CHASE–ESCAPE IN DYNAMIC DEVICE-TO-DEVICE NETWORKS

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Abstract

We feature results on global survival and extinction of an infection in a multi-layer network of mobile agents. Expanding on a model first presented in Cali *et al.* (2022), we consider an urban environment, represented by line segments in the plane, in which agents move according to a random waypoint model based on a Poisson point process. Whenever two agents are at sufficiently close proximity for a sufficiently long time the infection can be transmitted and then propagates into the system according to the same rule starting from a typical device. Inspired by wireless network architectures, the network is additionally equipped with a second class of agents able to transmit a patch to neighboring infected agents that in turn can further distribute the patch, leading to chase–escape dynamics. We give conditions for parameter configurations that guarantee existence and absence of global survival as well as an in-and-out of the survival regime, depending on the speed of the devices. We also provide complementary results for the setting in which the chase–escape dynamics is defined as an independent process on the connectivity graph. The proofs mainly rest on percolation arguments via discretization and multiscale analysis.

Keywords: Random segment process; Poisson point process; Cox point process; random waypoint model; continuum percolation; geostatistical Boolean model; non-Markovian dynamics

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1. Introduction and setting

In recent decades we have seen a tremendous increase in demand for data exchange on a global scale, creating significant pressure on network operators to maintain a good quality of service. One particularly interesting concept in this development is *device-to-device (D2D) communications*. The idea is that network components can exchange data in a peer-to-peer

fashion over a wireless channel and thereby constitute a decentralized ad hoc network. Among the many potential benefits of these networks, such as robustness, communication speed, cost efficiency, etc., one of the key challenges is their high complexity and unpredictability.

In order to cope with these challenges, since the early 1960s probabilistic modeling and analysis has been developed and employed to provide qualitative and quantitative insights for D2D networks; see, for example, [2, 3, 16, 20] and many more. In this context, one of the most fruitful approaches is based on *stochastic geometry*, where network components are conceived as point processes in space [10, 26]. Such a random point cloud is then often seen as the vertex set of a random spatial graph in which edges are drawn according to some rule that reflects the peer-to-peer communication paradigm. In its purest form this is the classical *Poisson–Gilbert graph* [18], which serves as a prototypical model for the spatial clustering behavior of a D2D network in a non-dynamical setting. In particular, we can already observe in this model the celebrated *phase transition of percolation* in the following sense. Let $\lambda > 0$ be the intensity of the underlying homogeneous Poisson point process X, and $r > 0$ the connectivity threshold, i.e. the Poisson–Gilbert graph has vertex set X and any pair of vertices is connected by an edge if and only if their Euclidean distance is less than r. Then, there exists a finite and positive $\lambda_c = \lambda_c(r)$ such that for $\lambda > \lambda_c$ the graph does not contain any infinite connected component almost surely, and for $\lambda > \lambda_c$ such an infinite connected component exists with probability 1. In the context of D2D networks, the parameter regime in which an infinite component exists, the so-called *percolation regime*, is a regime in which data can in principle be transmitted over long (infinite) distances via multiple hops.

In the past 60 years since the first proof of continuum percolation in the Poisson–Gilbert graph, this field has expanded tremendously and the presence as well as absence of percolation has been established in a great variety of static spatial models that extend the classical setting in various ways. Keeping the application area of D2D communication systems in mind, we highlight percolation results in so-called signal-to-interference-to-noise-ratio graphs [14, 28, 40] (SINR graphs), where the existence of an edge not only depends on the mutual distance but also on the density of other devices in the vicinity. Notably, in SINR graphs we can observe an in-and-out of percolation in the intensity parameter, since for large intensities the interference reduces the connectivity. On the other hand, let us mention Cox-percolation results [24, 25, 29], where the underlying point cloud is a Poisson point process with a random intensity measure that can, for example, be used to model urban environments such as street systems. Going further in this direction, in [11] sub- and supercritical regimes of percolation are established in a static spatial random graph model where the edge-drawing mechanism is designed to reflect realistic features of a connectivity network of moving devices in an urban environment. In brief, any pair of devices given by a Cox point process in which the environment is a planar random segment process is connected by an edge if and only if, on their path through the environment, they spend enough time on the same segment in sufficiently close proximity. A variety of phase transitions in the different parameters can be observed, where again the speed parameter for the device movement, in certain settings, features an in-and-out of percolation.

Going beyond the setting of static random graphs and their percolation behavior, the next natural step is to investigate data transmission on random graphs. Here, the starting point is to consider growth processes in which a message is present at an initial time at some (typical) vertex and then crosses the edges of the graph according to some passage times. This is the setting of *first-passage percolation* (FPP) [1, 30] and results often concern the speed of the data transmission over large (Euclidean) distances and the associated (asymptotic) shape of the set

of vertices that have received the message up to some fixed time. In [12], such a shape theorem is presented for FPP on the supercritical Poisson–Gilbert graph. Conceiving the message as some infection, these growth models are often presented in the context of spatial probabilistic epidemiology, and then it is natural to generalize the pure growth process of FPP to processes in which infected devices can also spontaneously heal (or messages can be dropped), giving rise to *contact processes* on random graphs [32, 33]. In this setting the main concern usually becomes showing the survival and extinction of the infection, and results are also available for contact processes on random graphs in the continuum [9, 35, 36]. In the context of D2D networks, however, another antagonist of the pure growth mechanism is important. Regarding the infection as some malware, there can be special devices in the network that have the property of removing the malware from infected devices, but are otherwise indistinguishable from the infected or susceptible devices. This gives rise to *chase–escape processes* [4, 7, 15, 21, 39] in which malware is transmitted from infected to susceptible devices and infected devices are patched. Again, the main results are concerned with exhibiting regimes of survival and extinction of the malware in the system, and such regimes have also been established in the continuum [22, 23].

We present two sets of results for chase–escape dynamics on supercritical percolation models in the continuum. In the first set of results we use the construction of [11], as mentioned above, and consider a system of moving particles in an environment of line segments in $\mathbb{R}^2$. However, we now go substantially beyond the setup in [11] and not only consider the resulting connectivity graph but introduce a chase–escape dynamics that respects the mobility and connectivity behavior of the network components; see Section 1.1 for further discussions. Controlling the long-range dependencies, we prove results on survival and extinction of malware in certain parameter regimes, and in particular again feature a phenomenon of in-and-out of survival with respect to the speed parameter. In the second set of results we complement these findings and present similar results in hybrid models in which the infection and patching processes are defined as an independent process on top of the (static) percolation model from [11].

The paper is organized as follows. In the remainder of this section we reproduce and extend our connectivity model from [11], starting with a dedicated comparison in Section 1.1. In Section 1.2 we construct a random street system as the first network layer via a stationary segment process that features decaying spatial correlations in the sense of stabilization plus good connectivity properties, and give examples. In Section 1.3, we place devices on the streets and describe the system of initial device positions. We then introduce the paradigmatic mobility model for the devices given by the random waypoint model, where devices individually pick a destination and velocity and commute back and forth on the shortest route on the street system. We also give the example of a uniformly chosen destination in a ball. Next, in Section 1.4, we establish our notion of connectivity where devices have to spend a sufficient amount of time in close proximity on the same street. Under these definitions, in Section 1.5 we establish the property of local connectedness in the sense that almost surely all devices have a finite degree. This concludes the construction of the connectivity network. Further, in Section 2 we introduce the chase–escape dynamics as a model for the spread of infections in the presence of a decentralized countermeasure respecting the dynamics in the connectivity model. For this system we exhibit our main results for global survival and absence of global survival of the malware, plus a statement on the in-and-out of survival with respect to the velocity parameter. Next, Section 3 features a hybrid model where the chase–escape dynamics is defined as an independent process on the connectivity graph, and we present statements about global survival

and absence of global survival under a variety of parameter regimes. Finally, in Section 4 we present the proofs.

1.1. Percolation and chase escape

Percolation analysis, in the context of random graphs modeling communication systems, can be considered a first and fundamental step towards an understanding of the possibility of transmitting data over long distances in a multi-hop fashion. In that sense [11] provides initial statements for the absence and presence of percolation in a simplified version of the model presented in this manuscript. Let us highlight that in [11] only a static graph is analyzed, where, however, the edge-drawing mechanism is based on the underlying dynamic system that we pick up here as well. The first generalization of the model presented here is that we use random transition times, independent and identically distributed (i.i.d.) for each pair of devices, instead of a globally fixed transition time, reflecting fluctuation in the individual pair-connectivity. The key innovation, however, is that we now introduce a second type of device with patching capability and that we now trace the set of susceptible, infected, and patched devices over time in a non-trivial manner, leading to notions of local and global survival of the infection. Interestingly, also in the presence of the chase–escape dynamics, some of the phase transitions in the percolation analysis in [11] have a reincarnation in terms of transition of global survival and extinction, most noteworthy with respect to the velocity parameter.

On the technical side, the refined dynamical analysis makes it necessary to impose certain stronger conditions on the connectivity graph, such as exponential moments or reachability properties of the street system as well as the waypoint kernel, and to provide corresponding non-trivial proofs for the implied local connectedness and large-time connectedness results. Here, the large-time connectedness is the crucial ingredient for the proofs for global survival, which also leverage discretization and comparison arguments for the local percolation of areas that support the spread of the infection, already partially present in [11]. On the other hand, the local connectedness is crucial for the proofs of absence of global survival, which leverage multiscale arguments first presented in the context of the heavy-tailed Boolean model in [19]. The main innovations of the arguments presented in this manuscript are due to the incorporation of the mentioned random transition times between pairs of devices in the context of geostatistical markings. Finally, let us mention that our arguments for survival of the infection in the setting of chase–escape on the connectivity graph are entirely new.

1.2. Street systems

As a first layer of the system, we consider a *stationary random planar segment process* $S = \bigcup_{i \in I} \{S_i\} \subset \mathbb{R}^2$ where each S_i is a line segment in $\mathbb{R}^2$ of finite length. We say that a process is stationary if its distribution is translation invariant. The line segments may intersect, and we call each maximal unintersected interval in S a *street*. The endpoints of each street are called *crossings* (even if they are dead-ends). In order to have something specific in mind, S can be, for example, some planar tessellation process like the *Poisson–Voronoi tessellation* (PVT) or the *Poisson–Delaunay tessellation* (PDT); see Figure 1 for an illustration.

We think of S as an urban street system and note that, for example, the PVT indeed shares some common characteristics with medieval city topologies; see [13].

