COVERING A GROUP WITH ISOLATORS OF FINITELY MANY SUBGROUPS

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Dedicated to Professor B. H. Neumann for his 80th birthday

1. Introduction. In **[6]** B. H. Neumann proved the following beautiful result: *if a group G is covered by finitely many cosets, say $G = \bigcup_{i=1}^{n} x_iH_i$, then we can omit from the union any x_iH_i for which $|G:H_i|$ is infinite. In particular, $|G:H_j|$ is finite, for some $j \in \{1, \ldots, n\}$.*

In an unpublished result R. Baer characterized the groups covered by finitely many abelian subgroups, they are exactly the centre-by-finite groups **[8]**. Coverings by nilpotent subgroups or by Engel subgroups and by normal subgroups have been studied, for example, by R. Baer (see **[8]**), L. C. Kappe **[2, 1]**, M. A. Brodie and R. F. Chamberlain **[1]**, and recently by M. J. Tomkinson **[9]**.

In this paper we study groups covered by finitely many isolators of subgroups.

If H is a subgroup of the group G, the *isolator* of H in G is, by definition, the subset

$$I_G(H) = \{x \in G \mid x^n \in H \text{ for some } n > 0\}.$$

We denote by $\mathfrak{X}$ the class of groups G such that, whenever $G = \bigcup_{i=1}^{n} I_G(H_i)$, *then* $G = I_G(H_j)$ for some $j \in \{1, \ldots, n\}$.

We prove the following results:

THEOREM A. *Let A be a normal abelian subgroup of G. If $G/A \in \mathfrak{X}$, then $G \in \mathfrak{X}$. If G is locally soluble, then $G \in \mathfrak{X}$.*

From Theorem A, using a result of J. C. Lennox **[4]**, it follows that if G is a finitely generated soluble group and $G = \bigcup_{i=1}^{n} I_G(H_i)$, then $|G:H_j|$ is finite, for some $j \in \{1, \ldots, n\}$.

THEOREM B. *Let $G = \bigcup_{i=1}^{n} I_G(H_i)$, where $H_1, \ldots, H_n$ are abelian subgroups of G. Then $G = I_G(H_j)$ for some $j \in \{1, \ldots, n\}$.*

The same conclusion of Theorem B holds if $G = \bigcup_{i=1}^{n} I_G(H_i)$, with $H_1, \ldots, H_n$ subnormal subgroups of G (Theorem C and Corollary 3.2).

Most of the standard notation used comes from **[8]**.

We say that a group G has the *isolator property* (G has *I.P.*) if the isolator of every subgroup of G is itself a subgroup of G.

A subgroup H is called *isolated* if $I_G(H) = H$.

Finally, if H, K are subgroups of G, then we write $H \sim K$ to mean $I_G(H) = I_G(K)$.

2. Proof of Theorem A. We begin with some preliminary results:

LEMMA 2.1. *If every two generator subgroup of G is in $\mathfrak{X}$, then so is G.*

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Proof. Suppose false, and let $G = \bigcup_{i=1}^{n} I_G(H_i)$, where $n \geq 2$ is minimal subject to $G \neq I_G(H_i)$ for any $i \in \{1, \ldots, n\}$. By minimality of n, there exists $h_1 \in H_1 - \left(\bigcup_{i=2}^{n} I_G(H_i)\right)$. Similarly there exists $h_2 \in H_2$ such that $h_2 \in I_G(H_1) \cup I_G(H_3) \cup \cdots \cup I_G(H_n)$. Let $J = \langle h_1, h_2 \rangle$. Then $J = \bigcup_{i=1}^{n} I_i(H_i \cap J)$, and by the hypothesis $J = I_J(H_i \cap J)$ for some i.

Hence $J \subseteq I_G(H_i)$ for some i, a contradiction.

Lemma 2.2. $\mathfrak{X} = Q\mathfrak{X}$.

Proof. Easily verified.

