** have shown that if every point of **P** is the centre, and every line the axis, of a non-trivial perspectivity in Γ, then Γ contains a subgroup of order n^2 which consists entirely of elations. It then follows that either **P** or its dual is a translation plane with respect to at least one line; in fact if Γ has no fixed elements, then **P** is desarguesian and Γ contains all elations of **P**. It was shown by Piper **(7)** and Cofman **(4)** that the hypotheses of Gleason and Wagner can be relaxed in certain cases, while the same conclusions hold. It is natural to ask what can be said about the case where every point (or line) of **P** is fixed by some perspectivity $\neq 1$ in Γ. One of the results of this paper is that then the above conclusions no longer follow. We shall characterize here those collineation groups Γ of a projective plane **P** of order $n > 2$ which are minimal with respect to the property just mentioned, in the sense that they satisfy the following condition:

(I) *Every point of* **P** *is fixed by some non-trivial perspectivity in* Γ, *and with one exception each point is fixed by only one non-trivial collineation in* Γ.

Since any two distinct perspectivities have a common fixed point, the words "with one exception" cannot be avoided.

The main part of this paper (§§3–6) consists in establishing three other characterizations of the collineation groups satisfying (I). The first of these is:

(II) *There exists a point z and a line Z such that:*

(II.1) Γ *fixes z and Z.*

(II.2) $z \mathrel{I} Z$ *if n is odd, and* $z \not\mathrel{I} Z$ *if n is even.*

(II.3) *Every point $\neq z$ on Z is the centre, and every line $\neq Z$ through z the axis, of an involutorial perspectivity in* Γ.

(II.4) *The set Δ of those collineations in Γ which are not among the perspectivities of* (II.3) *is an abelian normal subgroup of* Γ, *having the same orbits as* Γ.

Actually, it will be seen that Γ acts as a Frobenius group, with complement of order 2, on each orbit $\neq \{z\}, \{Z\}$, and the subgroup Δ is the Frobenius kernel of Γ. Note that for $n = 2$ the group of all elations with a given fixed

Received March 13, 1967. Work supported by the National Science Foundation, grant GP-6539.

point as centre satisfies (I) but not (II). This shows the necessity of the general hypothesis $n > 2$, which will also be needed in other places.

The two other characterizations are in a sense more interesting, since perspectivities are not mentioned in them. One of them is:

(III) Γ *has an orbit* $\mathfrak{A}$ *of lines such that:*
(III.1) *All lines of* $\mathfrak{A}$ *pass through one point* z.
(III.2) $|\mathfrak{A}| < |\Gamma|$.
(III.3) $|\mathfrak{A}| \geqslant |\mathfrak{x}|, |\mathfrak{X}|$, *for all point orbits* $\mathfrak{x}$ *and line orbits* $\mathfrak{X}$ *of* Γ.

Notice that (III.3) is not implied by (III.1), (III.2) and either $|\mathfrak{A}| \geqslant |\mathfrak{x}|$ or $|\mathfrak{A}| \geqslant |\mathfrak{X}|$ alone: In a finite desarguesian plane, the group of all elations with fixed axis satisfies $|\mathfrak{A}| \geqslant |\mathfrak{X}|$ but not $|\mathfrak{A}| \geqslant |\mathfrak{x}|$, and the group of all perspectivities with fixed centre c and axes through a fixed point $z \neq c$ satisfies $|\mathfrak{A}| \geqslant |\mathfrak{x}|$ but not $|\mathfrak{A}| \geqslant |\mathfrak{X}|$. Nevertheless, condition (III) appears to be rather more general than (II), and accordingly the main part of the proofs below will consist in showing that (III) implies (II).

The last characterization to be given here is the following:

(IV) *Let* m *denote the odd one of the integers* n *and* $n + 1$. *Then:*
(IV.1) $|\Gamma| = 2m$.
(IV.2) *With one exception, all point orbits of* Γ *have length* m.

Actually, we shall prove rather more than (IV.2) about the Γ-orbits, namely:

(V) *Let* $\mathfrak{p}$ *be a point orbit of* Γ, *with* $|\mathfrak{p}| > 1$. *Then, for some point* z:

(V.1) *If* n *is odd, then* $|\mathfrak{p}| = n$, *and either* $\mathfrak{p}$ *consists of the points* $\neq z$ *of some line through* z, *or* $\mathfrak{p}$ *and* z *together form an oval.*

(V.2) *If* n *is even, then* $|\mathfrak{p}| = n + 1$, *and either* $\mathfrak{p}$ *consists of all points of some line not through* z, *or* $\mathfrak{p}$ *is an oval with knot* z.

Here we mean by an *oval* any set of $n + 1$ points in $\mathbf{P}$ of which no three are collinear. It is well known (see, for example, Qvist (**8**, Theorem **5**) that if n is even, all tangents of an oval $\mathfrak{o}$ pass through one point; this point is called the *knot* of $\mathfrak{o}$.

Note that condition (II) is self-dual, whereas (I), (III), (IV), (V) are not. The equivalence of (I)–(IV) will therefore show that the duals (I′), (III′), (IV′), (V′) of (I), (III), (IV), (V) are likewise satisfied, and in fact (I′), (III′), (IV′) are equivalent to (I)–(IV). We can now summarize the main result of this paper as follows:

THEOREM. *If the collineation group* Γ *of the projective plane* $\mathbf{P}$ *of finite order* $n > 2$ *satisfies one of the conditions* (I), (I′), (II), (III), (III′), (IV), *or* (IV′), *then it satisfies them all, and* (V) *and* (V′) *hold also.*

In the final section, §7, we discuss the existence problem for pairs $\mathbf{P}$, Γ satisfying conditions (I)–(IV). Among other things, we shall see that there is essentially only on possibility for Γ in the desarguesian case, for odd n. We shall also see that (I)–(IV) do not imply Desargues' theorem, or self-duality, for **P**. These results will be obtained as consequences of theorems on (possibly infinite) Pappus planes.