In order to control spatial dependencies in S, we will almost exclusively work with systems that satisfy a quantitative mixing property called *stabilization* [31, 37, 38]. Additionally, we will assume that the system is connected in large regions with high probability and call this *asymptotic essential connectedness* [24, 29]. More precisely, let us define the square with
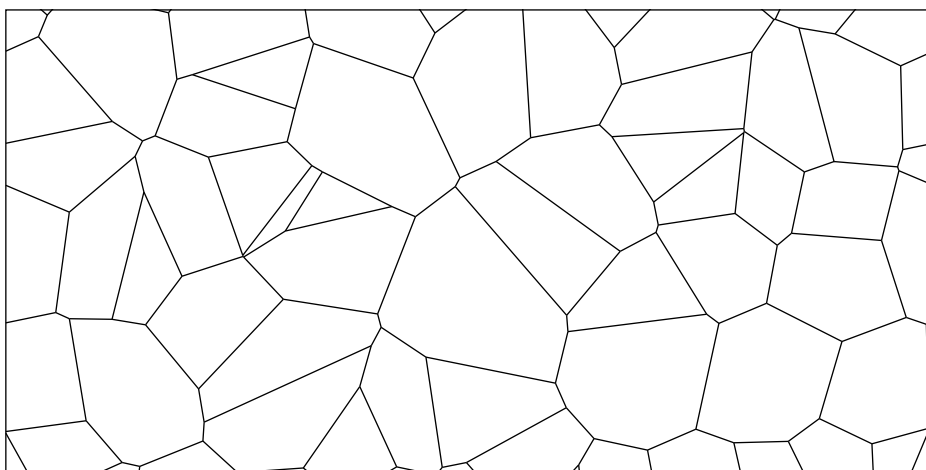
FIGURE 1. Realization of a street system given by a Poisson–Voronoi tessellation.

side length $n \geq 1$ centered at $x \in \mathbb{R}^2$ by $Q_n(x) = x + [-n/2, n/2]^2$, abbreviate $Q_n = Q_n(o)$, and denote by $\text{Sdist}(\varphi, \psi) = \inf\{|x - y| : x \in \varphi, y \in \psi\}$ the distance between sets $\varphi, \psi \subset \mathbb{R}^2$.

Definition 1. *(Stabilization and asymptotic essential connectedness.)* A stationary random segment process S is called *stabilizing* if there exists a random field of *stabilization radii* $R = \{R_x\}_{x \in \mathbb{R}^2}$ that is measurable with respect to S, and (S, R) is jointly stationary. Moreover, for $R(Q_n) = \sup_{y \in Q_n \cap \mathbb{Q}^2} R_y$ we have $\lim_{n \uparrow \infty} \mathbb{P}(R(Q_n) < n) = 1$ and, for all $n \geq 1$, the random variables $\left\{f(S \cap Q_n(x))\mathbf{1}\{R(Q_n(x)) < n\}\right\}_{x \in \varphi}$ are independent for all non-negative bounded measurable functions f and finite $\varphi \subset \mathbb{R}^2$ with $\text{Sdist}(x, \varphi \setminus \{x\}) > 3n$ for all $x \in \varphi$. Furthermore, S is called *asymptotically essentially connected* if it is stabilizing and, for all sufficiently large $n \geq 1$, whenever $R(Q_{2n}) < n/2$, we have $|S \cap Q_n| > 0$ and $S \cap Q_n$ is contained in one of the connected components of $S \cap Q_{2n}$. Here, $|\cdot|$ represents the total edge length, i.e. the one-dimensional Hausdorff measure.

We note that PVTs and PDTs are stabilizing [24] but Manhattan grids or (rectangular) Poisson-line tessellations are not stabilizing.

Finally, in what follows we will always assume that the expected number of streets and the expected number of crossings in the unit square is finite, and that the intensity of the street system is normalized, i.e. $\mathbb{E}[|S \cap Q_1|] = 1$.

1.3. Mobile devices

As a second layer of our model we represent *initial positions of devices* via *Cox point processes* $X^\lambda = \{X_i\}_{i \geq 1}$. That is, X^λ is a Poisson point process with a random intensity measure $\Lambda_S(A) = \lambda|S \cap A|$, $A \subset \mathbb{R}^2$ measurable, where $\lambda \geq 0$ is a parameter that represents the expected number of devices per unit of street length; see Figure 2. Note that if S is stationary, so is X^λ.

Having defined the initial positions of the devices via a Cox point process on the street system, we now introduce the random waypoint model as our destination-driven mobility scheme. Each device independently picks its destination and commutes back and forth on a shortest path.
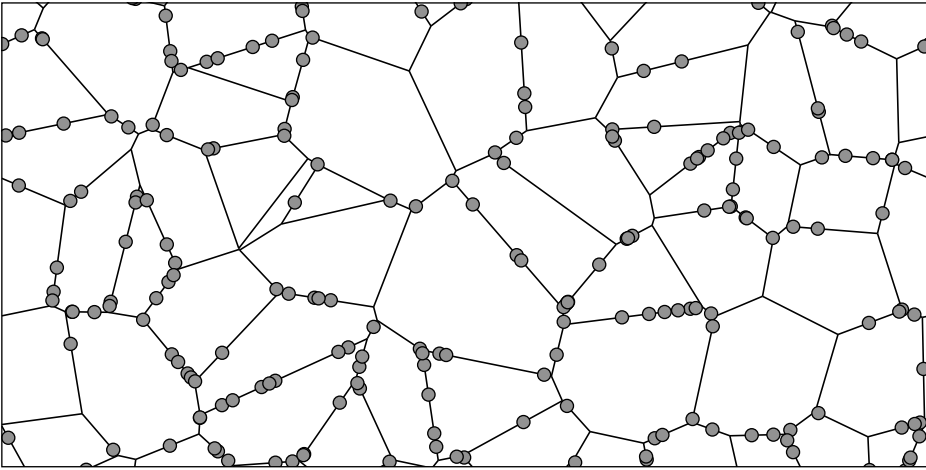
FIGURE 2. Realization of initial device positions confined to a street system given by a Poisson–Voronoi tessellation.

We start by considering the probability kernel $\kappa^S(x, \mathrm{d}y)$, $x \in S$, as a probability measure on S, and assume that the *support* of $\kappa^S(x, \mathrm{d}y)$, defined via

$$\mathrm{supp}\big(\kappa^S(x, \mathrm{d}y)\big) = \big\{y \in S \colon \kappa^S(x, B_\varepsilon(y)) > 0 \text{ for all } \varepsilon > 0\big\},$$

is contained in S for any $x \in S$, where $B_\varepsilon(y)$ denotes the disk with radius $\varepsilon > 0$ centered on y. We say that κ has (uniformly) *bounded support* if there exists $K > 0$ such that $|\mathrm{supp}(\kappa^S(x, \mathrm{d}y))| < K$ for almost all S. Moreover, we will always assume that κ is *translation covariant*, i.e. $\kappa^S(x, A) = \kappa^{S-x}(o, A - x)$ for almost all S, measurable $A \subset \mathbb{R}^2$, and $x \in \mathbb{R}^2$. The kernel $\kappa^S(x, \mathrm{d}y)$ serves as a *waypoint kernel* [8]. Furthermore, we require that the kernel is *finitely dependent*, i.e. that there is a constant $H > 0$ such that, for almost all S and $x \in S$, $\kappa^S(x, \mathrm{d}y) = \kappa^{S'}(x, \mathrm{d}y)$ for all S' with $S \cap B_H(x) = S' \cap B_H(x)$.

It will become useful to consider waypoint kernels that allow for very small displacements. In order to formalize this, we call the kernel *c-well-behaved* if $B_c(x) \cap S \subset \mathrm{supp}(\kappa^S(x, \mathrm{d}y))$ for almost all S, $x \in S$, where $\kappa^S(x, y)$ is the density, and say that it is *well-behaved* if it is *c-well-behaved* for some $c > 0$. An example that fulfills all those conditions is given by

$$\kappa^S(x, \mathrm{d}y) = \big|S \cap B_L(x)\big|^{-1}\big|\mathrm{d}y \cap S \cap B_L(x)\big|,$$

the uniform distribution on $S \cap B_L(x)$ for some $L > 0$; see also [11].

Then, each device $X_i \in X^\lambda$ independently picks a *target location* $Y_i \in S$ according to $\kappa^S(X_i, \mathrm{d}y)$. Further, we equip each device $X_i \in X^\lambda$ with an i.i.d. velocity V_i, drawn from the distribution μ, for which we assume that $0 < v_{\min} \leq v_{\max} < \infty$, where $v_{\min} := \sup\{v \colon \mu([v, \infty)) = 1$ and $v_{\max} := \inf\{v \colon \mu([0, v]) = 1\}$. Then, the path of the device X_i is defined iteratively as follows. X_i moves with speed V_i to Y_i along the shortest route in S that connects X_i and Y_i. If there are multiple shortest routes, the device chooses one such route independently and uniformly at random. It then immediately returns to its starting position with the same velocity along the same shortest path. This procedure is then repeated indefinitely; see Figure 3 for an illustration.
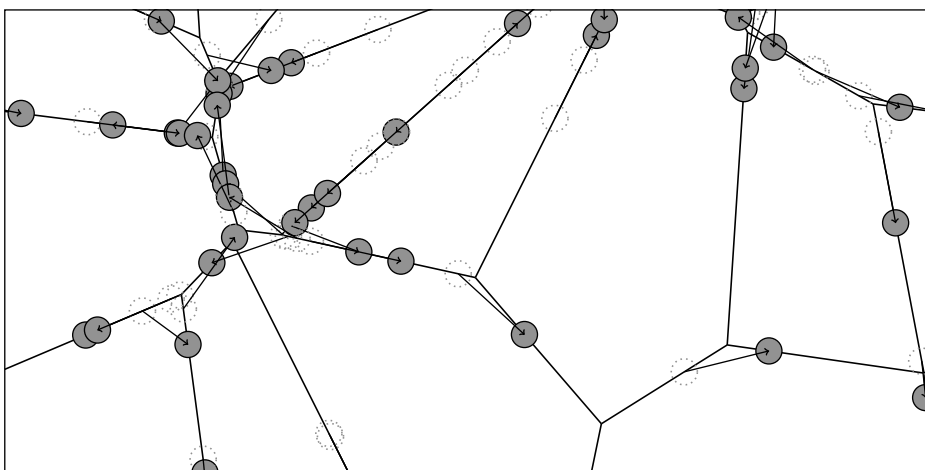
FIGURE 3. Realization of initial device positions (dotted) confined to a street system given by a Poisson–Voronoi tessellation and their respective positions at a fixed positive time (filled), with arrows indicating the corresponding displacement.

1.4. Connectivity of mobile devices

By a slight abuse of notation, we also denote by $X_i = (X_{i,t})_{t \geq 0}$ the trajectory of device X_i in S. For any pair of devices X_i and X_j, we then consider the set of *contact times*

$$Z(X_i, X_j) = \left\{ t \geq 0 \colon |X_{i,t} - X_{j,t}| < r \text{ and } X_{i,t}, X_{j,t} \text{ are on the same street} \right\}$$

as the times where the devices are on the same street and r-close together, where $r > 0$ is another parameter in the model, the *connectivity threshold*. Let us note that the constraint that contact times require the devices to be on the same street can be seen as a very strict *shadowing* assumption [17].

Next, we assume that any pair of devices X_i and X_j has an i.i.d. *infection time* $\rho(X_i, X_j)$, drawn from the distribution ϱ on $(0, \infty)$. Then, we say that a *transmission from X_i to X_j is possible at time $t \geq 0$* if $[t - \rho(X_i, X_j), t] \subset Z(X_i, X_j)$; see Figure 4 for an illustration of the resulting space–time connectivity network.

Let us note that the introduction of random transition times is a substantial generalization compared to the setup in [11], where only the case $\varrho = \delta_\rho$ is considered for some fixed $\rho \geq 0$.

