Geometry of the Isosceles Trapezium and the Contra-parallelogram, with applications to the geometry of the Ellipse.

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1. In Figure 11 let ABCD be a contraparallelogram having AB = CD, AD = BC.

Let O, P, Q, R be the mid-points of AB, BC, AD, DC respectively. They obviously lie in the line parallel to AC and to BD, and equidistant from them.

Let U be the mid-point of AC and V that of BD.

OURV is a rhombus each of whose sides is half of AD or BC, and parallel to AD or BC. PUQV is a rhombus each of whose sides is half of AB or CD, and parallel to AB or CD.

$$OQ = BV = VD = PR,$$

$$\text{and} \quad OP = AU = UC = QR.$$

Let X be the point of intersection of AD and BC, and Y that of AB and CD. Then X, Y, U, V are collinear.

Let M be the intersection of UV and OR.

Let $BP = a$ and $OB = c$.

Then
$$\begin{aligned} a^2 - c^2 &= BP^2 - OB^2 \\ &= OV^2 - PV^2 \\ &= OM^2 - PM^2 \\ &= OP \, . \, OQ \\ &= AU \, . \, BV. \end{aligned}$$

Now it is clear that a linkage formed of the jointed contra-parallelogram ABCD and the two rhombuses OVRU, PVQU, jointed at their common points, will have one internal freedom of motion, and that if AB be fixed every point not in AB will describe a definite curve, which will be a circle except for points on CD.

https://doi.org/10.1017/S0013091500032892 Published online by Cambridge University Press

Let X be the intersection of AD and BC.

It is clear that AX + XB = AX + XD = $2a$.

Thus if AB is fixed, X describes an ellipse whose foci are A and B and whose major axis $= 2a$.

2. Assuming for the moment that the tangent bisects the external angle between the focal distances, we see that the tangent is UXV, and AU, BV the perpendiculars on it from the foci. We proved

$$\text{AU . BV} = a^2 - c^2$$

$$= \text{square on minor semi-axis,}$$

which is a well-known proposition.

Again if the normal at X meets AB in G, we have

$$\frac{\text{OP}}{\text{GX}} = \frac{\text{BP}}{\text{BX}} = \frac{\text{OU}}{\text{BX}} = \frac{\text{OM}}{\text{BV}} = \frac{\text{OM}}{\text{OQ}}.$$

Thus OP . OQ = GX . OM.

Thus writing b^2 for $a^2 - c^2$ we have

$$b^2 = \text{GX . OM},$$

another well-known proposition.

It may be noted that if BC is held fixed, since BY − CY = BA, the locus of Y would be a hyperbola with B and C as foci, and YUV would be the tangent at Y.

3. The following proof of the proposition assumed above has the merit of proving a fundamental property of the tangent directly from the constancy of the sum of the focal distances, without using infinitesimals except in going to the limit.

In Figure 12 let A, B be the foci of an ellipse.

Let X, X′ be points on it, and let AX and AX′ be produced to C and C′ making AC = AC′ = $2a$.

Let the bisectors of the angles BXC, BX′C′ meet in W. It is clear that the triangles BXW, CXW are equal in all respects, also the triangles BX′W, C′X′W, as well as the triangles AWC, AWC′. Hence $\angle$XAW = $\angle$X′AW and $\angle$XBW = $\angle$X′BW.

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Again if XW be produced to Z we have

$$\angle CWC' = \angle CWB - \angle C'WB = 2\{\angle ZWB - \angle X'WB\}$$
$$= 2\angle ZWX'.$$

Thus the angle ZWX′ is half of CWC′, and vanishes with it. Hence in the limit, when X′ comes into coincidence with X, the direction of XX′ will coincide with the exterior bisector of AXB, which proves the proposition.

4. It is interesting to note that, going back to the case when X and X′ are separate, the figure gives the two tangents WX, WX′ drawn from an external point to the ellipse, and shows that they subtend equal angles at either focus.

5. The linkage indicated in Figure 11 is a well-known one, showing the relation between Peaucellier's linkage for drawing a straight line, and that of Hart. If, for example, O is fixed, and P linked to a fixed point G by a link PG = OP, then Q will describe a straight line perpendicular to OG.

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