2. Preparations. The purpose of this section is to establish the terminology and notation to be used, and to collect some known results which will be needed for the proofs later on. We begin with a well-known group theoretical result.

(2.1) *If a Frobenius group G has complement C of order 2, then its kernel K is abelian.*

This is a special case of a theorem of Burnside (**3**, pp. 172–173), but the proof is so easy that we can give it here. Let c be the unique element $\neq 1$ in C, and $x \in K$. Then $xc \notin K$; hence xc is conjugate to c and therefore an involution. Thus $xx^c = (xc)^2 = 1$, or $x^c = x^{-1}$. But the anti-automorphism $x \to x^{-1}$ of K can coincide with the automorphism $x \to x^c$ only if K is abelian.

Let Γ be a group of permutations of the finite set S. For $x \in S$ and $\gamma \in \Gamma$, the image of x under γ is denoted by $x\gamma$. The set $x\Gamma = \{x\gamma|\, \gamma \in \Gamma\}$ is the Γ-*orbit* of x. The *stabilizer* of $x \in S$ is the subgroup Γ_x of all $\gamma \in \Gamma$ for which $x\gamma = x$. It is easily proved (Wielandt (**11**, p. 5)) that

(2.2) $|\Gamma| = |\Gamma_x|\, |x\Gamma|$ *for all* $x \in S$.

The following result of Gleason (**6**, Lemma 1.7) will be useful later on:

(2.3) *If for every $x \in S$ there exists a permutation of the same prime order p, fixing x but no other element of S, then Γ is transitive on S.*

Now let **P** be a finite projective plane and Γ a collineation group of **P**. Then Γ can be regarded as a permutation group of the points as well as of the lines of **P**. We shall denote points and point orbits by lower-case letters—Latin and German, respectively—and lines and line orbits by the corresponding capitals. The number of points in the point orbit $\mathfrak{p}$ which are incident with the line L of the line orbit $\mathfrak{L}$ depends only on $\mathfrak{p}$ and $\mathfrak{L}$, not on the choice of $L \in \mathfrak{L}$. We denote this number by $(\mathfrak{p}\mathfrak{L})$, and $(\mathfrak{L}\mathfrak{p})$ means, dually, the number of lines in $\mathfrak{L}$ through an arbitrary point of $\mathfrak{p}$. This is the notation of (**5**, p. 62); of the relations between the numbers (pL) and (Lp) proved in (**5**) we need here only

(2.4) $(\mathfrak{p}\mathfrak{L})|\mathfrak{L}| = (\mathfrak{L}\mathfrak{p})|\mathfrak{p}|$ *for all* Γ-*orbits* $\mathfrak{p}, \mathfrak{L}$.

A collineation in Γ is a *perspectivity* if it fixes a line (its *axis*) pointwise or, equivalently, a point (its *centre*) linewise. The identical collineation is a perspectivity in this sense, and it is in fact the only perspectivity with more than one centre and more than one axis. A perspectivity $\neq 1$ is called an *elation* or a *homology* according as its centre and axis are incident or not. For any point-line pair p, L, we denote by $\Gamma(p, L)$ the set of all perspectivities in Γ that have centre p and axis L. Clearly, $\Gamma(p, L)$ is a subgroup of Γ, and so are the sets

$$\Gamma(K, L) = \cup_{pIK}\, \Gamma(p, L) \quad \text{and} \quad \Gamma(L) = \cup_p\, \Gamma(p, L).$$

In particular, $\Gamma(L, L)$ is the group of all elations with axis L in Γ; it is easily seen to be a normal subgroup of the stabilizer Γ_L and a characteristic subgroup of $\Gamma(L)$, which in turn is normal in Γ_L. We shall need the following facts about these groups:

(2.5) *$\Gamma(L, L)$ is transitive on those $p \not\mathrm{I}\, L$ for which $\Gamma(p, L) \neq 1$.*

Proof. André **(1)**.

(2.6) *If the groups $\Gamma(p, L)$, for $p \mathrm{~I~} L$, all have the same order $h > 1$, then $h = n$.*

Proof. Gleason **(6**, Lemma 1.6**)**.

The order of $\Gamma(p, L)$ divides n or $n - 1$ according as $p \mathrm{~I~} L$ or $p \not\mathrm{I}\, L$. This shows:

(2.7) *Involutorial perspectivities are elations if n is even and homologies if n is odd.*

This is contained in Baer **(2)**, who also determined the nature of all other involutorial collineations:

(2.8) *If the involutorial collineation γ of the projective plane* **P** *of order n is not a perspectivity, then its fixed elements form a subplane* **Q** *of* **P** *such that every point of* **P** *is on a line of* **Q**. *The order of* **Q** *is $\sqrt{n}$, so that n must be a square.*

Proof. Baer **(2**, p. 285**)**.

Finally, we list two results on ovals in projective planes of odd order.

(2.9) *Through every point of an oval $\mathfrak{o}$ in* **P** *there passes exactly one tangent (i.e. a line having no second point with $\mathfrak{o}$ in common). If n is odd, then the number of tangents of $\mathfrak{o}$ through an arbitrary point $\notin \mathfrak{o}$ is either* 0 *or* 2.

Proof. Qvist **(8**, Theorem 3**)**.

(2.10) *Let q be an odd prime power and* **P** *the desarguesian projective plane over* GF(q). *Then every oval in* **P** *is a conic, i.e. the set of self-conjugate points of a projective polarity.*

Proof. Segre **(9)**.

3. (I) **implies** (II). Suppose that the projective plane **P** of order $n > 2$ and the collineation group Γ of **P** satisfy condition (I). Call z the exceptional point mentioned in (I); then clearly

(3.1) *z is fixed by Γ.*

Furthermore, it is immediate that

(3.2) *If a collineation $\neq 1$ in Γ fixes a point $\neq z$, then it is an involutorial perspectivity.*

Let π and π' be two of these perspectivities. Any common point of their axes is fixed by both π and π', hence must be z. This shows that:

(3.3) *Distinct perspectivities* $\neq 1$ *in* Γ *have distinct axes, and all these axes pass through* z.