Lemma 2.3. *Let $H \leq G$ be such that $G = I_G(H)$. Then $G \in \mathfrak{X}$ if and only if $H \in \mathfrak{X}$.*

Proof. Assume $G \in \mathfrak{X}$. If $H = \bigcup_{i=1}^{n} I_H(K_i)$, then $G = \bigcup_{i=1}^{n} I_G(K_i)$, so that $I_H(K_i) = H$ for some i.

Conversely, let $H \in \mathfrak{X}$ and suppose $G = \bigcup_{i=1}^{n} I_G(H_i)$. Then $H = \bigcup_{i=1}^{n} I_H(H \cap H_i)$ and $H = I_H(H \cap H_i)$ for some i. Hence $G = I_G(H_i)$ for some i.

We prove now a weaker version of Theorem A.

Lemma 2.4. *Let $G = \langle a_1, \ldots, a_m, h \rangle$, where $A = \langle a_1, \ldots, a_m \rangle^G$ is abelian. Then $G \in \mathfrak{X}$.*

Proof. If G/A is finite, the result follows easily from 2.3.

Assume $G/A \cong \langle h \rangle$ infinite. We prove, by induction on n, that if $G = I_G(H_1) \cup I_G(H_2) \cup \ldots \cup I_G(H_n) \cup A$, then $G = I_G(H_j)$, for some $j \in \{1, \ldots, n\}$. Obviously we can assume $H_iA > A$, for every i, and so $|G : H_iA|$ is finite. Without loss of generality, we may assume $G = H_1A = H_2A = \ldots = H_nA$. Then $H_i \cap A \lhd G$, for every $i \in \{1, \ldots, n\}$.

We show that $G/(A \cap H_1 \cap \ldots \cap H_n)$ is polycyclic; then $G/(A \cap H_1 \cap \ldots \cap H_n)$ is almost I. P. by a result of Rhemtulla and Wehrfritz [**7**], and $G \in \mathfrak{X}$.

By a theorem of Lennox and Wiegold [**5**, Theorem B], it suffices to prove that $(\langle a, h \rangle (A \cap H_1 \cap \ldots \cap H_n))/(A \cap H_1 \cap \ldots \cap H_n)$ is polycyclic for every $a \in A$. Hence, without loss of generality, we can assume $A = \langle a \rangle^G$.

First, we show that $G/(H_j \cap A)$ is polycyclic, for some $j \in \{1, \ldots, n\}$.

For every $i \in \mathbb{N}$ there exists $\alpha \in \mathbb{N}$ such that $(ah^i)^\alpha \in H_1 \cup H_2 \cup \ldots \cup H_n$. Then there are i, $s \in \mathbb{N}$, $s > 1$, such that $h^ia \in I_G(H_j)$, $h^{is}a \in I_G(H_j)$ for the same $j \in \{1, \ldots, n\}$. Hence, for a suitable $\beta \in \mathbb{N}$, $(h^ia)^{\beta s} = h^{i\beta s}a^{h^{i(\beta s-1)}} \ldots a^{h^i}a \in H_j$ and $(h^{is}a)^\beta = h^{is\beta}a^{h^{is(\beta-1)}} \ldots a^{h^{is}}a \in H_j$, from which $a^{-1}a^{-h^i} \ldots a^{-h^{i(\beta s-1)}}a^{h^{is(\beta-1)}} \ldots a^{h^{is}}a \in A \cap H_j$. But $s > 1$, and so $i(\beta s - 1) > is(\beta - 1)$. Therefore we have $a^{h^{i(\beta s-1)}}a^{h^{\alpha_1}} \ldots a^{h^{\alpha_l}}a^{h^i} \in H_j \cap A$, with α_l suitable integers, $i < \alpha_l < i(\beta s - 1)$, from which $a^{h^{i(\beta s-2)}} \ldots a^{h^{\alpha_l - i}}a \in H_j \cap A$ and $a^{f(h)} \in H_j \cap A$, where $f(h)$ is a polynomial over $\mathbb{Z}$ with leading coefficient and constant term equal to 1. Therefore $G/(A \cap H_j)$ is polycyclic [**3**]. Assume $j = 1$; then $G/(A \cap H_1)$ is polycyclic.

