GEOMETRIC REALIZATIONS FOR FREE QUOTIENTS

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(Received 6 March 1970)

Communicated by G. E. Wall

1. Introduction

In [7] Lyndon introduced the concept of inner rank for groups. He defined the *inner rank* of an arbitrary group G to be the upper bound of the ranks of free homomorphic images of G. Both Lyndon and Jaco have shown that the inner rank of the fundamental group of a closed 2-manifold with Euler characteristic $2 - p$, $p \geqq 0$, is $[p/2]$ where $[p/2]$ is the greatest integer $\leqq p/2$. The proof given by Lyndon [8] uses algebraic techniques; whereas, the proof by Jaco [4] is geometrical.

If the free group F is a homomorphic image of the group G, we call F a free quotient of G. The purpose of this paper is to give a geometrical interpretation to free quotients of finitely presented groups.

In section 3 we show that whenever the group G can be expressed as a free product $G \approx G_1 * G_2$ where both G_1 and G_2 are finitely presented groups, then the inner rank of G is the sum of the inner ranks of G_1 and G_2.

2. Geometric realizations for free quotients

By *n-manifold* we shall mean a compact connected combinatorial n-manifold possibly with boundary [3, p. 26]. We denote the *interior* of an n-manifold M^n by $\mathrm{Int}\, M^n$. The *boundary* of an n-manifold defined as M^n-$\mathrm{Int}\, M^n$ is denoted δM^n. If $\delta M^n = \emptyset$, then M^n is said to be *closed*. A *k-submanifold*, N^k, of the n-manifold M^n is a k-manifold embedded as a subcomplex in some subdivision of M^n. If N^k is a k-submanifold of M^n, we say N^k is *properly embedded* in M^n if

$$N^k \cap \delta M^n = \delta N^k.$$

By a *surface* in the n-manifold M^n, we mean an $(n-1)$-submanifold, N^{n-1}, of M^n where N^{n-1} is properly embedded in M^n.

* The author was supported in part by NSF contract GP-11533.

Let I denote the interval $[-1,1]$. A surface N^{n-1} in M^n is said to have a *product neighborhood* $P(N^{n-1})$ if there is a *PL* embedding

$$h:(N^{n-1}\times I,\ \delta N^{n-1}\times I)\rightarrow(M^n,\delta M^n)$$

so that $h(N^{n-1}\times I)=P(N^{n-1})$ is a neighborhood of N^{n-1} and $h(s\times 0)=s$ for each $s\in N^{n-1}$. The embedding h is called a *parametrization* of $P(N^{n-1})$.

The collection $N_1^{n-1},\cdots,N_k^{n-1}$ of surfaces in the n-manifold M^n is said to be a *system of surfaces in* M^n if

a) $N_i^{n-1}\cap N_j^{n-1}=\varnothing,\ i\neq j$, and

b) each N_i^{n-1} has a product neighborhood

$$P(N_i^{n-1}) \text{ in } M^n.$$

A system of surfaces $N_1^{n-1},\cdots,N_k^{n-1}$ is called *independent* if

$$M^n-\bigcup_{i=1}^{k}N_i^{n-1}$$

is connected. Note that whenever $N_1^{n-1},\cdots,N_k^{n-1}$ is a system of surfaces in M^n, then the product neighborhoods $P(N_1^{n-1}),\cdots,P(N_k^{n-1})$ guaranteed by condition b) may be chosen so that

$$P(N_i^{n-1})\cap P(N_j^{n-1})=\varnothing,\quad i\neq j.$$

A group G has a *free quotient of rank* r if there is a homomorphism ϕ of G *onto* a free group of rank r.

THEOREM 2.1. *Let M^n be an n-manifold and let G denote the fundamental group of M^n. Then G has a free quotient of rank r if and only if there is an independent system of surfaces $N_1^{n-1},\cdots,N_r^{n-1}$ in M^n.*

PROOF. Suppose $N_1^{n-1},\cdots,N_r^{n-1}$ is an independent system of surfaces in M^n. Choose product neighborhoods $P(N_1^{n-1}),\cdots,P(N_r^{n-1})$, one for each N_i^{n-1}, so that

$$P(N_i^{n-1})\cap P(N_j^{n-1})=\varnothing,\quad i\neq j.$$

Then

$$M_1^n=M^n-\bigcup_{i=1}^{r}P^0(N_i^{n-1})$$

($Y\subset X$, then Y^0 denotes the point set interior of Y in X) is a connected n-manifold and δM_1^n contains two copies of N_i^{n-1} for each $i=1,\cdots,r$.