1.5. Degree of devices

In order to avoid scenarios in which a single device can transmit to infinitely many devices in the infinite time horizon that we consider, let us finally introduce a convenient condition for the *local connectedness* of the connectivity graph. For this, let $\ell_S(x, y) \subset \mathbb{R}^2$ be the shortest path between x and y on S, and write

$$\ell_S(x) = \sup \left\{ |\ell_S(x, y)| \colon y \in \operatorname{supp}\big(\kappa^S(x, \mathrm{d}y)\big) \right\} + r/2$$

for the length of the shortest path starting in x towards any reachable target on S plus half the connectivity range. Then, consider the following *Cox–Boolean model with geostatistical markings* in which $X_i, X_j \in X^\lambda$ are connected whenever $|X_i - X_j| \leq \ell_S(X_i) + \ell_S(X_j)$, and note
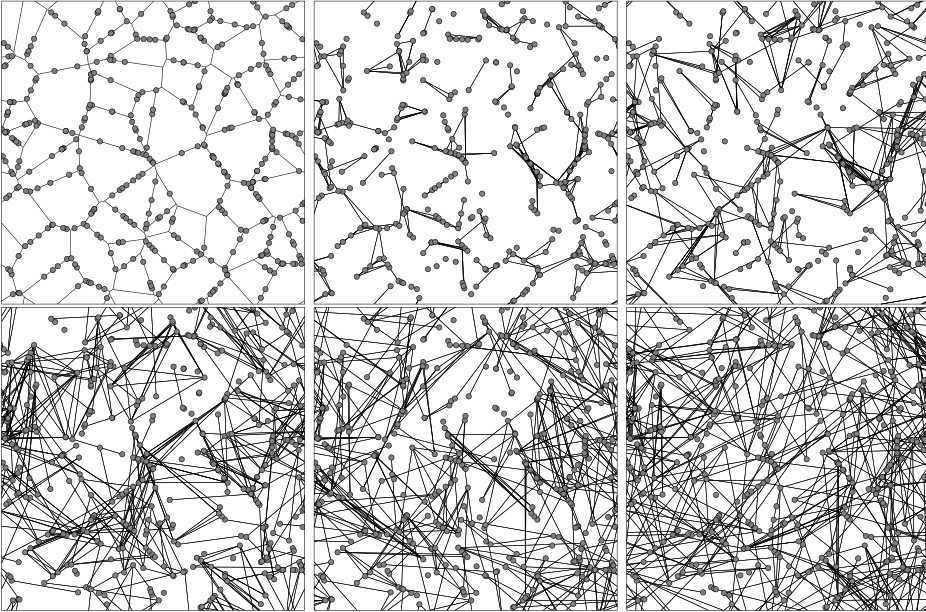
FIGURE 4. Realization of initial device positions (gray) on a street system given by a Poisson–Voronoi tessellation (only drawn in the initial picture, visually suppressed otherwise). Between two devices X_i and X_j an edge is drawn if $t \in Z(X_i, X_j)$ for $t = 0, 1, \ldots, 5$. We highlight that the total number of edges in this sequence of realizations stays roughly constant; however, the edges become longer. This is because we plot the initial positions, but the devices move and hence, over time, connect to more distant devices.

that this can be seen as a Boolean model in which the (random) disks associated to each Cox point depend on the underlying environment S. In particular, if X_i can transmit to X_j, there also exists an (undirected) edge between X_i and X_j in the Cox–Boolean model with geostatistical markings. Using this, for $X_i \in X^\lambda$, we define its degree,

$$\deg(X_i) = \#\big\{X_j \in X^\lambda \setminus X_i \colon |X_i - X_j| \le \ell_S(X_i) + \ell_S(X_j)\big\},$$

which serves as an upper bound for the degree of X_i in the original model. Now, we call the model *locally connected* if

$$\mathbb{P}\big(\text{there exists } X_i \in X^\lambda \text{ such that } \deg(X_i) = \infty\big) = 0,$$

and we will always assume that our system is locally connected. Let us also mention that, in order to ease notation, we use generic symbols $\mathbb{P}$ and $\mathbb{E}$ for the distribution and expectation of our model, even under changing parameters.

The following proposition establishes a condition under which the network is locally connected.

Proposition 1. (Local connectedness.) *If κ has bounded support, S is exponentially stabilizing, and $|S \cap Q_1|$ has exponential moments, then the network is locally connected.*

The proof rests on a contradiction that follows from a first-moment method for the expected degree of a typical point using the good control on stabilization and moment properties of

the street system. We present the details in Section 4.1. Let us note that the conditions are satisfied for our standard examples for street systems given by PVT and PDT, since they are exponentially stabilizing [24] and have exponential moments; see [27]. Also, as can be seen from the proof, the moment condition can be substantially relaxed.

Having now defined our system of moving devices in an urban street system and their connectivity relations, in the following section we introduce *malware* into the network that propagates like an *infection* starting from a single typical device.

2. Chase–escape on the dynamic graph

In the final model layer we introduce an infection into the system, which is carried by one device X_o that is chosen at random and will be referred to in the sequel as the *typical device*. The remaining devices are, at time zero, *susceptible* to the infection. As a counter measure we introduce an independent (Cox) point process of so-called *white knights*, which is, at time zero, a homogeneous Poisson point process Y^{λ_W} with linear intensity $\lambda_W \geq 0$ conditioned on S and following the same mobility scheme as devices in X^λ. From a modeling perspective, we are guided here by the idea that white knights are also regular devices and their general behavior is indistinguishable from other devices. However, white knights are not susceptible to the infection. Moreover, each white knight carries *patching* software that allows it to remove the infection from any device that wants to infect it with malware and update the defensive mechanism on the attacking device. Additionally to purging the infection, the attacking device obtains all the capabilities of a white knight. Let us highlight that although there is no inter-action between susceptible devices and white knights, their mere presence offers additional protection in the case of an infection.

Moreover, the process of susceptible devices is decreasing and the process of white knights is increasing, while the process of infected devices is non-monotone. More formally, let us introduce the *Palm version of susceptible devices* via

$$\mathbb{E}^*\big[f\big(X^\lambda, Y^{\lambda_W}\big)\big] = \frac{1}{\lambda}\mathbb{E}\Bigg[\sum_{X_i \in X^\lambda \cap Q_1} f\big(X^\lambda - X_i, Y^{\lambda_W} - X_i\big)\Bigg] \tag{1}$$

for bounded, measurable test functions f, and note that under $\mathbb{P}^*$ there exists a device at the origin with probability 1. We call this the typical device X_o and assume that it is *infected* at time zero.

In order to describe the infection and patching mechanism, instead of a single infection time distribution ϱ, we consider two distributions ϱ_I, respectively ϱ_W, for the *infection times*, respectively the *patching times*, and assume that they both have a density with respect to the Lebesgue measure on $(0, \infty)$. We denote by

$$\varrho_{I,min} = \inf\big\{x \geq 0 \colon x \in \mathrm{supp}(\varrho_I)\big\}$$

the minimal infection time, and analogously by $\varrho_{W,min}$ the minimal patching time. Now, the infection will be transmitted from an infected device X_i to a susceptible device X_j after completion of the first infection time $\rho_I(X_i, X_j)$, drawn from ϱ_I. Analogously, the patch will be transmitted from a white knight device X_i to an infected device X_j after completion of the first infection time $\rho_W(X_i, X_j)$, drawn from ϱ_W. In order to formalize the whole process, let S_t, I_t, and W_t denote the sets of susceptible, infected, and white knight devices at time $t \geq 0$, where
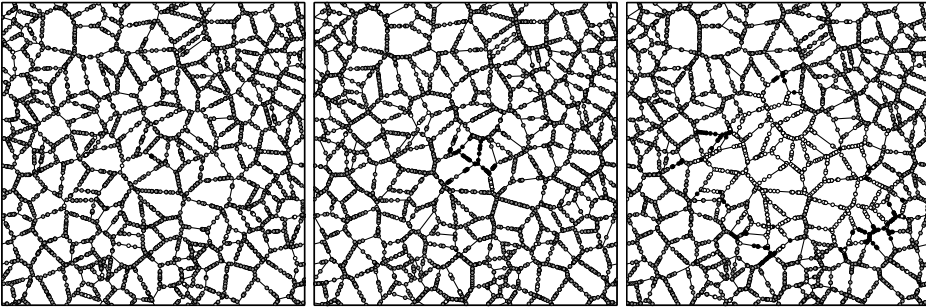
FIGURE 5. Propagation of infected devices (black) on a street system given by a Poisson–Voronoi tessellation. In the initial state (left) there is exactly one infected device present in the center and the remaining devices are either susceptible (gray) or white knights (white). At some small positive time (middle) further devices in the vicinity of the initially infected device have become infected and have started to make contact with white knights. At some later time (right) infected devices are only present along the boundary of the set of white knights in the center region.

$S_0 = X^\lambda \setminus X_o$, $I_0 = X_o$, and $W_0 = Y^{\lambda_W}$. In particular, $\big((I_s)_{0 \le s < t}, (W_s)_{0 \le s < t}\big)$ fully describes the state of the system up to time t. Then, at time t, we add new infected devices,

$$I_t \setminus \bigcup_{0 \le s < t} I_s = \left\{ X_i \in \bigcup_{0 \le s < t} S_s : \text{there exists } X_j \in \bigcap_{s \in [t - \rho_I(X_j, X_i), t)} I_s \right.$$
$$\left. \text{such that } [t - \rho_I(X_j, X_i), t) \subset Z(X_j, X_i) \right\},$$

and then add also new white knights,

$$W_t \setminus \bigcup_{0 \le s < t} W_s = \left\{ X_i \in X^\lambda : \text{there exists } X_j \in W_{t - \rho_W(X_j, X_i)} \text{ such that} \right.$$
$$\left. [t - \rho_W(X_j, X_i), t) \subset Z(X_j, X_i) \text{ and } X_i \in I_{t - \rho_W(X_i, X_j)} \setminus \bigcup_{0 \le s < t} W_s \right\}.$$

With this construction the process is well-defined since ϱ_I and ϱ_W put no mass on zero and the number of devices that are infected is finite for any finite time by the local connectedness condition. Note that we have indirectly implemented a tie-breaker rule that favors the infection. This can be seen by considering a situation where a device X_j becomes a white knight at time t; however, it can still finish a transmission of the infection at time t. We present a realization of the chase–escape dynamics in Figure 5.

   We say that the infection *survives locally* if $|I_t| > 0$ for all $t \ge 0$, it *survives globally* if $|I_t| > 0$ for all $t \ge 0$ and $\big|\bigcup_{t \ge 0} I_t\big| = \infty$, and the infection *goes extinct* if $|I_t| = 0$ for some $t \ge 0$. Note that, even if the street system is connected with positive probability, there can be isolated sets of devices that are not able to form connections to the rest of the devices. If the typical device X_o is contained in such a cluster and the cluster contains no white knight, the infection survives indefinitely, but no global outbreak can be observed.

Our goal is to isolate regimes of global survival and extinction of the infection using the transmission mechanism we just described. Let us start with regimes that ensure global survival of the infection even in the presence of white knights. For this, we guarantee the existence of a sequence of streets on which the infection can be transmitted to infinity with positive probability.

Note that, due to the transmission mechanism, streets need a certain minimal length in order to be useful for transmissions. We thus define a thinned system $S^a = \{s \in S : |s| \geq a\}$ that only contains streets of length greater or equal than $a \geq 0$. Let us further denote by R_x^a the distance of the furthest point from $x \in \mathbb{R}^2$ that is reachable without crossing S^a, i.e. we define the cell of x by

$$C_x^a = \sup \{y \in \mathbb{R}^2 : \text{there exists a continuous function}$$
$$f : [0, 1] \mapsto \mathbb{R}^2 \setminus S^a \text{ with } f(0) = x, f(1) = y\}$$

and define $R_x^a = \sup\{|y - x| : y \in C_x^a\}$. We say that the thinned graph is R^a-*connected* if $\lim_{n \uparrow \infty} \mathbb{P}\big(\sup_{x \in Q_n \cap \mathbb{Q}^2} R_x^a < n \big) = 1$, and define

$$a_{\mathrm{c}} := \sup \{a > 0 : S^a \text{ is } R^a\text{-connected}\}. \tag{2}$$

In words, for $a < a_{\mathrm{c}}$ the thinned street system satisfies a slightly stronger version of asymptotically essentially connectedness, and we have some control over the cell sizes. Note that $a_{\mathrm{c}} > 0$ for asymptotically connected street systems; see [11, Theorem 2.8].