We call $\mathfrak{A}$ the set of these axes. For any $A \in \mathfrak{A}$, let $c(A)$ be the centre of the unique perspectivity in Γ with axis A. Also, the set of all these centres $c(A)$ will be denoted by $\mathfrak{c}$. Result (2.7) shows that $c(A)$ I A if and only if n is even; we treat the easier case where n is odd first.

(3.4) *If* n *is odd, then* $\mathfrak{c}$ *consists of the* n *points* $\neq z$ *of some line* Z I z, *and* $\mathfrak{A}$ *of the* n *lines* $\neq Z$ *through* z. *Also,* $\mathfrak{c}$ *and* $\mathfrak{A}$ *are single* Γ-*orbits.*

Proof. Let $c \in \mathfrak{c}$ be the centre of the perspectivity $\pi \neq 1$ in Γ. As n is odd, π is a homology, whence $c \neq z$ by (3.3). Thus the line cz is well defined; we call it Z. If Z were in $\mathfrak{A}$, then c would be fixed by π and some non-trivial homology $\neq 1$ in Γ, contradicting (I). Hence $Z \notin \mathfrak{A}$, and (I) shows that each of the n points $\neq z$ on Z must be in $\mathfrak{c}$. The corresponding n axes then comprise all lines $\neq Z$ through z, and there can be no centre not on Z. This shows that

$$\mathfrak{A} = \{X \mid z \text{ I } X \neq Z\}$$

and $\mathfrak{c} = \{x \mid Z \text{ I } x \neq z\}$. Finally, a simple application of result (2.3), with $p = 2$, yields that $\mathfrak{A}$ and $\mathfrak{c}$ are Γ-orbits.

Parts (II.1)–(II.3) of condition (II) are now proved for odd n. Before turning to the even case, we show that:

(3.5) *If* n *is odd, then* $|\Gamma| = 2n$, *and all* Γ-*orbits* $\neq \{z\}, \{Z\}$ *have length* n.

Proof. If $c \in \mathfrak{c}$, then $|\Gamma_c| = 2$ by (I) and $|c\Gamma| = |\mathfrak{c}| = n$ by (3.4). Hence result (2.2) gives $|\Gamma| = 2n$. If $\mathfrak{x}$ is any point orbit $\neq \{z\}, \mathfrak{c}$, then there exists a point $x \in \mathfrak{x}$ on some line $A \in \mathfrak{A}$, whence

$$|\mathfrak{x}| = (\mathfrak{A}\mathfrak{x})|\mathfrak{x}| = (\mathfrak{x}\mathfrak{A})|\mathfrak{A}| = (\mathfrak{x}\mathfrak{A})n,$$

by (2.4) and (3.4). But $|\Gamma_x| = 2$ by (I), and now (2.2) gives

$$2n = |\Gamma| = |\Gamma_x|\,|\mathfrak{x}| = 2(\mathfrak{x}\mathfrak{A})n,$$

proving that $(\mathfrak{x}\mathfrak{A}) = 1$ and $|\mathfrak{x}| = n$. This proves (3.5) for point orbits; the proof for line orbits is dual.

Now we turn to the slightly more difficult case where n is even.

(3.6) *If* n *is even, then* $\mathfrak{c}$ *and* $\mathfrak{A}$ *are* Γ-*orbits of length* $n + 1$.

Proof. First, observe that $\mathfrak{A}$ must consist of all lines through z, by (I) and (3.3). Next, assume that $z \in \mathfrak{c}$. If there were a point $c \neq z$ in $\mathfrak{c}$, then c could not be on the axis corresponding to z; hence the elation with centre z moves c onto some other centre $c' \neq c$, and the line cz would be the axis for two distinct centres, contradicting (3.3). Thus, if $z \in \mathfrak{c}$, then $\mathfrak{c} = \{z\}$ and $c(A) = z$ for all $A \in \mathfrak{A}$. Consequently, $|\Gamma(z, A)| = 2$ for every A I z, so that the dual of result

https://doi.org/10.4153/CJM-1967-085-0 Published online by Cambridge University Press

(2.6) gives $n = 2$, against the general hypothesis $n > 2$. This proves that $z \notin \mathfrak{c}$, and it follows now that $A \neq A'$ implies $c(A) \neq c(A')$ for any $A, A' \in \mathfrak{A}$. Hence $|\mathfrak{c}| = |\mathfrak{A}| = n + 1$. Finally, that $\mathfrak{A}$ and $\mathfrak{c}$ are single Γ-orbits follows again from result (2.3). This proves (3.6), and a simple consequence of this and (2.2) is:

(3.7) *If n is even, then* $|\Gamma| = 2(n + 1)$.

For the proof of (II.1)–(II.3) in the even case, it is now sufficient to show that

(3.8) *If n is even, then the points of* $\mathfrak{c}$ *are collinear.*

Proof. Let Z be the line joining two distinct points of $\mathfrak{c}$, and $\mathfrak{Z}$ the line orbit containing Z. As $(\mathfrak{c}\mathfrak{A}) = 1 < (\mathfrak{c}\mathfrak{Z})$, we have $\mathfrak{Z} \neq \mathfrak{A}$, whence in particular Z is not incident with z. Let Σ be the subgroup generated by the $(\mathfrak{c}\mathfrak{Z})$ elations in Γ whose centres are on Z; clearly $\Sigma \subseteq \Gamma_Z$. Also, result (2.3) shows that Σ is transitive on the $(\mathfrak{c}\mathfrak{Z})$ points of $\mathfrak{c}$ on Z; this and the obvious $|\Sigma_c| = 2$ show that $|\Sigma| = 2(\mathfrak{c}\mathfrak{Z})$.