If $n = 1$, the result follows. Assume $n > 1$. Let $1 \leq l \leq n$ be maximum such that $G/(A \cap H_1 \cap \ldots \cap H_l)$ is polycyclic. Assume for a contradiction $l < n$. Write $B = A \cap H_1 \cap \ldots \cap H_l$ and let $g \in G - (A \cup I_G(H_1) \cup \ldots \cup I_G(H_{n-1}))$. Thus $g = ch^s$, for some

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$c \in A$, $s \in \mathbb{Z}$, $s \neq 0$. Put $K = B\langle g \rangle$, then from $B \leq H_i$ it follows $K \cap I_G(H_i) = B$ for every $1 \leq i \leq l$, and $K = B \cup I_K(H_{i+1} \cap K) \cup \ldots \cup I_K(H_n \cap K)$. Notice that B is finitely generated as a K-group. By induction, $K = I_K(H_j \cap K)$ for some $j \leq n$, and $K = I_K(H_n \cap K)$ since $g \notin A \cup I_G(H_1) \ldots \cup I_G(H_{n-1})$. Arguing as before we get $(\langle b, g \rangle (B \cap H_n))/(B \cap H_n)$ polycyclic for every $b \in B$, and then $(\langle b, dh \rangle (B \cap H_n))/(B \cap H_n)$ is polycyclic for every $d \in A$. Hence $(\langle b, x \rangle (B \cap H_n))/(B \cap H_n)$ is polycyclic for every $b \in B$, $x \in G$ and $G/(B \cap H_n)$ is polycyclic by a theorem of Lennox and Wiegold [**5**, Theorem B], contradicting the maximality of l.

Now we can prove Theorem A.

Proof of Theorem A. Suppose $A \leq G$, A abelian, $G/A \in \mathfrak{X}$, and for a contradiction $G \notin \mathfrak{X}$.

Let n be the least integer >1 such that $G = \bigcup_{i=1}^{n} I_G(H_i)$, $H_i \leq G$, but $G \neq I_G(H_i)$ for any $i \in \{1, \ldots, n\}$.

First remark that we may assume

(I) $G = \bigcup_{i=1}^{n} I_G(H_i)$, $G \neq I_G(H_i)$ *for any* $i \in \{1, \ldots, n\}$, $G = AH_1 = \ldots = AH_l$, $A \leq \bigcap_{i=1}^{n-l} H_{l+1}$, *where* $1 \leq l \leq n$. *Moreover* $A \cap H_i \lhd G$, *for any i.*

For, if $I_G(AH_1) \neq G$, then replace H_1 by AH_1; if $I_G(AH_1) = G$, then replace G by AH_1 and for $i \neq 1$, replace H_i by $AH_1 \cap H_i$. Observe that $\bigcup_{i=1}^{n} I_{AH_1}(AH_1 \cap H_i) = AH_1 \cap \bigcup_{i=1}^{n} I_G(H_i) = AH_1$, and, by our minimal choice of n, $I_{AH_1}(AH_1 \cap H_i) \neq AH_1$ for any i. Furthermore the given normal abelian subgroup A is still contained in the new G, $H_1 \geq A$ or $AH_1 = G$, and, in both cases. $A \cap H_1 \lhd G$. There exists $i \in \{1, \ldots, n\}$ such that $I_G(AH_i) = G$, because $G/A \in \mathfrak{X}$. We may assume $i = 1$ and $G = AH_1$.

Now suppose we have made the adjustment for the first r subgroups $H_1, \ldots, H_r$ and for the group G such that:

$(*)$ *A is contained in the new* G, *either* $H_i \geq A$ *or* $AH_i = G$, *for any* $1 \leq i \leq r$.

Remark that then $A \cap H_i \lhd G$, for any $1 \leq i \leq r$.