In order to exhibit a regime of global survival, we will furthermore require a slightly stronger version of well-behavedness. We say that a kernel κ is *c-continuous* if, for almost-all S and $x \in S$, the uniquely defined absolutely continuous part of $\kappa^S(x, \mathrm{d}y)$ is non-trivial and its density $\kappa^S(x, y)$ with respect to the Lebesgue measure on S satisfies $\inf_{y \in B_c(x)} \kappa^S(x, y) > 0$. Note that *c*-continuous implies *c*-well-behaved.

Using these definitions, we can state our first main result on global survival of the infection.

Theorem 1. (Global survival.) *Let κ be c-continuous for some $c > 0$ and $\varrho_{\mathrm{W,min}} > \varrho_{\mathrm{I,min}} > 0$. Then, for all sufficiently small v_{min} such that $\min(a_{\mathrm{c}}/2, r, c/2) > v_{\mathrm{min}} \varrho_{\mathrm{I,min}} > 0$, there exists λ_{c} such that, for all $\lambda > \lambda_{\mathrm{c}}$ and all $\lambda_{\mathrm{W}} \geq 0$, $\mathbb{P}(\text{infection survives globally}) > 0$.*

The punchline of the preceding statement is the fact that the critical device intensity is independent of the white knight intensity. In other words, no matter how densely we can pack white knights into the system, survival is guaranteed as long as we have sufficiently many devices in the system. Roughly, the reason for this phenomenon is that the white knights simply act too slowly since $\varrho_{\mathrm{W,min}} > \varrho_{\mathrm{I,min}} > 0$ when there are sufficiently many devices that can transmit the infection to their neighbors before white knights are able to eliminate it. The idea of the proof is to establish an infinite cluster of good streets that have the property that, if the malware enters the street on one side, it must also exit the street on the other side. We present the details in Section 4.2.

Next, we present our result on the extinction of the infection in infinite components.

Theorem 2. (Extinction.) *Consider a street system given by a Poisson–Voronoi or Poisson–Delaunay tessellation. Let κ be well-behaved with bounded support, and assume that $r > v_{\mathrm{max}} \varrho_{\mathrm{W,min}}$ and $\varrho_{\mathrm{I,min}} > \varrho_{\mathrm{W,min}} > 0$. Then, for all $\lambda \geq 0$, there exists $\lambda_{\mathrm{W,c}}$ such that, for all $\lambda_{\mathrm{W}} > \lambda_{\mathrm{W,c}}$, $\mathbb{P}(\text{infection survives globally}) = 0$.*

The proof rests on a multi-scale argument and will be given in Section 4.3.

Finally, we present a result that features the remarkable non-monotone behavior of our model with respect to the speed parameter. More precisely, the following result states that both too-high and too-low velocities are detrimental to the propagation of the infection. We state the result only for fixed velocities and infection and patch times, and note that a corresponding result for general μ_v, ϱ_I, and ϱ_W also holds.

Theorem 3. (Speed-dependent survival and extinction.) *Consider a street system given by a Poisson–Voronoi or Poisson–Delaunay tessellation. Let κ have bounded support and be c-well-behaved for some $c > 0$. Assume further that $\varrho_W = \delta_{\rho_W}$ and $\varrho_I = \delta_{\rho_I}$ with $\rho_W > \rho_I$, and let $0 < v_o < \min(a_c/2, r, c/2)/\rho_I$. Then, there exist $\lambda_c > 0$ and $\lambda_{W,c} > 0$ (independent of λ_c and v_o) such that, for all $\lambda > \lambda_c$ and all $\lambda_W > \lambda_{W,c}$,*

   (i)   there exists $v_o > v_c(\lambda, \lambda_W) > 0$ such that, for all $\mu = \delta_v$ with $v < v_c(\lambda, \lambda_W)$, $\mathbb{P}$(infection survives globally) $= 0$;

   (ii) for $\mu = \delta_{v_o}$, $\mathbb{P}$(infection survives globally) > 0; and

   (iii) there exists $\infty > v^c > v_o$ (independent of λ and λ_W) such that, for all $v > v^c$, $\mathbb{P}$(infection survives globally) $= 0$.

In words, the choice $\lambda > \lambda_c$ puts us in a parameter regime in which survival of the infection is possible, as described in Theorem 1. However, in the presence of sufficiently many white knights and sufficiently large or small velocity we are again in regimes of extinction. One important aspect here is that sufficiently small velocities and large white knight intensities also lead to absence of global survival in the case where the infection is faster than the patching. We present the proof in Section 4.4.

3. Chase–escape on the connectivity graph

In Section 2 we exhibited rigorous results for chase–escape on the dynamic graph. One important feature here is that device mobility is intimately linked to the transmission mechanism, both of the infection and the patching, leading to a highly correlated non-Markovian space–time process. As a consequence, it currently seems out of reach to derive approximate values for the various critical parameters predicted by our rigorous analysis in ways other than computer simulations as presented, for example, in [5, 6]. As it turns out, a major part of the computational effort has to go into simulating the movement of the devices in the environment. In an effort to produce large numbers of sample paths of the chase–escape dynamics it is therefore desirable to uncouple the connectivity graph and the epidemiological process and derive corresponding results in this *mean-field type* setting. This is our primary goal in this section.

To start, recall the connectivity rule, introduced in Section 1.4, where for every pair of devices X_i, X_j the set of connection times is denoted $Z(X_i, X_j)$. Now, we want to fix the infection time $\rho \geq 0$, i.e. assume that $\varrho = \delta_\rho$ and define the *connectivity graph* $g_{\rho,r}(S, X^\lambda \cup Y^{\lambda_W})$, in which any two vertices $X_i, X_j \in X^\lambda \cup Y^{\lambda_W}$ are connected by an edge if there exists a $t \geq 0$ such that $[t, t + \rho] \subset Z(X_i, X_j)$. Let us highlight that this connectivity graph is *static* even though the edge-drawing mechanism is based on an underlying space–time process. The main computational advantage of this connectivity graph lies in the fact that the connections can be determined purely from the device trajectories and speeds without the need to simulate in which connection interval the connection is realized. This avoids the computationally expensive collective movement of the devices.

Now, we consider the chase–escape model as an independent space–time process on realizations of the connectivity graph. This also brings us much closer to the existing literature on chase–escape dynamics on fixed and random graphs; see, for example, [4, 7, 15, 21–23, 39], as chase–escape has not been studied on a dynamic graph so far. More precisely, we consider $g_{\rho,r}(S, X^\lambda \cup Y^{\lambda_W})$ under the Palm distribution with respect to the X^λ described in (1) and let $N(X_i)$ denote the set of direct neighbors of vertex X_i. Further, we consider two general transmission-time distributions ϱ_I and ϱ_W on $(0, \infty)$ that govern the i.i.d. times $\rho_I(X_i, X_j)$, respectively $\rho_W(X_i, X_j)$, needed to pass an infection, respectively a patch, from device X_i to X_j. Then, at time zero, the set of infected devices I_0 is given by the typical device X_o, which is positioned at the origin $o \in \mathbb{R}^2$ with probability 1, i.e. $I_0 = \{X_o\}$. For the initial set of white knights we have $W_0 = Y^{\lambda_W}$. Then, as in Section 2, we define the process iteratively by considering the system $\big((I_s)_{0 \le s < t}, (W_s)_{0 \le s < t}\big)$ up to time $t > 0$ and then defining the newly infected devices at time t by

$$I_t \setminus \bigcup_{0 \le s < t} I_s = \left\{ X_i \in \bigcup_{0 \le s < t} S_s \colon N(X_i) \cap \bigcap_{s \in [t - \rho_W(X_j, X_i), t)} I_s \ne \emptyset \right\},$$

where $S_s = X^\lambda \setminus I_s$, and the newly patched devices at time t by

$$W_t \setminus \bigcup_{0 \le s < t} W_s = \Bigg\{ X_i \in X^\lambda \colon \text{there exists } X_j \in W_{t - \rho_W(X_i, X_j)} \cap N(X_i),$$
$$X_i \in I_{t - \rho_W(X_i, X_j)} \setminus \bigcup_{0 \le s < t} W_s \Bigg\}.$$

Let us comment on the construction. As mentioned above, the model can be seen as an approximation of chase–escape on the dynamical graph where we associate to every edge an independent random time that represents the time that it takes to transmit the infection. The distribution of this transfer time can be, for example, sampled by considering the transmissions of a typical device towards its neighbors in an independent simulation step. In other words, we insert information about the typical transmission behavior and apply it independently to every edge of the static graph. Let us mention that this approach is not unlike the approach used in [6, Section V], where the spatial graph was replaced by a sequence of edges which had the length distribution of a typical edge in the PVT.

We are again interested in the conditions under which global survival is possible or impossible. Let us start with the preliminary observation in the case where the transmission times are fixed, i.e. $\varrho_I = \delta_{T_I}$ and $\varrho_W = \delta_{T_W}$ for some $T_W, T_I > 0$. In this case, survival and extinction essentially depend on the ordering of T_W and T_I. We say that the graph $g_{\rho,r}(S, X^\lambda)$ is in the *supercritical percolation regime* if there is an unbounded connected component, and we note that [11] exhibits non-trivial criteria that guarantee the existence of a supercritical percolation regime. We first formulate a result for constant transmission times. Extended results with respect to general ϱ_I and ϱ_W are presented in Theorems 4, 5, and 6.

Proposition 2. *Assume that $g_{\rho,r}(S, X^\lambda)$ is in the supercritical percolation regime. Then,*

$$\mathbb{P}(\textit{infection survives globally}) \begin{cases} > 0 & \textit{for } T_I \le T_W, \lambda_W \ge 0, \\ = 0 & \textit{for } T_I > T_W, \lambda_W > 0. \end{cases}$$

The proof is based on a robust descending-chain argument that in fact generalizes to arbitrary supercritical graphs, and is presented in Section 4.5.

Let us now present our main results for general ϱ_I and ϱ_W. In order to prove global survival of the infection we need to require stronger notions of percolation of the connectivity graph. We present two alternative conditions in Theorems 4 and 5. First, in Theorem 4 we require a parameter regime for the street system and waypoint kernel that guarantee a strongly percolating structure similar to Theorem 1. Recall the definition of a_c from (2), and let us denote, for any $b > 0$, by ϱ_I^b the shifted version of ϱ_I defined via $\int f(x/b)\varrho_I^b(\mathrm{d}x) = \int f(x)\varrho_I(\mathrm{d}x)$ for all measurable functions with bounded support f.

Theorem 4. (Mean-field survival I.) *Let κ be c-well-behaved for some $c > 0$, and assume that $v_{\min}$ satisfies $0 < v_{\min}\rho < \min(a_c/2, r, c/2)$ and that $\varrho_{W,\min} > 0$. Then, there exists $\lambda_c < \infty$ such that, for all $\lambda > \lambda_c$ and $\lambda_W \geq 0$, there exists $b_c < \infty$ such that, for ϱ_I^b with $b > b_c$, $\mathbb{P}(\text{infection survives globally}) > 0$.*

The proof rests on percolation arguments similar to the ones used for the proof of Theorem 1. The details are presented in Section 4.6.