Now assume that $(\mathfrak{c}\mathfrak{Z}) < n + 1$. Then there exist points $\notin \mathfrak{c}$ on Z, and for any such point p we have $|p\Sigma| = |\Sigma| = 2(\mathfrak{c}\mathfrak{Z})$, since otherwise some $\sigma \neq 1$ in Σ fixes p and Z, whence $\sigma \in \Gamma(p, Z)$ by (I), and $p \in \mathfrak{c}$. It follows that Σ permutes the points $\notin \mathfrak{c}$ of Z in orbits of length $2(\mathfrak{c}\mathfrak{Z})$, so that

$$\text{(a)} \qquad n + 1 = (2k + 1)(\mathfrak{c}\mathfrak{Z}) \qquad \text{for some } k \geqslant 1.$$

Furthermore, there are $n + 1 - (\mathfrak{c}\mathfrak{Z})$ points of $\mathfrak{c}$ not on Z. These are on $2k(\mathfrak{c}\mathfrak{Z})$ distinct lines in $\mathfrak{A}$; hence the corresponding $2k(\mathfrak{c}\mathfrak{Z})$ elations $\neq 1$ in Γ map Z onto $2k(\mathfrak{c}\mathfrak{Z})$ distinct lines $\neq Z$ in $\mathfrak{Z}$. Hence

$$\text{(b)} \qquad |\mathfrak{Z}| \geqslant 1 + 2k(\mathfrak{c}\mathfrak{Z}).$$

On the other hand, $\Sigma \subseteq \Gamma_Z$ implies that $2(\mathfrak{c}\mathfrak{Z})|\mathfrak{Z}| = |\Sigma||\mathfrak{Z}|$ divides

$$|\Gamma_Z||\mathfrak{Z}| = |\Gamma|.$$

But $|\Gamma| = 2(n + 1)$; hence $|\mathfrak{Z}|$ divides $(n + 1)(\mathfrak{c}\mathfrak{Z})^{-1} = 2k + 1$; cf. (a). Therefore, $|\mathfrak{Z}| \leqslant 2k + 1$. Comparing this with (b), we obtain $(\mathfrak{c}\mathfrak{Z}) \leqslant 1$. But $(\mathfrak{c}\mathfrak{Z}) > 1$ by definition; hence the above assumption is false, and $(\mathfrak{c}\mathfrak{Z}) = n + 1$. This shows that $\mathfrak{Z} = \{Z\}$ and $\mathfrak{c} = \{x \mathrm{I} Z\}$, proving (3.8).

This completes the proof of (II.1)–(II.3) also for even n. For the proof of (II.4), we first note that:

(3.9) *If n is even, then every Γ-orbit $\neq \{z\}, \{Z\}$ has length $n + 1$.*

This is shown in the same fashion as the corresponding fact in (3.5) for odd n; we omit this verification.

Finally, we prove (II.4), simultaneously for odd and even n. Let $\mathfrak{x}$ be any point orbit $\neq \{z\}$; then $|\mathfrak{x}| = |\Gamma|/2$, by (3.5), (3.7), (3.9). If $x \in \mathfrak{x}$, then the only collineation in Γ fixing x is an involutorial perspectivity which fixes no other point in $\mathfrak{x}$. This shows that Γ acts faithfully as a Frobenius group on $\mathfrak{x}$, and the Frobenius kernel is precisely the set Δ of (II.4), consisting of 1 and the non-perspectivities in Γ. By Frobenius' theorem, Δ is a transitive regular normal subgroup of Γ; furthermor, Δ is abelian by result (2.1). This proves (II.4) for point orbits, and a dual argument yields the proof for line orbits.

4. (II) **implies** (III). Let **P** and Γ satisfy condition (II), and again let $\mathfrak{A}$ be the set of all lines $\neq Z$ through z. By (II.3) and (II.4), $\mathfrak{A}$ is precisely the set of axes of non-trivial perspectivities in Γ, and by result (2.3), with $p = 2$, $\mathfrak{A}$ is a single Γ-orbit. We show that $\mathfrak{A}$ satisfies condition (III).

(III.1) is obvious, and (III.2) follows immediately from (2.2), since every $A \in \mathfrak{A}$ is the axis of a perspectivity $\neq 1$ in Γ. Thus it remains to prove (III.3).

The subgroup Δ mentioned in (II.4) has the same orbits as Γ, whence

(4.1) $|\Delta| \geqslant |\mathfrak{x}|, |\mathfrak{X}|$, *for all* Γ-*orbits* $\mathfrak{x}, \mathfrak{X}$.

On the other hand, as Δ is abelian, the permutation group induced by Δ on $\mathfrak{A}$ is regular. This means that if $\delta \in \Delta$ fixes the line $A \in \mathfrak{A}$, then δ fixes every line of $\mathfrak{A}$, and therefore every line through z. But then δ is a perspectivity with centre z, and the definition of Δ shows that $\delta = 1$. Consequently, $\Delta_A = 1$ for every $A \in \mathfrak{A}$, and (2.2) gives

$$|\Delta| = |\mathfrak{A}|. \tag{4.2}$$

Condition (III.3) now follows from (4.1) and (4.2).

5. (III) **implies** (IV) **and** (V). Suppose that **P** and Γ satisfy condition (III).

(5.1) *If* $\mathfrak{p}$ *is a point orbit* $\neq \{z\}$ *such that* $(\mathfrak{p}\mathfrak{A}) \neq 0$, *then* $|\mathfrak{p}| = |\mathfrak{A}|$ *and* $(\mathfrak{p}\mathfrak{A}) = (\mathfrak{A}\mathfrak{p}) = 1$.

Proof. That $(\mathfrak{A}\mathfrak{p}) = 1$ follows immediately from (III.1) and $\mathfrak{p} \neq \{z\}$. But then (III.3) and (2.4) give $|\mathfrak{p}| \leqslant |\mathfrak{A}| \leqslant (\mathfrak{p}\mathfrak{A})|\mathfrak{A}| = (\mathfrak{A}\mathfrak{p})|\mathfrak{p}| = |\mathfrak{p}|$, implying that $|\mathfrak{p}| = |\mathfrak{A}|$ and $(\mathfrak{p}\mathfrak{A}) = 1$.

(5.2) *For any line* $A \in \mathfrak{A}$, *the stabilizer* Γ_A *consists entirely of perspectivities with axis* A.