If $I_G(AH_{r+1}) \neq G$, then replace H_{r+1} by AH_{r+1} and observe that $(*)$ is satisfied for H_{r+1} as well. If $I_G(AH_{r+1}) = G$, then replace G by $G_1 = AH_{r+1}$ and H_i by $H_i \cap AH_{r+1}$ for all i. If $i = r+1$, then H_{r+1} satisfies $(*)$; if $i \leq r$ and $AH_i = G$, then $AH_i \cap AH_{r+1} = A(H_i \cap AH_{r+1}) = G$; if $i \leq r$ and $A \leq H_i$, then $A \leq H_i \cap AH_{r+1}$. Hence $(*)$ holds for H_i, for any $i \leq r+1$.

Thus we have made the adjustment for the first $r+1$ subgroups $H_1, \ldots, H_{r+1}$ to satisfy $(*)$. Continue this process until $r = n$. As a result of the above adjustment we may assume (I).

Write $M = A \cap \bigcap_{i=1}^{n} H_i$.

Passing, if necessary, to the quotient group G/M, we have, without loss of

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generality,

(II) $A \cap \bigcap_{i=1}^{n} H_i = 1.$

The next step is to show that

(III) *A is periodic.*

If not, then let $\langle a \rangle$ be infinite, $a \in A$. By (II), $\langle a \rangle \cap H_i = 1$ for some i, say $i = 1$. Also, by minimality of n, there exists $h \in H_1$ such that $h \notin \bigcup_{i=2}^{n} I_G(H_i)$. Let $H = \langle a, h \rangle$. Clearly $H = \bigcup_{i=1}^{n} I_H(H_i \cap H)$ and $H \neq I_H(H_i \cap H)$ for any i. But, by Lemma 2.4, $H \in \mathfrak{X}$, a contradiction.

Now, let T be a subset of $\{1, \ldots, n\}$ of largest cardinality such that $AK \sim G$, where $K = \bigcap_{i \in T} H_i$. For any $j \notin T$, let $K_j = K \cap H_j$. By (I), $|T| \geq 1$. Pick any $a \in A$.

For each $g \in K - \bigcup_{i \notin T} I_G(K_i)$ some power g^m of g centralizes a modulo $H_j \cap A$ for some $j \in T$. For, if $|(\langle a^{\langle g \rangle} \rangle (H_j \cap A))/(H_j \cap A)| = \infty$, then $ag^r \notin I_G(H_j)$ for any non-zero integer r. If this happens for all $j \in T$, then $ag^r, ag^s \in I_G(H_i)$ for some $i \notin T$, $r, s \in \mathbb{N}$, $r \neq s$. From this we get a contradiction to $g \notin \bigcup_{i \notin T} I_G(K_i)$.

Let $C_j = \langle g \in K \mid [a, g] \in H_j \rangle$. Then $K \sim \bigcup_{i \notin T} K_i \cup \bigcup_{j \in T} C_j$, and $K \cap A \leq C_j$ for all $j \in T$. Since $AK \sim G$, $AK/A \sim G/A \in \mathfrak{X}$ and so $K/(K \cap A) \simeq AK/A \in \mathfrak{X}$. Hence either $(A \cap K)K_i \sim K$ for some $i \notin T$ (alternative $(\mathscr{A})$) or $C_j \sim K$ for some $j \in T$ (alternative $(\mathscr{B})$).

If $(\mathscr{A})$ holds, then $A(A \cap K)K_i \sim AK \sim G$, so that $AK_i \sim G$, contradicting the maximality of the set T.