Next, we present an alternative result for global survival which requires that a subgraph of the connectivity graph is in the supercritical percolation regime. More precisely, we define the random subgraph $g_{\rho,r}^{M,p}(S, X^\lambda) \subset g_{\rho,r}(S, X^\lambda)$ via the following thinning procedure. Let $(\mathrm{Ber}_p(X_i))_{X_i \in X^\lambda}$ be an i.i.d. field of Bernoulli random variables with parameter p, and define the vertex set of $g_{\rho,r}^{M,p}(S, X^\lambda)$ by

$$V^{M,p} = \left\{ X_i \in X^\lambda : N(X_i) \leq M \text{ and } \mathrm{Ber}_p(X_i) = 1 \right\},$$

where $N(X_i)$ denotes the number of neighbors in the graph $g_{\rho,r}(S, X^\lambda, Y^{\lambda_W})$ that also includes the white knights. In words, vertices with more than M neighbors are eliminated, and additionally each of the remaining vertices is kept independently with probability p. The edge set of $g_{\rho,r}^{M,p}(S, X^\lambda)$ is then inherited from the edge set of $g_{\rho,r}(S, X^\lambda)$, but only those that connect $X_i, X_j \in V^{M,p}$. It is not hard to see that

$$\lim_{M\uparrow\infty, p\uparrow 1} g_{\rho,r}^{M,p}(S, X^\lambda) = g_{\rho,r}(S, X^\lambda)$$

weakly; however, for global observables such as percolation this is non-trivial. We have the following result.

Theorem 5. (Mean-field survival II.) *Assume that there exist $M > 0$ and $p > 0$ such that $g_{\rho,r}^{M,p}(S, X^\lambda)$ is in the supercritical percolation regime. Then, for all $\lambda_W \geq 0$, there exists $b_c < \infty$ such that, for all ϱ_I^b with $b > b_c$, $\mathbb{P}(\text{infection survives globally}) > 0$.*

The proof rests on the idea that, for a sufficiently fast infection rate, the process of nodes with maximal degree, which also transmit the infection faster than being patched, is already in the supercritical percolation regime. A motivation for this thinning lies in an application of the SINR: In this connection model, communication is only possible if the incoming signal strength is high enough compared to the total interference. If too many devices are transmitting simultaneously in a small area, communication is not possible due to background noise. We present the details in Section 4.7.

We conclude this section with our result on the absence of global survival.

Theorem 6. (Mean-field extinction.) *Consider a street system given by a Poisson–Voronoi or Poisson–Delaunay tessellation and assume that $\varrho_{I,\min} > \varrho_{W,\min}$. Then, there exists $\lambda_{W,c} < \infty$ such that, for all $\lambda_W > \lambda_{W,c}$, $\mathbb{P}$(infection survives globally) $= 0$.*

The proof is based on the idea that, for a sufficiently large intensity of white knights, any connection between susceptible devices is accompanied by many white knight connections. In this case, it becomes highly unlikely that the infection is passed on before being eliminated by the white knights. In order to make this precise, we have to leverage local independence properties of the propagation model. We present the details in Section 4.8.

4. Proofs

4.1. Proof Proposition 1

Proof Proposition 1. First, let us assume that $\mathbb{P}$(there exists $X_i \in X^\lambda$ such that $\deg(X_i) = \infty$) > 0. Then, by continuity of measures, there exists $R > 0$ such that $\mathbb{P}$(there exists $X_i \in X^\lambda \cap Q_R$ such that $\deg(X_i) = \infty$) > 0 and hence, by translation invariance, also $\mathbb{P}$(there exists $X_i \in X^\lambda \cap Q_1$ such that $\deg(X_i) = \infty$) > 0. But the last inequality implies that $\mathbb{E}\left[\sum_{X_i \in X^\lambda \cap Q_1} \deg(X_i)\right] = \infty$. However, in order to derive a contradiction, we can use the Mecke–Slivnyak theorem twice to see that

$$\mathbb{E}\left[\sum_{X_i \in X^\lambda \cap Q_1} \deg(X_i)\right] \leq \mathbb{E}\left[\sum_{X_i \in X^\lambda \cap Q_1} \sum_{X_j \in X^\lambda \setminus X_i} \mathbf{1}\left\{|X_i - X_j| \leq \ell_S(X_i) + \ell_S(X_j)\right\}\right]$$

$$= \lambda^2 \mathbb{E}\left[\int_{S \cap Q_1} dx \int_S dy\, \mathbf{1}\left\{|x - y| \leq \ell_S(x) + \ell_S(y)\right\}\right].$$

Next, we bound the maximal reach ℓ_S by the street length in a stabilized region. For this, let $K := \sup\{|x - y|: x \in S, y \in \text{supp}(\kappa^S(x, dy))\} < \infty$ denote the maximal reach of the kernel κ, which exists due to the assumption of a uniformly bounded support of κ. Using this, we can bound the length of any shortest path starting in $x \in S$ to any $y \in S \cap B_K(x)$ by the total street length of the box in which there is a connection between x and y, due to asymptotic essential connectedness. More precisely, note that if $R(Q_{2n}(x)) < n/2$ then $\ell_S(x) \leq |S \cap Q_{2n}(x)| + r/2$, for any $n > 2K$, and we can define

$$N(x) := \min_{n \in \mathbb{N}, n > K} R(Q_{2n+1}(x)) < n/2.$$

In particular, for all $y \in Q_1(z)$ we have $\ell_S(y) \leq |S \cap Q_{2N(z)+1}(z)| + r/2$, and hence

$$\mathbb{E}\left[\int_{S \cap Q_1} dx \int_S dy\, \mathbf{1}\left\{|x - y| \leq \ell_S(x) + \ell_S(y)\right\}\right]$$

$$\leq \mathbb{E}\left[\sum_{n > K} \mathbf{1}\{N(o) = n\}|S \cap Q_1| \int_S dy\, \mathbf{1}\left\{|y| \leq \ell_S(y) + 1/2 + r/2 + |S \cap Q_{2n+1}|\right\}\right]$$

$$\leq \mathbb{E}\left[\sum_{n > K} \sum_{m > K} \sum_{z \in \mathbb{Z}^2} \mathbf{1}\{N(o) = n\}\mathbf{1}\{N(z) = m\}|S \cap Q_1|\right.$$

$$\left. \int_{S \cap Q_1(z)} dy\, \mathbf{1}\left\{|y| \leq \ell(y) + 1/2 + r/2 + |S \cap Q_{2n+1}|\right\}\right]$$

$$\leq \sum_{n>K} \sum_{m>K} \sum_{z\in\mathbb{Z}^2} \mathbb{E}\big[\mathbf{1}\{N(o)=n\}\mathbf{1}\{N(z)=m\}$$

$$|S\cap Q_1||S\cap Q_1(z)|\mathbf{1}\{|z|\leq |S\cap Q_{2m+1}(z)|+1+r+|S\cap Q_{2n+1}|\}\big],$$

where we used $|y|\leq |z|+1/2$ for $y\in Q_1(z)$ in order to eliminate the dependence of the integrals on x and y respectively and thus derive a uniform bound for the integrals. It remains to show that this bound is integrable. By Hölder's inequality we can bound each summand by

$$\mathbb{P}(N(o)=n)^{1/5}\mathbb{P}(N(o)=m)^{1/5}\mathbb{E}\big[|S\cap Q_1|^5\big]^{2/5}$$

$$\mathbb{P}\big(|z|\leq |S\cap Q_{2m+1}(z)|+1+r+|S\cap Q_{2n+1}|\big)^{1/5}$$

$$\leq C_1\exp(-c_1(n+m))\mathbb{P}(|z|\leq |S\cap Q_{2m+1}(z)|+1+r+|S\cap Q_{2n+1}|)^{1/5}$$

for some constants $C_1, c_1 > 0$ due to the existence of exponential moments for $|S\cap Q_1|$ as well as the assumption of exponential stabilization. Now, the last factor can be further bounded by Markov's inequality as

$$\mathbb{P}\big(|z|\leq |S\cap Q_{2m+1}(z)|+1+r+|S\cap Q_{2n+1}|\big)^{1/5}$$

$$\leq |z|^{-3}\mathbb{E}\big[(|S\cap Q_{2m+1}(z)|+1+r+|S\cap Q_{2n+1}|)^{15}\big]^{1/5}$$

$$\leq |z|^{-3}\big(\mathbb{E}\big[(3|S\cap Q_{2m+1}|)^{15}\big]+(3+3r)^{15}+\mathbb{E}\big[(3|S\cap Q_{2n+1}|)^{15}\big]\big)^{1/5}$$

$$\leq |z|^{-3}\big((3(2m+1)^2+3(2n+1)^2)^{15}\mathbb{E}\big[(|S\cap Q_1|)^{15}\big]+(3+3r)^{15}\big)^{1/5}$$

$$\leq C_2(mn)^{c_2}|z|^{-3},$$

for some constants $C_2, c_2 > 0$. Finally, we can use $|z| > |z|_\infty$, where $|\cdot|_\infty$ denotes the ℓ_∞-norm, combined with the fact that for $k\in\mathbb{N}_0$ we have $\sum_{z\in Z}\mathbf{1}\{|z|_\infty=k\}\leq 4k+1$, to see that

$$\sum_{n>K}\sum_{m>K}\sum_{k\in\mathbb{N}_0}(4k+1)C_2(mn)^{c_2}C_1\exp(-c_1(n+m))k^{-3}<\infty,$$

and hence the proof is finished. $\qquad\square$

4.2. Proof of Theorem 1

Proof of Theorem 1. For simplicity we only prove the case where $\mu=\delta_v$. The general case can be verified using a thinning argument where we ignore all devices that have a speed greater than $v_{\min}+\varepsilon$ for an arbitrarily small ε, only leading to a potentially larger value of λ_c.

First note that the assumption that κ is translation covariant and c-well-behaved for some $c>2\varrho_{I,\min}v$ ensures that it is possible for the kernel to generate a shortest path from one street to an adjacent street while spending a sufficient amount of time on both streets to allow an infection to be passed on to another device. Further, the requirement that $r>\varrho_{I,\min}v$ ensures that the communication radius is large enough such that an infection transmission can happen between devices with initial and target locations in the vicinity of the crossing. Also recall that we require that the streets of length larger than $2v\varrho_{I,\min}$ percolate with positive probability.

Since we assume λ_W to be arbitrary, in spirit, we think of white knights being everywhere. We consider open streets as well as open crossings. Let us start with the streets. Let $0<\varepsilon<\min(r/4, v(\varrho_{I,\min}-\varrho_{W,\min})/2)$ and define $n(s)=\lfloor |s|/(2r)\rfloor$, which will represent the number of devices on the street s on which the infection will be propagated. We call a street $s=(c_1, c_2)$ *open* if there are devices $X_0^s, \ldots, X_{n(s)}^s$ such that:

(S1) $|s| > v\varrho_{I,\min} + \varepsilon$;

(S2) for all t, $X_{i,t}^s \in B_\varepsilon \left(c_1 + \dfrac{ri}{2} \dfrac{c_2 - c_1}{|c_2 - c_1|} \right)$ for all $i \in \{1, \ldots, n(s)\}$;

(S3) $\rho_I\left(X_i^s, X_{i+1}^s \right) < \varrho_{W,\min}$ for all $i \in \{1, \ldots, n(s) - 1\}$; and

(S4) $\rho_I\left(X_i^s, X_{i-1}^s \right) < \varrho_{W,\min}$ for all $i \in \{2, \ldots, n(s)\}$.