Proof. Let $\gamma \in \Gamma_A$; we must show that γ fixes every point $p \mathrel{\mathrm{I}} A$. This is trivial if $p = z$; hence suppose that $p \neq z$. But then the orbit $\mathfrak{p} = p\Gamma$ satisfies the hypothesis of (5.1), whence $(\mathfrak{p}\mathfrak{A}) = 1$. This means that p is the only point of $\mathfrak{p}$ on A, so that γ must fix p with A.

(5.3) *Only the identical collineation in* Γ *fixes more than one line of* $\mathfrak{A}$.

This is an immediate consequence of (5.2): no perspectivity $\neq 1$ can have two axes. (5.3) implies, in particular, that no perspectivity in Γ has centre z, and:

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(5.4) *If* $\gamma \neq 1$ *is a homology in* Γ_A, *for* $A \in \mathfrak{A}$, *then the centre of* γ *is on no line of* $\mathfrak{A}$.

For the following, it will be convenient to use the following abbreviation. If p is any point $\neq z$, and if $\mathfrak{p} = p\Gamma$, let

$$r(p) = r(\mathfrak{p}) = \sum_{\substack{\mathfrak{X} \neq \mathfrak{A} \\ (\mathfrak{p}\mathfrak{X}) = 1}} (\mathfrak{X}\mathfrak{p});$$

in other words, $r(p) = r(\mathfrak{p})$ is the number of lines $\notin \mathfrak{A}$ through p which carry no point $\neq p$ of $\mathfrak{p}$.

(5.5) *Suppose that* $z \neq c \mathrm{\ I\ } A \in \mathfrak{A}$ *and* $r(c) > 1$. *Then* Γ_A *consists of elations with centre* c.

Proof. By (5.2), any $\gamma \neq 1$ in Γ_A is a perspectivity with axis A; assume that the centre of γ is not c. As $r(c) > 1$, there exists a line L with the following properties:

(a) L is not the axis of γ.

(b) L is not incident with the centre of γ.

(c) $L \mathrm{\ I\ } c$, and no point $\neq c$ of the orbit $\mathfrak{c} = c\Gamma$ is on L.

Denote by $\mathfrak{L}$ the line orbit containing L. By (a) and (b), $L\gamma \neq L$, so that $(\mathfrak{L}\mathfrak{c}) > 1$. On the other hand, $(\mathfrak{c}\mathfrak{L}) = 1$ by (c), and now (5.1), (2.4), and (III.3) give

$$|\mathfrak{A}| = |\mathfrak{c}| < (\mathfrak{L}\mathfrak{c})|\mathfrak{c}| = (\mathfrak{c}\mathfrak{L})|\mathfrak{L}| = |\mathfrak{L}| \leqslant |\mathfrak{A}|,$$

a contradiction. Hence c is the centre of γ, and (5.5) is proved.

(5.6) *An arbitrary line* $A \in \mathfrak{A}$ *carries at most one point* $c \neq z$ *with* $r(c) > 1$.

This follows immediately from (5.5): no elation $\neq 1$ can have two distinct centres.

(5.7) $|\mathfrak{A}| = n$ *or* $n + 1$, *and* $|\mathfrak{A}| = n + 1$ *if and only if* Γ_A (*for* $A \in \mathfrak{A}$) *contains non-trivial elations*.

Proof. (5.6) shows that every $A \in \mathfrak{A}$ carries at least one point p with $r(p) \leqslant 1$. This means that at least $n - 1$ of the n lines $\neq A$ through p carry points $\neq p$ of the orbit $\mathfrak{p} = p\Gamma$. These points are all distinct, whence $n - 1 \leqslant |\mathfrak{p}| - 1$ and $|\mathfrak{A}| \geqslant n$, by (5.1). On the other hand, (III.1) clearly gives $|\mathfrak{A}| \leqslant n + 1$.

If $|\mathfrak{A}| = n + 1$, then every point of $\mathbf{P}$ is on a line of $\mathfrak{A}$, whence (5.4) shows that Γ_A, for any $A \in \mathfrak{A}$, consists of elations. Conversely, if $|\mathfrak{A}| = n$, then there is a unique line $Z \notin \mathfrak{A}$ through z, which must be fixed by Γ. In this case, if γ is an elation in Γ_A, then $Z\gamma = Z$, showing that the centre of γ is $A \cap Z = z$. But then (5.3) gives $\gamma = 1$, and (5.7) is proved.

(5.8) *Suppose the point orbit* $\mathfrak{p} \neq \{z\}$ *satisfies* $r(p) \leqslant 1$ *and* $(\mathfrak{p}\mathfrak{A}) \neq 0$. *Then* $\mathfrak{p}$ *contains no three collinear points*.

Proof. Assume the contrary. Then there exists a line orbit $\mathfrak{W}$ with $(\mathfrak{p}\mathfrak{W}) \geqslant 3$; let $W \in \mathfrak{W}$ and p be one of the points of $\mathfrak{p}$ on W. Also, let A be the unique line of $\mathfrak{A}$ through p. As $r(\mathfrak{p}) \leqslant 1$, there is at most one line $\neq A$ through p which carries no point $\neq p$ of $\mathfrak{p}$. If there were no such line, there would be at least $n-1$ points of $\mathfrak{p}$ not incident with W, whence $|\mathfrak{p}| > n+1$, contradicting the fact that $|\mathfrak{p}| = |\mathfrak{A}| \leqslant n+1$ which follows from (5.1). Hence there is a unique line $L \neq A$ through p such that p is the only point of p on L. Also, each line $X \neq A, L, W$ through p carries precisely one point $x \neq p$ of $\mathfrak{p}$.

Now let γ be a perspectivity $\neq 1$ in Γ_A. The number of points in $\mathfrak{p}$ on any line is invariant under γ; hence L and W must be fixed by γ, so that their intersection point p is the centre of γ. By the general hypothesis $n > 2$, there is a line $X \neq A, L, W$ through p. As observed above, X carries precisely one point $\neq p$ of p; this point must clearly be fixed by γ, but is not on the axis A of the elation γ. This gives $\gamma = 1$, a contradiction, proving (5.8).