So assume $(\mathscr{B})$. For each $a \in A$, let T_a be the subset of T such that $C_i = C_i(a) \sim K$ for all $i \in T_a$. Then $T_a \neq \emptyset$. For each $j \in T$, let $E_j = \{a \in A$ such that $j \notin T_a\}$. Observe that if a, $b \in E_j$, then $ab \in E_j$, for $T_{ab} \supseteq T_a \cap T_b$. Also $a \in E_j$ if and only if $a^{-1} \in E_j$. Thus $E_j \leq A$, and $A = \bigcup_{j \in T} E_j$. Furthermore $E_j \lhd G$, for any $j \in T$. By B. H. Neumann's result $|A : E_j| \leq |T|$, for some $j \in T$, say $|A : E_1| \leq |T|$ (and $1 \in T$). Then for any $g \in K$, $a \in A$, we have $[a, g^s] \in E_1$, for some $s > 0$, and, for a suitable $r > 0$, $[a, g^r, g^r] \in H_1 \cap A$: thus, if $|a| = k$, then $[a, g^{rk}] \in H_1 \cap A$. Therefore $E_1 = A$, so that for any $a \in A$, any $g \in K$, $g^r \in C_1(a)$ for some $r > 0$, and hence $[g^r, a] \in H_1 \cap A$, so that some suitable power of ag lies in H_1. This gives $AK \subseteq I_G(H_1)$ and $G = I_G(H_1)$, a contradiction.

Then $G \in \mathfrak{X}$.

Now assume G locally soluble, we prove that $G \in \mathfrak{X}$. By Lemma 2.1 it suffices to show that every 2-generator subgroup of G is in $\mathfrak{X}$. Thus, without loss of generality, we can assume G soluble, and the result follows easily by induction on the derived length.

Corollary 2.5. *Let G be a finitely generated soluble group.*

If $G = I_G(H_1) \cup I_G(H_2) \cup \ldots \cup I_G(H_n)$, with $H_1, H_2, \ldots, H_n$ subgroups of G, then $|G : H_i|$ is finite for some $i \in \{1, \ldots, n\}$.

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Proof. We have $G = I_G(H_i)$, for some $i \in \{1, \dots, n\}$, and, by a result of J. Lennox **[4]**, $|G : H_i|$ is finite.

3. Groups covered by isolators of finitely many abelian subgroups.

Proof of Theorem B. We argue by induction on n. Obviously the result is true for $n = 1$; assume $n > 1$, and, for a contradiction, $I_G(H_i) \not\subseteq \bigcup_{j \neq i} I_G(H_j)$, for any i.

First we show that we may assume

(I) $H_i \cap H_j = 1$, *for* $i \neq j$.

For, if $T \leq G$ and $T \not\subseteq I_G(H_i)$ for any i, then for every (h, k), $h \neq k$, $T \cap \langle H_h, H_k \rangle \not\subseteq I_G(H_i)$ for any i. In fact, if $T \cap \langle H_h, H_k \rangle \subseteq I_G(H_i)$ for some i, then $T \cap H_h$, $T \cap H_k \subseteq I_G(H_i)$ with either $i \neq h$ or $i \neq k$. Assume for example $i \neq h$. Then $T = \bigcup_{j \neq h} I_T(T \cap H_j)$ and, by induction, $T = I_T(T \cap H_s) \subseteq I_G(H_s)$ for some s, a contradiction.

Now write $X = \bigcap_{1 \leq i \neq j \leq n} \langle H_i, H_j \rangle$. Then it is easy to see that $X \not\subseteq I_X(H_i \cap X)$ for any i, and we can assume $G = X$, so that $H_i \cap H_j \lhd G$ for any $i \neq j$. Put $Y = \prod_{1 \leq i \neq j \leq n} (H_i \cap H_j)$. Then $Y \lhd G$ and Y is soluble. If $G/Y \subseteq I_{G/Y}(H_jY/Y)$ for some $j \in \{1, \dots, n\}$, then $G \sim H_jY$. But H_jY is soluble, thus, by Theorem A, $H_jY \sim H_s \cap H_jY$ for some $s \in \{1, \dots, n\}$ and $G \sim H_s$, a contradiction. Then we can assume $Y = 1$ and (I) holds.

Now we prove that

(II) *for every* $i \in \{1, \dots, n\}$ *and for every* $g \in G$, *there exists* $\alpha = \alpha(i, g) \in \mathbb{N}$ *such that* $\langle H_i, H_i^{g^\alpha} \rangle \subseteq I_G(H_i)$.