Let

$$h_i:N_i^{n-1}\times I\rightarrow P(N_i^{n-1})$$

denote a parametrization of $P(N_i^{n-1})$ guaranteed by the definition of $P(N_i^{n-1})$.

Choose $s_i \in N_i^{n-1}$ and let $s_{ij} = h_i(s_i \times j)$ for $j = -1, 1$. Choose $s_0 \in \operatorname{Int} M_1^n$. There are arcs α_{ij}, $i = 1, \cdots, r$; $j = -1, 1$, embedded in M_1^n as subcomplexes of some subdivision of M_1^n so that

a) α_{ij} is an arc from s_0 to s_{ij},
b) $\alpha_{ij} \cap \alpha_{kl} = \{s_0\}$, $(i, j) \neq (k, l)$, and
c) $\alpha_{ij} - \{s_{ij}\} \subset \operatorname{Int} M_1^n$.

Let $\Gamma' = \bigcup_{i,j} \alpha_{ij}$. Then Γ' is a wedge at s_0 of the arcs α_{ij}. There is a retraction f_1 of $P(N_i^{n+1})$ onto $h_i(s_i \times I)$. Let f_2' denote the retraction of

$$\Gamma' \cup \bigcup_{i,j} h_i(N_i^{n-1} \times \{j\}), \quad j = -1, 1$$

onto Γ' defined as

$$f_2' \mid \Gamma' = \mathrm{id} \mid \Gamma', \text{ and}$$

$$f_2' \mid \bigcup_{i,j} h_i(N_i^{n-1} \times \{j\}) = f_1 \mid \bigcup_{i,j} h_i(N_i^{n-1} \times \{j\}).$$

Then by Tietze's Theorem [2], there is an extension f_2 of f_2' retracting M_1^n onto Γ'.

Let Γ be the wedge of r simple closed curves defined as

$$\Gamma = \Gamma' \cup \bigcup_i h_i(s_i \times I).$$

The map $f: M^n \to \Gamma$ defined as

$$f \mid \bigcup_i P(N_i^{n-1}) = f_1 \mid \bigcup_i P(N_i^{n-1}) \quad \text{and}$$

$$f \mid M_1^n = f_2 \mid M_1^n$$

is a retraction of M^n *onto* Γ. Hence, G has a free quotient of rank r.

Choose a point $s_0 \in \operatorname{Int} M^n$. Suppose ψ is a homomorphism of $\pi_1(M^n, s_0)$ onto F, the free group of rank r. We are interested in the case $r \geqq 1$.

Let T denote a wedge at t_0 of r simple closed curves $T_1, \cdots, T_r$. Then there is a homomorphism ϕ of $\pi_1(M^n, s_0)$ *onto* $\pi_1(T, t_0)$. Let f denote a simplicial map of some subdivision of M^n to some subdivision of T taking s_0 to t_0 so that the homomorphism f_* of $\pi_1(M^n, s_0)$ to $\pi_1(T, t_0)$ induced by f is equal to ϕ.

Choose points t_i, $1 \leqq i \leqq r$, so that t_i is interior to some 1-simplex, Δ_i, of T_i in the subdivision of T for which f is simplicial. Let δ_i: $t_i \times I$ be a linear embedding of $t_i \times I$ into Δ_i^0 so that $\delta_i(t_i \times 0) = t_i$. Then $\bigcup_i f^{-1}(t_i)$ is a system of surfaces in M^n. Furthermore, for any surface $N_{ik}^{n-1} \in f^{-1}(t_i)$ a product neighborhood $P(N_{ik}^{n-1})$ of N_{ik}^{n-1} may be chosen so that $P(N_{ik}^{n-1})$ has a parametrization

$$h_{ik}: N_{ik}^{n-1} \times I \to P(N_{ik}^{n-1})$$

where for each $s \in N_{ik}^{n-1}$, h_{ik} $(s \times I)$ is carried by the map f both homeomorphically and linearly onto $\delta_i(t_i \times I)$ and

https://doi.org/10.1017/S1446788700011034 Published online by Cambridge University Press

$$P(N_{ik}^{n-1}) \cap P(N_{lm}^{n-1}) = \phi, \quad \text{if } (i,k) \neq (l,m).$$

Let S_i denote the system of surfaces $f^{-1}(t_i)$.