(5.9) *If* $\mathfrak{p}$ *is a point orbit* $\neq \{z\}$ *for which either* $r(\mathfrak{p}) > 1$ *or* $(\mathfrak{p}\mathfrak{A}) = 0$, *then the points of* $\mathfrak{p}$ *are collinear.*

Proof. Suppose first that $(\mathfrak{p}\mathfrak{A}) = 0$. Then (5.7) gives $|\mathfrak{A}| = n$, so that $\mathfrak{p}$ must consist of points of the unique line $Z \notin \mathfrak{A}$ through z. Hence we need only consider the case where $r(\mathfrak{p}) > 1$ and $(\mathfrak{p}\mathfrak{A}) \neq 0$. By (5.5), if p is the unique point of $\mathfrak{p}$ on the line $A \in \mathfrak{A}$, then $\Gamma_A = \Gamma(p, A)$. Consider two distinct points $p, p' \in \mathfrak{p}$ and the line $Z = pp'$, and assume that Z contains a points x with $r(x) \leqslant 1$. As x is not fixed by either one of the groups Γ_{zp}, $\Gamma_{zp'}$, there must exist a second point $x' \neq x$ of the orbit $\mathfrak{x} = x\Gamma$ on Z. But then (5.8) shows, as $r(\mathfrak{x}) \leqslant 1$, that there is no third point of $\mathfrak{x}$ on Z. Therefore, x and x' constitute a full orbit under either of the groups Γ_{zp}, $\Gamma_{zp'}$, whence $|\Gamma_{zp}| = |\Gamma_{zp'}| = 2$. Let γ, γ' be the unique non-trivial elations in Γ_{zp} and $\Gamma_{zp'}$, respectively. Then $\gamma\gamma'$ fixes x and x' and therefore zx and zx'. Hence (5.3) gives $\gamma\gamma' = 1$ or $\gamma = \gamma'$, which is clearly false since γ and γ' have distinct centres. Thus $Z = pp'$ carries no point x with $r(x) \leqslant 1$, and (5.6) shows that all points of $\mathfrak{p}$ are on Z.

(5.10) $|\Gamma_A| = 2$ *for every* $A \in \mathfrak{A}$.

Proof. Put $|\Gamma_A| = k$. We consider two cases.

Case 1. Γ_A *contains a non-trivial elation.* Then (5.7) gives $|\mathfrak{A}| = n+1$, so that every point of **P** is on some line of $\mathfrak{A}$. Next, (5.4) shows that Γ_A contains no homologies, whence Γ_A consists of elations with axis A. For any one of the n lines $X \neq A$ in $\mathfrak{A}$, the orbit $A\Gamma_X$ consists of $k = |\Gamma_X|$ lines, by (5.3). If two distinct orbits of this kind, say $A\Gamma_X$ and $A\Gamma_Y$ with $A \neq X, Y \in \mathfrak{A}$, had a line $A' \neq A$ of $\mathfrak{A}$ in common, then there would exist $\xi \in \Gamma_X$ and $\eta \in \Gamma_Y$ with $A\xi\eta = A'\eta = A$. Hence $\xi\eta$ would be a perspectivity, by (5.2). But ξ and η are elations with different centres and axes, and the product of two such elations is never a perspectivity. This contradiction proves that

$$A\Gamma_X \cap A\Gamma_Y = \{A\} \qquad \text{for } X \neq Y.$$

Consequently, the union $\bigcup_{A \neq X \in A} A\Gamma_X$ consists of $1 + n(k-1)$ lines, and we can conclude that

$$n + 1 \geqslant 1 + n(k-1),$$

which implies that $k \leqslant 2$. Since $k \geqslant 2$ is an obvious consequence of (III.2), the proof of (5.10) is complete in this case.

Case 2. Γ_A *contains no non-trivial elation.* Then Γ_A consists of homologies, by (5.2), and result (2.5) shows that $\Gamma_A = \Gamma(c, A)$, for some fixed point $c \not I A$. Consider a line $A' \neq A$ in $\mathfrak{A}$ and a point p with $r(p) \leqslant 1$ on A'; this exists by (5.6). Then $p \neq c$, because of (5.4), and the orbit $p\Gamma_A$ consists of precisely $|\Gamma_A| = k$ points. But these points are collinear, and now (5.8) implies that $k \leqslant 2$ again.

This proves (5.10), and condition (IV.1) now follows from (5.7) and results (2.2), (2.7). It remains to prove condition (V), which clearly implies (IV.2).

First, let n be odd. Then the only non-trivial perspectivity in Γ_A is a homology, and $|A| = n$ by (5.7). Hence there is a unique line $Z \notin \mathfrak{A}$ through z which carries all the centres of the homologies in Γ. Let $\mathfrak{p}$ be any point orbit of length >1. Then $\mathfrak{p} \neq \{z\}$, and if $(\mathfrak{p}\mathfrak{A}) \neq 0$, then $|\mathfrak{p}| = |\mathfrak{A}| = n$ by (5.1). Also, as there are no elations $\neq 1$ in Γ, we have $r(\mathfrak{p}) \leqslant 1$, by (5.5). But then (5.8) shows that $\mathfrak{p} \cup \{z\}$ is an oval. If $(\mathfrak{p}\mathfrak{A}) = 0$, then $\mathfrak{p}$ consists of points $\neq z$ on Z. Let $\mathfrak{o}$ be one of the ovals just found; then Z is one of its tangents. Let $\mathfrak{T}$ be the set of the n remaining tangents of $\mathfrak{o}$; this is an orbit since $\mathfrak{o} - \{z\}$ is an orbit. Hence the intersection points $T \cap Z$, with $T \in \mathfrak{T}$, also form an orbit, and by result (2.9) these are precisely the n points $\neq z$ of Z. This implies again that $|\mathfrak{p}| = n$, and (V.1) is proved.