Let $a \in H_i - \left(\bigcup_{j \neq i} I_G(H_j) \right)$. Then, for some h, k, $h < k$, a^{g^h} and a^{g^k} are in $I_G(H_s)$ for a suitable $s \in \{1, \dots, n\}$. Hence, for some $\gamma \in \mathbb{Z} - \{0\}$, we have $(\alpha^\gamma)^{g^h}$, $(a^\gamma)^{g^k} \in H_s$, and $\langle (a^\gamma)^{g^h}, (a^\gamma)^{g^k} \rangle$ is abelian. Thus $\langle a^\gamma, (a^\gamma)^{g^{h-k}} \rangle$ is abelian, so that there exists $j \in \{1, \dots, n\}$ for which $\langle a^\gamma, (a^\gamma)^{g^{h-k}} \rangle \subseteq I_G(H_j)$. Obviously $j = i$, since $a^\gamma \in I_G(H_i)$ and $H_i \cap H_j = 1$ for $i \neq j$, by (I). Then $a^{g^{h-k}} \in I_G(H_i)$ and obviously $a^{g^{h-k}} \notin \bigcup_{j \neq i} I_G(H_j)$. For any $a_1 \in H_i$, $\langle a^{g^{h-k}}, a_1^{g^{h-k}} \rangle$ abelian it follows, arguing as before, $\langle a^{g^{h-k}}, a_1^{g^{h-k}} \rangle \subseteq I_G(H_i)$; hence the group $H_i/(H_i \cap H_i^{g^{k-h}})$ is periodic.

Write $X = \langle H_i, H_i^{g^{k-h}} \rangle$, then $H_i \cap H_i^{g^{k-h}} \lhd X$ and writing $\bar{X} = X/(H_i \cap H_i^{g^{k-h}})$, we have $\bar{X} \subseteq \bigcup_{j \neq i} I_{\bar{X}}(\overline{H_j \cap X})$. It follows by induction that $X \subseteq I_X((H_j \cap X)(H_i \cap H_i^{g^{k-h}}))$ for some $j \in \{1, \dots, n\}$. From $(H_j \cap X)(H_i \cap H_i^{g^{k-h}})$ soluble it follows, by Theorem A, $(H_j \cap X)(H_i \cap H_i^{g^{k-h}}) \sim H_t \cap (H_j \cap X)(H_i \cap H_i^{g^{k-h}})$ for some t, hence $X \sim H_t \cap X$. Obviously the only possibility is $t = i$ and (II) holds.

Now take $a \in H_1 - \bigcup_{i \neq 1} I_G(H_i)$, $b \in H_2 - \bigcup_{i \neq 2} I_G(H_i)$. Then, by (II), there is $\alpha \in \mathbb{Z} - \{0\}$ such that $\langle H_1, H_1^{b^\alpha} \rangle \subseteq I_G(H_1)$, so that, for some $r \in \mathbb{Z} - \{0\}$, $[a^r, b^\alpha] \in H_1$ and, for every $s \in \mathbb{Z}$, $[a^r, b^\alpha]^s = [a^{rs}, b^\alpha] \in H_1$. Also, by (II), there exists $k \in \mathbb{Z} - \{0\}$ such that $\langle H_2, H_2^{a^{rk}} \rangle \subseteq I_G(H_2)$, hence $[a^{rk}, b^\alpha] \in I_G(H_2)$ and, for some $s \in \mathbb{Z} - \{0\}$, $[\alpha^{rk}, b^\alpha]^s \in H_2$.

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Thus $[a^{rks}, b^{\alpha}] = [a^{rk}, b^{\alpha}]^{s} = [a^{r}, b^{\alpha}]^{ks} \in H_1 \cap H_2 = 1$, and $\langle a^{rks}, b^{\alpha}\rangle$ is abelian. Then $\langle a^{rks}, b^{\alpha}\rangle \subseteq I_G(H_s)$, for some $s \in \{1, \ldots, n\}$; from $a \in I_G(H_s)$ it follows $s = 1$ and from $b \in I_G(H_s)$, $s = 2$, the final contradiction.