In words, a street is open if, by Condition (S1), the street is long enough that an infection transmission is possible. By Condition (S2), there is a chain of devices reaching from the crossing c_1 to the crossing c_2 such that each pair of neighboring devices has distance less than r. In particular, if device X_0^s is infected, Condition (S3) ensures that the infection cannot be stopped by white knights as they are too slow and hence the infection propagates from X_0^s to $X_{n(s)}^s$. Similarly, Condition (S4) ensures that an infection also propagates from $X_{n(s)}^s$ to X_0^s. Let us call $X_{c_1}^s = X_1^s$ and $X_{c_2}^s = X_{n(s)}^s$ the *boundary devices* of an open street $s = (c_1, c_2)$.

Next, for a street $s = (c_1, c_2)$, denote by $[a, b]_{s,c_1}$ the segment

$$\left\{ x \in s \colon x = c_1 + d \frac{c_2 - c_1}{|c_2 - c_1|} \text{ for some } d \in [a, b] \right\}$$

on s such that $\{0\}_{s,c_1}$ is identified with the crossing c_1. Consider again $0 < \varepsilon < \min(r/4, v(\varrho_{I,\min} - \varrho_{W,\min})/2)$ to be small enough that $v_{\min}\varrho_{I,\min} + \varepsilon < c/2$. We call a crossing c *open at time t* if, for any ordered pair of open streets s_1, s_2 that share the crossing c, there exists a *bridging device* X^{s_1,s_2} such that:

(C1) $X_0^{s_1,s_2} \in A_{s_1} = [v\varrho_{I,\min}/2 + \varepsilon/2, v\varrho_{I,\min}/2 + \varepsilon]_{s_1,c}$;

(C2) X^{s_1,s_2} has its target location in $B_{s_2} = [v\varrho_{I,\min}/2, v\varrho_{I,\min}/2 + \varepsilon]_{s_2,c}$;

(C3) $X_t^{s_1,s_2} \in J_{s_1} = [0, \varepsilon/2]_{s_1,c}$ and is on its way back to its starting location;

(C4) $\rho_I\left(X^{s_1,s_2}, X_c^{s_1} \right) < \varrho_{I,\min} + \varepsilon/(2v)$; and

(C5) $\rho_I\left(X^{s_1,s_2}, X_c^{s_2} \right) < \varrho_{I,\min} + \varepsilon/v$.

These conditions roughly guarantee that, for sufficiently long streets, the bridging device X^{s_1,s_2} propagates the infection from s_1 to s_2. The time the bridging device stays on each of the streets is insufficiently long to allow patching by white knights. This is ensured by Conditions (C1) and (C2), as the distance the devices travel on each street before entering the crossing is at most $v\varrho_{I,\min} + 2\varepsilon < v\varrho_{W,\min}$, since ε is sufficiently small. Together then, since s_2 is open, due to the configuration of the devices on the open street and the fact that $r > v\varrho_{I,\min}$, X^{s_1,s_2} will pass the infection on to the boundary device $X_c^{s_2}$ of the chain of transmitting devices on s_2. This is ensured by Condition (C5). Finally, Conditions (C3) and (C4) ensure that X^{s_1,s_2} becomes infected if $X_c^{s_1}$ becomes infected at time t, as it is on its way back to its starting location and will spend at least a time of $\varrho_{W,\min} + \varepsilon/(2v)$ in the vicinity of $X_c^{s_1}$.

The following statement guarantees that there exists a uniform bound for the probability that a given crossing is open for all sufficiently large times. In particular, this bound converges to 1 as λ tends to infinity. Recall that we denoted by H the range of dependence of κ, and define $L = H + v\varrho_{I,\min}/2 + \varepsilon$.

Lemma 1. *For almost all S we have that* $(\{c \text{ is open at time } t\})_{c \in S}$ *is an independent family of events, conditioned on S. Further, there exists* $T_0 > 0$ *such that, for almost all S and all crossings* $c \in S$, *there exists a constant* $C_c = C_c(S \cap B_L(c), T)$ *such that* $\mathbb{P}(c \text{ is open at time } t \mid S) \geq 1 - \exp(-\lambda C_c)$ *for all* $t \geq T_0$.

We present the proof of the lemma later in this section.

Consider $\tau_{s_1, c}$, the time that a boundary device associated with the crossing c is infected; then, for $\tau_{s_1, c} > T_0$ and conditioned on S, we can couple the event that c is open at time $\tau_{s_1, c}$ to a Bernoulli random variable $\mathcal{B}_c$ with parameter $1 - \exp(-\lambda C_c)$, and note that the $\mathcal{B}_c$ are independent conditioned on S and only depend on S in the region $S \cap B_L(c)$. For $\tau_{s_1, c} = \infty$, i.e. if the boundary device never becomes infected, we assume that the crossing is open nonetheless. We say that a crossing is *open* if $\mathcal{B}_c = 1$. In order to make use of this observation, we need to ensure that the infection survives up until time T_0. However, note that survival for long but finite times has a positive probability and can be constructed locally by an appropriate isolation procedure for the typical device.

Now we consider the model at time $t \geq T$ and show that the graph of open streets and open crossings percolates with positive probability via stabilization arguments. For this, let us fix an a such that $\varrho_{\text{I,min}} < a/2 < a_c/2$ and recall that we have the random field of stabilizing radii $\{R_x\}_{x \in \mathbb{R}^2}$ for the street system at our disposal, in addition to the random field $\{R_x^a\}_{x \in \mathbb{R}^2}$ that acts as a replacement for the asymptotically essentially connectedness of the thinned graph S^a. Then, we say that $z \in \mathbb{Z}^2$ is *n-open* if

(a) $R(Q_{3n}(nz)) < n$;

(b) $R^a(Q_n(nz)) < n/2$;

(c) every street fully contained in $Q_{3n}(nz) \cap S^a$ is open; and

(d) every crossing in $Q_{3n}(nz)$ is open.

Condition (b) limits the size of cells of points in $Q_n(nz)$ to a diameter of at most $n/2$ with respect to S^a. Further, under the condition, there exists a unique giant component in $Q_n(nz) \cap S^a$, built by the boundaries of the cells, that is connected in $Q_{3n}(nz)$. If, now, adjacent sites z_1 and z_2 both satisfy Condition (b), they form a joint connected component, since there exist cells that are simultaneously in $Q_n(nz_1)$ and in $Q_n(nz_2)$. Due to Conditions (c) and (d), the infection is able to propagate through open streets and open crossings unpatched.

Now, we can use the domination by product measures theorem [34, Theorem 0.0] to establish Bernoulli site percolation of n-open sites on $\mathbb{Z}^2$ since Condition (a) guarantees 7-dependence of S^a, and under Condition (a) Conditions (b) and (c) can also be decided within the 7-dependent region. By choosing $n > L$, the crossing-dependent constant C_c from Lemma 1 is also 7-dependent like the street system. Hence, using translation invariance, it suffices to show that

$$\liminf_{n \uparrow \infty} \liminf_{\lambda \uparrow \infty} \mathbb{P}(o \text{ is } n\text{-open}) = 1.$$

To show this, we denote by $A(n)$, $B(n)$, $C(n, \lambda)$, and $D(n, \lambda)$ the events that Conditions (a), (b), (c), and (d) are not satisfied for $z = o$. Then, $\mathbb{P}(o \text{ is } n\text{-open}) \geq 1 - \mathbb{P}(A(n) \cup B(n)) - \mathbb{P}(C(n, \lambda)) - \mathbb{P}(D(n, \lambda))$. By assumption, $\limsup_{n \uparrow \infty} \mathbb{P}(A(n) \cup B(n)) = 0$ independently of λ. For fixed n, using Markov inequality and dominated convergence, we now also have $\limsup_{\lambda \uparrow \infty} (\mathbb{P}(C(n, \lambda)) + \mathbb{P}(D(n, \lambda))) = 0$, as λ increases, where we used that the expected number of streets and crossings is finite.

Finally, due to the translation invariance of the model, the existence of an infinite cluster of open sites implies that an infinite cluster of good streets and good crossings exists, and, thus, there is a positive probability that the typical device transmits the infection to infinity. □

Proof of Lemma 1. First note that the conditions are only dependent on the bridging devices, which have their initial positions and their destinations close to the crossing and move over the crossing, and are therefore independent for distinct crossings. This guarantees the independence of the events.

In order to bound the probability of openness at late times, we compute the intensity of devices that satisfy Conditions (C1), (C2), and (C3) at time t and then show that, as t tends to infinity, this intensity converges to a positive crossing-dependent value. In turn, the probability that no such devices exist can be seen as a Poisson void probability.

Let us start by deriving expressions for the intensity of devices at time t in the desired interval, i.e. devices that start on s_1 in the interval A_{s_1} and have a target location in B_{s_2}, on s_2. Recall that the streets s_1 and s_2 have a joint crossing c. Note that, by the displacement theorem for Poisson point processes, the number of devices that satisfy the conditions is a Poisson-distributed random variable and its parameter is given by $\lambda p_S(s_1, s_2)$, where

$$p_S(s_1, s_2) = \int_{A_{s_1}} \mathrm{d}x \int_{B_{s_2}} \kappa^S(x, \mathrm{d}z) \, \mathbf{1}\{0 \leq r_t(|x, z|_S) - |z, c|_S \leq \varepsilon/2\},$$

where $|z, x|_S$ denotes the shortest distance between x and z on S, and $r_t(|z, x|_S)$ is uniquely defined as $vt = |z, x|_S n_t + r_t(|z, x|_S)$ with n_t the largest odd number such that $0 \leq r_t(|z, x|_S) \leq 2|z, x|_S$. In words, $n_t(|z, x|_S)$ denotes the number of times that the device has traveled from x to z and back, and since we want the device to be on the way back, we require $n_t(|z, x|_S)$ to be odd. Then, $r_t(|z, x|_S)$ is the distance from z on the way back. Subtracting the distance from z to the joint crossing c, we obtain the traveled distance on s_1, which has to lie in the interval $(0, \varepsilon)_{s_1, c}$.

We will show that there is a time T_0 such that, for all $t \geq T_0$, we can bound $p_S(s_1, s_2)$ from below by a constant that only depends on $S \cap B_L(c)$.

Let us define for $x \in A_{s_1}$ by

$$0 < \eta_x = \eta_x\big(s_1, s_2, S \cap B_L(x), \kappa^S(x, \mathrm{d}y)\big) = \inf_{y \in B_{s_2}} \kappa^S(x, y)$$

the minimal density for the kernel, starting in x and having a destination in B_{s_2}. This value η_x is positive as the kernel is c-continuous, $B_{s_2} \subset B_c(x)$, and we use our assumption that $2v\varrho_{\mathrm{I,min}} + 2\varepsilon < c$.