Finally, let n be even. Then $|\mathfrak{A}| = n + 1$, and every point orbit $\mathfrak{p}$ satisfies $(\mathfrak{p}\mathfrak{A}) \neq 0$. In fact, if $|\mathfrak{p}| > 1$, then $\mathfrak{p} \neq \{z\}$, and (5.1) gives $|\mathfrak{p}| = n + 1$. If $r(\mathfrak{p}) \leqslant 1$, then $\mathfrak{p}$ is an oval with knot z, by (5.8), and if $r(\mathfrak{p}) > 1$, then the points of $\mathfrak{p}$ are those of a fixed line Z, by (5.9). This proves (V.2), and we have now completed the proof that (III) implies (IV) and (V).

6. (IV) **implies** (I). Suppose that **P** and Γ satisfy condition (IV). In particular, let m be n or $n + 1$, whichever of these integers is odd. Since the number of points in **P** is

$$n(n + 1) + 1 \equiv 1 \mod m,$$

it follows from (IV.2) that there is a unique fixed point of Γ; we call this point again z. From (IV.1) we infer that

$$|\Gamma_x| = 2 \qquad \text{for every point } x \neq z; \tag{6.1}$$

we put $\Gamma_x = \{1, \pi(x)\}$ for these x. Clearly, $\pi(x)$ is an involution. In view of (6.1), for the proof of (I), it remains to show that these involutions $\pi(x)$ are all perspectivities.

Let $\mathfrak{x}$ be an arbitrary point orbit $\neq \{z\}$, so that

$$|\mathfrak{x}| = m \geqslant n, \tag{6.2}$$

by (IV.2). Since the subgroups Γ_x, for $x \in \mathfrak{x}$, are conjugate, either all or none of the corresponding $\pi(x)$ are perspectivities. Consider the second alternative.

Then result (2.8) yields that

$$(6.3) \qquad n = t^2 \qquad \text{for some integer } t > 1,$$

and that each of the $\pi(x)$ in question fixes precisely $t^2 + t + 1$ points. On the other hand, $\pi(x)$ and $\pi(y)$, for $x \neq y$, cannot have a common fixed point $\neq z$, by (6.1). Hence the union of the sets of fixed points of all $\pi(x)$ with $x \in \mathfrak{x}$ consists of precisely $1 + |\mathfrak{x}|(t^2 + t)$ points of $\mathbf{P}$. This, together with (6.2) and (6.3), yields

$$t^4 + t^3 = t^2(t^2 + t) \leqslant |\mathfrak{x}|(t^2 + t) \leqslant n^2 + n = t^4 + t^2,$$

a contradiction. Hence the $\pi(x)$ are perspectivities, and (I) is proved.

This completes the proof of the theorem stated in the Introduction.

7. Examples. Let $\mathbf{P}$ be the projective plane over the commutative field F of characteristic $\neq 2$. Also, let $\mathfrak{c}$ be a conic in $\mathbf{P}$, v a fixed point of $\mathfrak{c}$, and W the tangent of $\mathfrak{c}$ in v. Let $\mathbf{A}$ be the affine plane with respect to W. By a suitable choice of $o = (0, 0)$ and $e = (1, 1)$ in $\mathbf{A}$, and of $u \neq v$ on W, we can introduce affine coordinates (x, y) in $\mathbf{A}$ with respect to which $\mathfrak{c}$ is given by the equation

$$(7.1) \qquad y = x^2.$$

One verifies that the projective collineations of $\mathbf{P}$ which leave $\mathfrak{c}$ and v invariant are precisely those given by

$$(7.2)\ \pi(a, c)\colon (x, y) \to (ax + c, 2acx + a^2y + c^2) \qquad \text{with } a, c \in F \text{ and } a \neq 0.$$

These collineations are multiplied according to the rule

$$(7.3) \qquad \pi(a, c)\pi(a', c') = \pi(aa', a'c + c').$$

This shows that the $a \neq 0$ involved in a subgroup of the group of all collineations (7.2) must themselves form a subgroup of the multiplicative group F^* of F. Let Γ be the collineation group obtained by restricting a to the subgroup $\{1, -1\}$ of F^*. Then the Γ-orbit of an arbitrary point $(x_0, y_0) \in \mathbf{A}$ is easily seen to be the conic $y = x^2 + b$, with $b = y_0 - x_0^2$. After these preparations, we can prove:

THEOREM 1. *In a finite desarguesian projective plane of odd order, a collineation group* Γ *satisfies one, and hence all, of the conditions* (I)–(IV) *if, and only if, it can be represented by the mappings* (7.2) *with* $a = \pm 1$.

Proof. If $F = \mathrm{GF}(q)$ with odd q, then the group Γ of the $\pi(\pm 1, c)$ given by (7.2) has order $2q$. Also, every point orbit in $\mathbf{A}$ consists of the q points $\neq v$ of a conic $y = x^2 + b$. As conics are ovals, (2.9) shows that the ideal points $\neq v$ also form a Γ-orbit of length q. Hence Γ satisfies (IV) and therefore also (I)–(III).

Conversely, let Γ satisfy (I)–(IV). By the main theorem, Γ also satisfies (V).

https://doi.org/10.4153/CJM-1967-085-0 Published online by Cambridge University Press

Hence there is an oval $\mathfrak{c}$ and a point $v \in \mathfrak{c}$ fixed by Γ. Using Segre's theorem (2.10), we can identify $\mathfrak{c}$ with the conic (7.1). Also, Γ is generated by perspectivities, by (II); hence Γ consists of projective collineations, and therefore of collineations of the form $\pi(a, c)$ given by (7.2). Since $(0, 0)\pi(a, c) = (c, c^2)$, every $c \in F$ must actually occur in some $\pi(a, c) \in \Gamma$. It follows that $|\Gamma| = qt$, with t the order of the multiplicative subgroup S of the elements $a \neq 0$ involved in the $\pi(a, c)$ which make up Γ. But $|\Gamma| = 2q$ by condition (IV.I); hence $t = 2$, and since F^* is cyclic, S must be the unique subgroup $\{1, -1\}$ of order 2. Therefore, Γ consists precisely of the $\pi(\pm 1, c)$, and Theorem 1 is proved.