In order to prove Theorem C, we need the following easy Lemma:

Lemma 3.1. *Let G be a group,* $G = \bigcup_{i=1}^{n} I_G(H_i)$, *where* $H_i \leq G$, $i = 1, \ldots, n$. *Assume* $G = H_j \times H$, *for some j and some* $H \leq G$. *Then either* $G = \bigcup_{i \neq j} I_G(H_i)$, *or* $G = I_G(H_j)$.

Proof. If $G \neq \bigcup_{i \neq j} I_G(H_i)$, there exists $b \in H_j$, $b \notin \bigcup_{i \neq j} I_G(H_i)$. For any $a \in H$, consider the elements $a^m b$, $m \in \mathbb{N}$. Then there exists $s \in \{1, \ldots, n\}$ such that $a^h b, a^k b \in I_G(H_s)$ for $h, k \in \mathbb{N}$, $h \neq k$. Then $(a^h b)^{\beta} = a^{h\beta} b^{\beta} \in H_s$ and $(a^k b)^{\beta} = a^{k\beta} b^{\beta} \in H_s$, for a suitable $\beta \in \mathbb{N}$ and $a^{\beta(h-k)} \in H_s$. Thus $a \in I_G(H_s)$ and $b \in I_G(H_s)$, since $a^h b \in I_G(H_s)$, then $s = j$ and $G = I_G(H_j)$, as required.

Theorem C. *Let G be a group,* $H_1, \ldots, H_n$ *normal subgroups of G such that* $G = \bigcup_{i=1}^{n} I_G(H_i)$.

Then $G = I_G(H_j)$ *for some* $j \in \{1, \ldots, n\}$.

Proof. By induction on n we may assume $G/H_1 \subseteq I_{G/H_1}(H_j H_1/H_1)$ for some j.

Let $l \geq 1$ be maximum such that

$$G/(H_1 \cap \ldots \cap H_l) \sim (H_t(H_1 \cap \ldots \cap H_l))/(H_1 \cap \ldots \cap H_l),$$

for some $t \in \{1, \ldots, n\}$.

If $l = n$, then the result follows. Assume for a contradiction $l < n$. Without loss of generality we can assume $t = l + 1$, so that $G/(H_1 \cap \ldots \cap H_l) \sim (H_{l+1}(H_1 \cap \ldots \cap H_l))/(H_1 \cap \ldots \cap H_l)$. Write $X = H_{l+1}(H_1 \cap \ldots \cap H_l)$, then $X/(H_1 \cap \ldots \cap H_{l+1}) = H_{l+1}/(H_1 \cap \ldots \cap H_{l+1}) \times (H_1 \cap \ldots \cap H_l)/(H_1 \cap \ldots \cap H_{l+1})$, by Lemma 3.1 and by induction, we have $X/(H_1 \cap \ldots \cap H_{l+1}) \sim ((H_s \cap X)(H_1 \cap \ldots \cap H_{l+1}))/(H_1 \cap \ldots \cap H_{l+1})$ for some $s \in \{1, \ldots, n\}$. Thus $G/(H_1 \cap \ldots \cap H_{l+1}) \sim (H_s(H_1 \cap \ldots \cap H_{l+1}))/(H_1 \cap \ldots \cap H_{l+1})$ because $G \sim X$, contradicting the maximality of l.

Corollary 3.2(†). *Let G be a group,* $H_1, \ldots, H_n$ *subnormal subgroups of G such that* $G = \bigcup_{i=1}^{n} I_G(H_i)$. *Then* $G = I_G(H_j)$ *for some* $j \in \{1, \ldots, n\}$.

Proof. Denote by m_i the subnormal defect of H_i, for any $i \in \{1, \ldots, n\}$. We argue by induction on the sum of the m_i's. By Theorem C, $G = I_G(H_j^G)$ for some j. But $H_j \lhd^{m_j - 1} H_j^G$, and $H_i \cap H_j^G \lhd^{m_i} H_j^G$ for $i \neq j$. So $H_j^G = I_{H_j^G}(H_i \cap H_j^G)$ for some i and $G = I_G(H_i)$, as required.

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