Let $U_i = \delta_i(t_i \times I)$. We call $U_i^+ = \delta_i(t_i \times [0,1])$ ($U_i^- = \delta_i(t_i \times [-1,0])$) the *positive side* (*negative side*) of U_i. We call

$$P_+(N_{ik}^{n-1}) = h_{ik}(N_{ik}^{n-1} \times [0,1])(P_-(N_{ik}^{n-1}) = h_{ik}(N_{ik}^{n-1} \times [-1,0]))$$

the *positive side* (*negative side*) of $P(N_{ik}^{n-1})$.

Let $\bar{t}_i$ denote an embedding of S^1 onto T_i so that if we think of the class of $\bar{t}_i$ in $\pi_1(T, t_0)$, $\bar{t}_i$ is oriented so as to 'cross' t_i from U_i^- to U_i^+. We also write $\bar{t}_i$ for the class of $\bar{t}_i$ in $\pi_1(T, t_0)$.

Since the homomorphism

$$f_*: \pi_1(M^n, s_0) \to \pi_1(T, t_0)$$

induced by f is *onto*, there is a loop l_i, $1 \leqq i \leqq r$, in M^n based at s_0 (l_i may be chosen as a simple closed curve if $n \geqq 3$) so that the element $[l_i]$ of $\pi_1(M^n, s_0)$ determined by l_i is carried onto $\bar{t}_i$ by f_*. We may assume that l_i is chosen in general position with respect to $\bigcup_i S_i$.

There is a procedure [4, p. 368] for reading the word $w(l_i)$ in the symbols $\bar{t}_1, \bar{t}_1^{-1}, \cdots, \bar{t}_r, \bar{t}_r^{-1}$ of the free group π_1 (T, t_0) which corresponds to $f_*[l_i]$ by observing the way l_i meets $\bigcup_i S_i$. Since $\pi_1(T, t_0)$ is a free group and $w(l_i)$ is equal to $\bar{t}_i$ in $\pi_1(T, t_0)$, it must be true that either l_i meets only one component of S_i or that there is a cancellation of the form $\bar{t}_j \bar{t}_j^{-1}$ (or $\bar{t}_j^{-1} \bar{t}_j$) in $w(l_i)$.

At this point it is convenient to consider the two cases $n < 3$ and $n \geqq 3$.

CASE 1. $n < 3$. If $n = 1$, then a separate, straightforward argument applies. If $n = 2$, then the desired conclusion is just Lemma 3.2. of [4].

CASE 2. $n \geqq 3$. We now have that l_i, $1 \leqq i \leqq r$, is a simple closed curve in M^n based at s_0 and $l_i \cap l_j = \{s_0\}$, $i \neq j$. A cancellation of the form $\bar{t}_j \bar{t}_j^{-1}$ (or $\bar{t}_j^{-1} \bar{t}_j$) in $w(l_i)$ has as its geometric counterpart in M^n a subarc α_i of l_i which meets $\bigcup_i S_i$ only in its end points which are both in S_j (possibly not the same component of S_j). We shall use this geometric interpretation of the reduction of $w(l_i)$ to $\bar{t}_i$ in $\pi_1(T, t_0)$ to obtain a system of surfaces $N_1^{n-1}, \cdots, N_r^{n-1}$ in M^n so that $l_i \cap N_i^{n-1}$ is precisely one point at which l_i pierces N_i^{n-1}; i.e. l_i meets both $P_+(N_i^{n-1})$ and $P_-(N_i^{n-1})$. Furthermore, $l_i \cap N_j^{n-1}$ will be void for $i \neq j$.