We remark at this point that the mappings $\pi(\pm 1, c)$ are collineations not only in the desarguesian case when F is a field, but also whenever F is a distributive quasi-field (satisfying all skew-field axioms except perhaps associativity of multiplication) of characteristic $\neq 2$. Hence there exist groups satisfying (I)–(IV) also in the non-desarguesian planes over distributive quasi-fields. Such a plane need not be self-dual.

We cannot prove a theorem as strong as Theorem 1 for even q since in this case the analogue to Segre's theorem (2.10) is false. However, there exist groups satisfying (I)–(V) also in Pappus planes of characteristic 2; we show this in a special case.

THEOREM 2. *Let F be a commutative field of characteristic 2, such that the multiplicative group F^* contains no element of order 3. (This condition is satisfied, for example, by all* GF(2^{2m+1}).) *Let* $\mathbf{A}$ *be the affine plane over F, and $\mathfrak{c}$ the point set defined by*

$$x^2 + xy + y^2 = 1. \tag{7.4}$$

Also, let Γ be the group of all projective collineations of $\mathbf{A}$ *which leave $\mathfrak{c}$ invariant and fix the point* $(0, 0)$. *Then Γ is non-regular on any one of its orbits, and the lines through* $(0, 0)$ *form a maximal* Γ-orbit $\mathfrak{A}$. *Hence if F is finite, then* Γ *satisfies condition* (III) *and hence also* (I), (II), (IV), *and* (V).

Proof. Any projective collineation fixing (0, 0) is of the form

$$\phi\colon (x, y) \to (ux + vy, sx + ty).$$

Since $(1, 0)$, $(1, 1)$, $(0, 1)$ are points of $\mathfrak{c}$, the condition that their images under ϕ are also on $\mathfrak{c}$ gives

$$u^2 + us + s^2 = v^2 + vt + t^2 = ut + vs = 1. \tag{a}$$

Consider first the case $u \neq 0$. Then (a) gives

$$\begin{aligned} u^{-2}(v^2 + uv + 1) &= u^{-2}[v^2(u^2 + us + s^2) + uv + 1] \\ &= v^2 + v(1 + vs)u^{-1} + (1 + v^2s^2)u^{-2} \\ &= v^2 + vt + t^2 = 1, \end{aligned}$$

whence $u^2 = v^2 + uv + 1$, or

$$\text{(b)} \qquad u^2 + uv + v^2 = 1.$$

This and (a) together give $us + s^2 = u^2 + 1 = uv + v^2$, or

$$\text{(c)} \qquad u(v + s) = (v + s)^2.$$

Now either $v = s$, and then $t = (1 + v^2)u^{-1} = (u^2 + uv)u^{-1} = u + v$, by (a) and (b); or $v \neq s$, and then $u = v + s$ by (c). This shows that if $u \neq 0$, then ϕ is one of the collineations

$$\text{(7.5)} \qquad \left.\begin{aligned} \gamma(u, v)&: (x, y) \to (ux + vy, (u + v)x + uy) \\ \delta(u, v)&: (x, y) \to (ux + vy, vx + (u + v)y) \end{aligned}\right\} \text{where } u, v \text{ satisfy (b).}$$

But if $u = 0$, then (a) gives $s = v = 1$ and $t = 0$ or 1; consequently, ϕ is either $\gamma(0, 1)$ or $\delta(0, 1)$. Hence ϕ is always of the form (7.6). Conversely, it is straightforward to verify that all $\gamma(u, v)$ and $\delta(u, v)$ fix $\mathfrak{c}$ and $(0, 0)$; hence the group Γ of Theorem 2 consists precisely of the collineations (7.5).

An easy consequence of (7.5) is

$$\delta(u, v)\delta(u', v') = \delta(uu' + vv', uv' + vu' + vv');$$

this implies that the $\delta(u, v)$ form a subgroup Δ of Γ. Also, $(1, 0) \in \mathfrak{c}$ is mapped onto $(u, v) \in \mathfrak{c}$ by precisely one collineation in Δ, namely by $\delta(u, v)$; hence Δ is transitive and regular on $\mathfrak{c}$. Next, we show that:

(7.6) *Every line through* $(0, 0)$ *meets* $\mathfrak{c}$ *in a unique point.*

This is clear for the line $x = 0$; hence we need only consider the lines $y = mx$. An intersection point (u, v) of $\mathfrak{c}$ with the line $y = mx$ must satisfy

$$u^2(m^2 + m + 1) = 1.$$

But the polynomial $x^2 + x + 1$ is irreducible in F (otherwise F^* would contain elements of order 3); hence $m^2 + m + 1 \neq 0$, and

$$u = (m^2 + m + 1)^{-2}$$

is uniquely determined by m.

This proves (7.6), and the transitivity of Δ on $\mathfrak{c}$ implies now that the set $\mathfrak{A}$ of all lines through $(0, 0)$ is a single Γ-orbit on which Δ acts transitively and regularly. Hence the only non-trivial collineations in Γ which fix a line in $\mathfrak{A}$ must be of the form $\gamma(u, v)$. Since

$$(x, 0)\gamma(u, v) = (ux, (u + v)x),$$

and since this equals $(ux, 0)$ if and only if $u + v = 0$, whence $u = v = 1$ by (b), the stabilizer of the line $y = 0$ in $\mathfrak{A}$ consists of 1 and $\gamma(1, 1)$ only. But $\gamma(1, 1)$ is an elation with axis $y = 0$; hence for every $A \in \mathfrak{A}$ the stabilizer Γ_A contains only one non-trivial collineation, viz., an elation with axis A. The centres of all these elations are all ideal points of **A**; hence every line is fixed by some elation in Γ. The same is true for every point of the projective plane determined by **A**, and we can now conclude that Γ is not regular on any orbit. Finally, $|\Gamma_A| = 2$ for $A \in \mathfrak{A}$ shows that $\mathfrak{A}$ is a Γ-orbit of maximal cardinality. This completes the proof.

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