In order to simplify the notation for A_{s_1} and B_{s_2} let us abbreviate $b = v\varrho_{\mathrm{W,min}}/2$ and associate bijectively the interval $A = [b + \varepsilon/2, b + \varepsilon]$ to A_{s_1} such that $x \mapsto x_{s_1, c}$. We associate $B = [b, b + \varepsilon]$ in a similar way to B_{s_2}. Then,

$$p_S(s_1, s_2) \geq \int_A \mathrm{d}x \, \eta_x \int_B \mathrm{d}z \, \mathbf{1}\{0 \leq r_t(x + z) - z \leq \varepsilon/2\},$$

where we also dropped the singular part in $\kappa^S(x, \mathrm{d}y)$. Now, let us define by

$$\phi_x(s) = \lim_{a \downarrow 0} \frac{1}{a} \int_B \mathrm{d}z \, \mathbf{1}\{0 \leq r_s(x + z) - z \leq a\}$$

the density of points started at x that are on their way back from their target location, and that are at time s precisely at the origin. Note that a device satisfies Condition (C3) at time t if and only if it passes the crossing on its way to its starting location in the time interval $[t - \varepsilon(2v)^{-1}, t]$. Therefore, we can rewrite

$$\int_A dx\, \eta_x \int_B dz\, \mathbf{1}\{0 \le r_t(x+z) - z \le \varepsilon/2\} = \int_{t-\varepsilon/(2v)}^t ds \int_A dx\, \eta_x \phi_x(s), \tag{3}$$

and we will prove that, $\mathbb{P}$-almost surely, conditioned on S, there exist constants T_0 and C that depend on v, $\varrho_{\mathrm{I,min}}$, and ε but not on S, x, or κ^S, such that $\inf_{t \ge T_0} \phi_x(t) > C > 0$. In order to derive this statement, the main tool is the following identity for $\phi_x(t)$:

$$\phi_x(t) = \sum_{n=1}^{\infty} \frac{1}{2\varepsilon n} \mathbf{1}\{x + 2b + (n-1)2(b+x) \le vt < x + 2b + 2\varepsilon + (n-1)2(x+b+\varepsilon)\}.$$

Let us explain this formula. Here, n represents the number of times that a device started at x reaches its destination, which is uniformly distributed in B. With every reflection in B, the density is distributed to an interval of length $2\varepsilon n$, where ε is the length of B, again uniformly at random. Now, $x + 2b$, respectively $x + 2b + 2\varepsilon$, are the shortest, respectively the longest, distance that a device started at x travels before it revisits the crossing c for the first time. For each consecutive visit to the origin an additional length of $2(b+x)$, respectively $2(x+b+\varepsilon)$, has to be added as the length of a complete cycle from the origin to its destination and back.

In other words, each indicator becomes non-zero in $x + 2b + (n-1)2(x+b)$ and adds a mass of $(2\varepsilon n)^{-1}$ over a length of $2n\varepsilon$. For $n < (x+b)/\varepsilon$ the indicator functions are disjoined, i.e. for $vt < x + 2b + ((x+b)/\varepsilon - 1)2(x+b)$ there is at most one indicator function that is non-zero. Now, for $n > (x+b)/\varepsilon$, the interval for vt in which the indicator function is non-negative is longer than the minimal cycle length. In other words, it is possible that at least two indicator functions are non-negative.

Now, for $(x+b)/\varepsilon < n < 2(x+b)/\varepsilon$ the indicator functions contribute for more than the length of a full cycle $2(x+b)$ and less than twice that length. Therefore, for each $vt \in [x + 2b + (n_1 - 1)2(b+x), x + 2b + (n_2 - 1)2(b+x)]$ there are either exactly one or two non-negative indicators, where $n_i = \lceil i(x+b)/\varepsilon \rceil$. As each indicator has a weight of at least $(4\varepsilon(x+b)/\varepsilon)^{-1}$ we can bound $\phi_x(t) > (4\varepsilon(x+b)/\varepsilon)^{-1}$.

More generally, in the interval $vt \in [x + 2b + (n_m - 1)2(b+x), x + 2b + (n_{m+1} - 1)2(b+x)]$ there are either m or $m+1$ overlapping indicator functions whose contributions can each be bounded by $(2n_{m+1}\varepsilon)^{-1}$. Therefore, we can bound

$$\phi_x(t) > \frac{m}{2(n_{m+1})\varepsilon} = \frac{m}{2\varepsilon\lceil(m+1)(x+b)/\varepsilon\rceil} \ge \frac{1}{2\varepsilon\lceil 2(x+b)/\varepsilon\rceil} \ge \frac{1}{4(2b+\varepsilon)} = C > 0,$$

where we used that the bound monotonically increases in m and $x \le b + \varepsilon$. Therefore, for $T_0 = (x + 2b + (n_1 - 1)2(b+x))v^{-1}$ the desired bound is realized and integration of (3) yields

$$p_S(s_1, s_2) \ge \varepsilon(2v)^{-1}(8b + 4\varepsilon)^{-1} \int_A dx\, \eta_x = \tilde{p}_S(s_1, s_2) > 0.$$

Noting that the requirements in Conditions (C4) and (C5) can be considered independently, respectively with probability $p_4 = \varrho_{\mathrm{I}}([\varrho_{\mathrm{I,min}}, \varrho_{\mathrm{I,min}} + \varepsilon/2])$ and $p_5 = \varrho_{\mathrm{I}}([\varrho_{\mathrm{I,min}}, \varrho_{\mathrm{I,min}} + \varepsilon])$, the number of devices that satisfy the criterion of a bridging device X^{s_1,s_2} can be bounded from

below by a Poisson-distributed random variable with expectation $\tilde{p}_S(s_1, s_2)p_4p_5$. As c is open if there is at least one bridging device for each combination of open adjacent streets, we can bound

$$\mathbb{P}(c \text{ is open at time } t \mid S) \geq \prod_{\substack{i \neq j, s_i, s_j \text{ open} \\ c \in s_i, c \in s_j}} \left(1 - \exp(-\lambda \tilde{p}(s_i, s_j)p_4p_5)\right) \quad \text{for all } t \geq T,$$

which proves the lemma. $\qquad\square$

4.3. Proof of Theorem 2

For the proof we adopt a multiscale argument in the spirit of the proof of existence of subcritical regimes for the Poisson–Boolean model with random radii in [19]. We note that this approach has also been applied in the setting of Cox–Boolean models with random radii in [29]. More specifically, we adapt the arguments for absence of percolation in the connectivity graph for sufficiently large velocities presented in [11]. Roughly speaking, we prove that a Cox–Boolean model in which any two devices that could transmit the infection somewhere on their path and are sufficiently close are connected, becomes subcritical if the white knight intensity is sufficiently large.

First, we call a street $s = (c_1, c_2)$ *blocked for (device) x* if either $|s| < v_{\min}\varrho_{\mathrm{I,min}}/2$ or there exist two white knights $Y_1, Y_2 \in Y^{\lambda_W}$ such that for $i \in \{1, 2\}$ we have $\varrho_{\mathrm{I,min}} > \rho_{\mathrm{W}}(Y_i, x)$, $Y_{i,t} \in s$, and $|c_i - Y_{i,t}| < r$ for all $t \geq 0$. Otherwise we call s *unblocked for x*. Then, we consider the thinned process

$$X^{\lambda,\mathrm{th}} = \Big\{X_i \in X^\lambda : \text{there exists } y \in \mathrm{supp}\big(\kappa^S(X_i, \mathrm{d}y)\big)$$
$$\text{and } s \in \ell_S(X_i, y) \text{ such that } s \text{ is unblocked for } X_i\Big\}.$$

In words, for devices in $X^\lambda \setminus X^{\lambda,\mathrm{th}}$ every possible path only contains streets that are either too short to perform any transmission of the infection or contain a white knight that patches the device before it can transmit the infection on the street. Let us note that here we use that $r > v_{\max}\varrho_{\mathrm{W,min}}$ is large enough to allow a successful patch. Using these definitions, we note that, if the infection survives globally, then this implies that the geostatistical Cox–Boolean model $\mathcal{C} = \bigcup_{X_i \in X^{\lambda,\mathrm{th}}} B_{\ell_S(X_i)}(X_i)$ contains an unbounded component. Hence, in order to prove absence of global survival, it suffices to prove the absence of percolation in this Cox–Boolean model, which is also well-defined by the local connectedness assumption. Let $\mathcal{C}_x(V)$ denote the connected component containing x of the geostatistical Cox–Boolean model, based on points in $V \subset \mathbb{R}^2$. Then, we define, for any $x \in \mathbb{R}^2$ and $\alpha > 0$, the event

$$G(x, \alpha) = \big\{\mathcal{C}_x(B_{10\alpha}(x)) \not\subset B_{8\alpha}(x)\big\}$$

that the cluster containing x, formed by disks whose centers lie in $B_{10\alpha}(x)$, reaches beyond $B_{8\alpha}(x)$. Consider the set of α-local points $A_S(\alpha) = \{x \in S : \ell_S(x) < \alpha\}$, let S^* denote the Palm version of S, and recall the stabilization radii R_y of the street system. Recall that $R(V) = \sup_{y \in V \cap \mathbb{Q}^2} R_y$ for any $V \subset \mathbb{R}^2$. Then, we can employ the following key lemma that establishes a scaling relation in the model.

Lemma 2. ([11, Lemma III.1].) *Consider the geostatistical Cox–Boolean model. There exists a constant $c > 0$ such that, for all $\alpha > 0$,*

$$\mathbb{P}(G(o, 10\alpha)) \leq c\mathbb{P}(G(o, \alpha))^2 + c\alpha^2\mathbb{P}\big(o \in A^{\mathrm{c}}_{S^*}(\alpha)\big) + c\mathbb{P}(R(Q_{10\alpha}) \geq \alpha).$$

In the next lemma we deviate from [11, Proof of Theorem II.3]. We show that the local percolation probability becomes zero for large white knight intensities.

Lemma 3. *There exists $c > 0$ such that*

$$\mathbb{P}(G(o, \alpha)) \le c\alpha^2 \mathbb{P}\big(\text{there exists } y \in \text{supp}\big(\kappa^{S^*}(o, \mathrm{d}y)\big)$$

$$\text{and at least one unblocked } s \in \ell_{S^*}(o, y)\big),$$

and $\mathbb{P}\big(\text{there exists } y \in \text{supp}\big(\kappa^{S^*}(o, \mathrm{d}y)\big)$ *and at least one unblocked* $s \in \ell_{S^*}(o, y)\big)$ *tends to zero as* λ_W *tends to infinity.*

Proof of Lemma 3. Note that

$$\mathbb{P}(G(o, \alpha)) \le \mathbb{P}\big(X^{\lambda,\text{th}}(B_{10\alpha}) > 0\big) \le \mathbb{E}\big[X^{\lambda,\text{th}}(B_{10\alpha})\big]$$

$$\le \lambda \mathbb{E}\bigg[\int_{S \cap B_{10\alpha}} \mathrm{d}x\, \mathbb{P}\big(\text{there exists } y \in \text{supp}\big(\kappa^S(x, \mathrm{d}y)\big) \text{ and } s \in \ell_S(x, y)$$

$$\text{such that } s \text{ is unblocked for } x\big)\bigg]$$

$$\le \lambda 10^2 \alpha^2 \mathbb{P}\big(\text{there exists } y \in \text{supp}\big(\kappa^{S^*}(o, \mathrm{d}y)\big) \text{ and } s \in \ell_{S^*}(o, y)$$

$$\text{such that } s \text{ is unblocked for } o\big).$$

But, by dominated convergence,

$$\mathbb{P}\big(\text{there exists } y \in \text{supp}\big(\kappa^{S^*}(o, \mathrm{d}y)\big) \text{ and } s \in \ell_{S^*}(o, y) \text{ such that } s \text{ is unblocked for } o\big)$$

tends to zero as λ_W tends to infinity, and hence we arrive at the desired result. □

As a final input for the proof, we recall the following essential result about convergence properties of functions satisfying some scaling inequality.