Consider the subarc α_i of l_i with its endpoints in S_j and otherwise α_i misses $\bigcup_i S_i$. There is a combinatorial n-cell $Q^n \subset M^n$ and parametrization of Q^n as a product of the combinatorial $(n-1)$-cell Q^{n-1} and the interval I so that

a) for some point $0 \in \operatorname{Int} Q^{n-1}$,

$$\alpha_i = 0 \times I,$$

b) $Q^n \cap l_i = \alpha_i$, $Q^n \cap l_j = \varnothing$, $i \neq j$,

c) $Q^n \cap S_i = \emptyset$, $i \neq j$,
d) Q^n meets only the components of S_j which α_i meets, and
e) $Q^n \cap S_j = Q^{n-1} \times -1 \cup Q^{n-1} \times 1$.

Let S_j' be the *system of surfaces* in M^n obtained from S_j and $\mathrm{Bd}\, Q$ by replacing the $(n-1)$-cells $Q^{n-1} \times -1$ and $Q^{n-1} \times 1$ by the closed annulus $\delta Q^{n-1} \times I$.

We now have a collection $S_1', \cdots, S_j', \cdots, S_r'$ where each S_i' is a system of surfaces in M^n, $S_i' = S_i$, $i \neq j$ and S_j' is described above. The word problem for the simple closed curve l_i has been reduced with respect to this collection since the cancellation $\bar{t}_j \bar{t}_j^{-1}$ (or $\bar{t}_j^{-1} \bar{t}_j$) has been eliminated as viewed geometrically. In other words we have reduced the number of components of $l_i \cap \bigcup_i S_i$ by looking at the number of components of $l_i \cap \bigcup_i S_i'$. In a finite number of steps, we obtain the system of surfaces $N_1^{n-1}, \cdots, N_r^{n-1}$ in M^n promised above.

Since the wedge of simple closed curves $\bigcup_i l_i$ has the property $l_i \cap N_i^{n-1}$ is precisely one piercing point and $l_i \cap N_j^{n-1} = \emptyset$, $i \neq j$, the system of surfaces $N_1^{n-1}, \cdots, N_r^{n-1}$ is independent in M^n. This concludes the proof of Proposition 2.1.

3. Additivity of inner rank

Suppose G_1 and G_2 are groups. We designate the free product of G_1 and G_2 by $G_1 * G_2$ [6].

To this author's knowledge a result like the following lemma first appeared in the literature in [9]. We include a brief outline of a proof for completeness of our argument.

LEMMA 3.1. *Suppose G is a finitely presented group. Then there is a closed, connected, combinatorial 4-manifiold M_G^4 so that $\pi_1(M_G^4) \approx G$.*

PROOF. There is a connected, finite, simplicial 2-complex K_G with $\pi_1(K_G) \approx G$ [1, Theorem 6.4.6]. Let h denote a simplicial embedding of K_G into a standard rectilinear subdivision of the 5-sphere, S^5.

If $N(K_G)$ denotes a regular neighborhood of $h(K_G)$ in S^5 [3, p. 59], then

$$M_G^4 = \delta N(K_G)$$

is the desired closed, connected, combinatorial 4-manifold.

Suppose G is a group. Let

$$IN(G) = \max_F r(F)$$

where F is a free quotient of G and $r(F)$ denotes the *rank* of F. We call $IN(G)$ the *inner rank* of G.

THEOREM 3.2. *Suppose G_1 and G_2 are both finitely presented groups.*

$$IN(G_1 * G_2) = IN(G_1) + IN(G_2).$$

PROOF. Let M_1^4, M_2^4 denote closed, 4-manifolds where $\pi_1(M_1^4) \approx G_1$ and $\pi_1(M_2^4) \approx G_2$. Let $r_1 = IN(G_1)$, $r_2 = IN(G_2)$.

There is a homomorphism ϕ of $G_1 * G_2$ onto the free group F of rank $r_1 + r_2$. Hence

$$IN(G_1 * G_2) \geqq IN(G_1) + IN(G_2).$$

Let M^4 denote the closed, connected combinatorial 4-manifold $M_1^4 \# M_2^4$ obtained from M_1^4 and M_2^4 via connected sum. Then by Van Kampen's Theorem we have

$$\pi_1(M^4) \approx G_1 * G_2.$$

If F is a free group of rank s and ϕ is a homomorphism of $G_1 * G_2$ onto F, then by Proposition 2.1 there is an independent system of surfaces $N_1^3, \cdots, N_s^3$ in M^4.

There is a 3-sphere $S^3 \subset M^4$ so that

a) $M^4 - S^3$ has precisely two components Q_1^4, Q_2^4 where the closure of Q_i^4 is PL homeomorphic with M_i^4 minus the interior of a 4-cell in M_i^4, $i = 1,2$, and

b) $\bigcup_{i=1}^s N_i^3 \cap S^3 = \bigcup_{j=1}^p F_j^2$ where $F_1^2, \cdots, F_p^2$ is a system of surfaces in S^3 (possible an empty collection).

Let p, the number of components of

$$\bigcup_i N_i^3 \cap S^3,$$

denote the *complexity of the system* $N_1^3, \cdots, N_s^3$ *relative to* S^3. If $p = 0$, then it follows that $s \leqq r_1 + r_2$ and thus

$$IN(G_1 * G_2) \leqq IN(G_1) + IN(G_2).$$

If $p > 0$, then we shall show that there is an independent system of surfaces $R_1^3, \cdots, R_s^3$ in M^4 where the complexity of the system $R_1^3, \cdots, R_s^3$ relative to S^3 is strictly less than p.

There is a product neighborhood $P(S^3)$ of S^3 in M^4 and a parametrization

$$h: S^3 \times I \to P(S^3)$$

so that

$$\bigcup_{i=1}^s N_i^3 \cap P(S^3) = \bigcup_{j=1}^p h(F_j^2 \times I).$$

Actually, we may have to move the system $N_1^3, \cdots, N_s^3$ by an ambient homeomorphism to obtain an independent system which satisfies this condition. Since such a homeomorphism may be chosen so as to leave S^3 invariant, we continue to use the same notation for this new system.

Each F_j^2 separates S^3. Choose the indexing of $F_1^2, \cdots, F_p^2$ so that F_p^2 is an innermost surface in S^3; i.e. one of the two domains complementary to F_p in S^3 meets no F_j^2. Let D denote the closure of this domain in S^3. Suppose $F_p^2 \subset N_k^3 \cap S^3$. There are two cases to consider:

CASE 1. F_p^2 does not separate N_k^3. Let

$$R_k^3 = (N_k^3 - h(F_p^2 \times I)) \cup (h(D \times \{1\}) \cup h(D \times \{-1\})).$$

Then $R_1^3, \cdots, R_k^3, \cdots, R_s^3$ defined as $R_i^3 = N_i^3$, $i \neq k$, is an independent system of surfaces in M^4 and the complexity of the system $R_1^3, \cdots, R_s^3$ is $p-1$.

CASE 2. F_p^2 separates N_k^3. Let $N_{k_1}^3$ and $N_{k_2}^3$ denote the closure in N_k^3 of the two components complementary to F_p^2 in N_k^3. Since $N_1^3, \cdots, N_k^3$ is an independent system of surfaces in M^4, there is a wedge T of a simple curves $T_1, \cdots, T_s$ in M^4 so that $T_i \cap N_i^3$ is a single piercing point and $T_i \cap N_j^3 = \emptyset$, $i \neq j$. Furthermore, $T \cap D = \emptyset$. Choose notation so that

$$T \cap N_k^3 = T \cap N_{k_1}^3.$$

Suppose

$$N_{k_1}^3 \cap P(S^3) \supset h(F_p^2 \times \{1\}).$$

Let

$$R_k^3 = (N_{k_1}^3 - h(F_p^2 \times I)) \cup h(F_p^2 \times \{1\}).$$

Then $R_1^3, \cdots, R_k^3, \cdots, R_s^3$ defined as $R_i^3 = N_i^3$, $i \neq k$ is an independent system of surfaces in M^4. For p' the complexity of the system $R_1^3, \cdots, R_s^3$ we have $p' \leqq p-1$.

We conclude that for $N_1^3, \cdots, N_s^3$ any independent system of surfaces in M^4, that there is an independent system of surfaces $R_1^3, \cdots, R_s^3$ in M^4 where the complexity of the system $R_1^3, \cdots, R_s^3$ relative to S is zero. By the remark above, this completes the proof of Theorem 3.2. Lyndon has shown the author an algebraic proof of Theorem 3.2.

Several applications of Theorem 3.2 are given in [5].

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