Lemma 4. ([19, Lemma 3.7].) *Let f and g be two bounded measurable functions from $[1, \infty]$ to $[0, \infty)$. Additionally, let f be bounded by $1/2$ on $[1, 10]$, g be bounded by $1/4$ on $[1, \infty]$, and assume that $f(\alpha) \le f(\alpha/10)^2 + g(\alpha)$ for all $\alpha \ge 10$. Then, $\lim_{\alpha \uparrow \infty} g(\alpha) = 0$ implies that $\lim_{\alpha \uparrow \infty} f(\alpha) = 0$.*

Now we are in the position to prove the theorem.

Proof of Theorem 2. Recall that S is assumed to be a PVT or PDT. In order to prove that $\lambda_{W,c} < \infty$, it suffices to show that $\lim_{\alpha \uparrow \infty} \mathbb{P}\big(X^{\lambda,\text{th}}(\mathcal{C}_o(\mathbb{R}^2)) > X^{\lambda,\text{th}}(B_{8\alpha})\big) = 0$ for all sufficiently large λ_W. But this is true if $\lim_{\alpha \uparrow \infty} \mathbb{P}(G(0, \alpha)) = 0$ since, by [11][Lemma III.4 and III.5],

$$\mathbb{P}\big(X^{\lambda,\text{th}}\big(\mathcal{C}_o(\mathbb{R}^2)\big) > X^{\lambda,\text{th}}(B_{8\alpha})\big) \le \mathbb{P}(G(o, \alpha)) + \lambda \mathbb{E}\bigg[\int_S \mathrm{d}x\, \mathbf{1}\big\{10\alpha \le |x| \le 9\alpha + \ell_S(x)\big\}\bigg],$$

where $\mathbb{E}\big[\int_S \mathrm{d}x\, \mathbf{1}\{10\alpha \le |x| \le 9\alpha + \ell_S(x)\}\big]$ tends to zero as α tends to infinity. Now, in order to show that $\lim_{\alpha \uparrow \infty} \mathbb{P}(G(0, \alpha)) = 0$ we apply Lemmas 2, 3, and 4 for proper choices of f and g.

For this, note that, in Lemma 2, since we assume stabilization, $\mathbb{P}(R(Q_{10\alpha}) \ge \alpha)$ tends to zero as α tends to infinity, and the same is true for $\alpha^2 \mathbb{P}(o \in A^c_{S^*}(\alpha))$ by [11, Lemma III.2]. Hence, we can define

$$\alpha_1 := \inf \left\{ s \geq 1 : \phi_1(\alpha) = \alpha^2 \mathbb{P}(o \in A_{S^*}^c) < (8c^2)^{-1} \text{ for all } \alpha \geq s \right\},$$

$$\alpha_2 := \inf \left\{ s \geq 1 : \phi_2(\alpha) = \mathbb{P}(R(Q_{10\alpha}) \geq \alpha) < (8c^2)^{-1} \text{ for all } \alpha \geq s \right\},$$

set $\alpha_c = \alpha_1 \vee \alpha_2$, and note that $\alpha_c < \infty$. Further, we let

$$\lambda_{W,c} = \inf \Big\{ \lambda_W \geq 1 : \mathbb{P}\big(\text{there exists } y \in \mathrm{supp}\big(\kappa^{S^*}(o, \mathrm{d}y)\big)$$

$$\text{and at least one unblocked } s \in \ell_{S^*}(o, y)\big) < \tfrac{1}{2}(100c\alpha_c)^{-2} \Big\},$$

where $\lambda_{W,c} < \infty$ by Lemma 3. Then, we write $f(\alpha) = c\mathbb{P}(G(o, 10\alpha_c\alpha))$ and $g(\alpha) = c^2(\phi_1(\alpha_c\alpha) + \phi_2(\alpha_c\alpha))$ and hence, again using Lemma 3, we have $f(\alpha) \leq \frac{1}{2}$ for all $1 \leq \alpha \leq 10$ and $\lambda_W > \lambda_{W,c}$, and, by definition, $g(\alpha) \leq \frac{1}{4}$ for all $1 \leq \alpha$. Finally, using Lemma 2,

$$f(\alpha) \leq c^2 (\mathbb{P}(G(o, \alpha_c\alpha))^2 + \phi_1(\alpha_c\alpha) + \phi_2(\alpha_c\alpha)) = f(\alpha/10)^2 + g(\alpha) \quad \text{for all } \alpha \geq 10.$$

Hence, since $\lim_{\alpha \uparrow \infty} g(\alpha) = 0$, an application of Lemma 4 gives the result. $\square$

4.4. Proof of Theorem 3

We prove the theorem in two parts.

Proof of Theorem 3 parts (ii) and (iii). Note that part (ii) is the statement of Theorem 1 which puts no restrictions on the white knight intensity. Further, part (iii) follows from [11][Theorem 3.3] where a subcritical percolation regime is established for sufficiently fast speeds, independent of the device intensities. This in particular implies that global survival is possible only with probability 0. $\square$

Hence, it only remains to prove part (i). The main idea is that small velocities make it impossible to transmit the infection through crossings if sufficiently many white knights are available. We say that a street $s = (c_1, c_2)$ is *closed* if:

(A) $|s| > 2v\rho_W$;

(B) there exist white knights $Y^s \subset Y^{\lambda_W}$ such that, for all $t \geq 0$ and $x \in s$, $|x - Y_{i,t}| < r/2$ for some $Y_i \in Y^s$; and

(C) if, at time $t \geq 0$, there exists an infected $X_i \in X^\lambda$ with $X_{i,t} = c_1$ (respectively $X_{i,t} = c_2$), then $\{X_j \in X^\lambda \cap s : |X_{j,u} - c_2| \leq v\rho_W \text{ for some } u \in \tau(t)\} = \emptyset$ (respectively $\{X_j \in X^\lambda \cap s : |X_{j,u} - c_1| \leq v\rho_W \text{ for some } u \in \tau(t)\} = \emptyset$). Here, $\tau(t)$ is the time interval in which the infection reaches $s \cap B_{v\rho_W + r}(c_2)$ (respectively $s \cap B_{v\rho_W + r}(c_1)$).

A street is called *open* if it is not closed. The key idea is that a closed street cannot be used for the transmission of the infection. Indeed, Condition (A) guarantees that the street is sufficiently long to allow the white knights to patch, and the probability that the condition is satisfied tends to 1 as v tends to 0. Further, Condition (B) guarantees the existence of white knights at all times close to every point on the street, and in particular close to the crossings. These white knights help to patch any incoming infected device, and the condition becomes likely to be satisfied for large white knight intensities. Now, for Condition (C), note first that a large device intensity λ makes it probable that an infection travels through a street unpatched since we assume that $\rho_W > \rho_I$. Also, due to the continuous movement of the devices, it is unavoidable that an infected device enters the street at some time. However, in order for an

infected device to leave the street unpatched, it needs to receive the infection at the opposite end of the street, sufficiently close to the crossing, i.e. in $s \cap B_{v\rho_W+r}(c_2)$ (respectively $s \cap B_{v\rho_W+r}(c_1)$). Only in this case can it exit the street unpatched from the blocking white knight that is positioned there. Very roughly, then, again for small v, Condition (C) becomes likely.

What makes the probability that Condition (C) is satisfied challenging to control is the fact that, in principle, a street at any given time could host devices from distant spatial areas. In order to cope with this we again employ a multi-scale argument very similar to the one presented in Section 4.3. Using the definitions for ℓ_S from there, we now consider the thinned point cloud

$$X^{\lambda,\text{th}} = \left\{ X_i \in X^\lambda : \text{there exists } y \in \text{supp}\big(\kappa^S(X_i, \text{d}y)\big) \text{ and } s \in \ell_S(X_i, y) \text{ such that } s \text{ is open} \right\}.$$

Again, the idea is that any device that is not in $X^{\lambda,\text{th}}$ only finds closed streets on any possible path and hence cannot contribute to the spread of the infection. Thus, it suffices to show subcriticality in the associated geostatistical Cox–Boolean model $\mathcal{C} = \bigcup_{X_i \in X^{\lambda,\text{th}}} B_{\ell_S(X_i)}(X_i)$. However, the situation is more complicated compared to the proof of Theorem 2 since the thinning procedure depends on the entire point cloud. Still, we can employ Lemmas 2 and 4, and only need to replace Lemma 3 with the following, which is presented in terms of $\mathbb{P}^*$, the Palm version of $\mathbb{P}$.

Lemma 5. *There exists $c > 0$ such that*

$$\mathbb{P}(G(o, \alpha)) \leq c\alpha^2 \mathbb{P}^*\big(\text{there exists } y \in \text{supp}\big(\kappa^S(o, \text{d}y)\big) \text{ and } s \in \ell_S(o, y) \text{ such that } s \text{ is open}\big)$$

and

$$\limsup_{\lambda_W \uparrow \infty} \limsup_{v \downarrow 0} \mathbb{P}^*\big(\text{there exists } y \in \text{supp}\big(\kappa^S(o, \text{d}y)\big) \text{ and } s \in \ell_S(o, y) \text{ such that } s \text{ is open}\big) = 0.$$

Proof of Lemma 5. First note that we can use the Slivnyak–Mecke formula and translation invariance to estimate

$$\begin{aligned}
\mathbb{P}(G(o, \alpha)) &\leq \mathbb{P}\big(X^{\lambda,\text{th}}(B_{10\alpha}) > 0\big) \leq \mathbb{E}\big[X^{\lambda,\text{th}}(B_{10\alpha})\big] \\
&\leq \lambda 10^2 \alpha^2 \mathbb{P}^*\big(\text{there exists } y \in \text{supp}\big(\kappa^S(o, \text{d}y)\big) \text{ and } s \in \ell_S(o, y) \text{ such that } s \text{ is open}\big).
\end{aligned}$$

We highlight here that, differently to Lemma 3, the probability also extends to the devices X^λ since openness is defined not only with respect to white knights and the street system.

In order to show the second statement of the lemma, we employ stabilization based on the stabilizing random field $\{R_x\}_{x \in \mathbb{R}^2}$. For all sufficiently large n, we have, for $\varepsilon' > 0$,

$$\mathbb{P}^*\big(\text{there exists } y \in \text{supp}\big(\kappa^S(o, \text{d}y)\big) \text{ and } s \in \ell_S(o, y) \text{ such that } s \text{ is open}\big)$$

$$\leq \mathbb{P}^*\big(R(Q_{2n}) < n/2, \text{ there exists } y \in \text{supp}\big(\kappa^S(o, \text{d}y)\big) \text{ and } s \in \ell_S(o, y) \text{ such that } s \text{ is open}\big) + \varepsilon'$$

$$\leq \mathbb{P}^*(\text{there exists an open } s \in Q_{2n}) + \varepsilon',$$

where we used that $\mathbb{P}(R(Q_{2n}) \geq n/2)$ tends to zero as n tends to infinity, as well as the fact that under the event $R(Q_{2n}) < n/2$ any shortest path starting in o must be contained in Q_{2n}. Now, by the Markov inequality,

$$\mathbb{P}^*(\text{there exists an open } s \in Q_{2n}) \leq \mathbb{E}^*\left[\sum_{s \in S \cap Q_{2n}} \mathbb{P}^*(s \text{ is open} \mid S) \right]$$

and, using dominated convergence and the fact that $\mathbb{E}^*[\#\{s \in S \cap Q_{2n}\}] < \infty$ for PVT and PDT, it suffices to show that, for all $s \in S \cap Q_{2n}$, $\limsup_{\lambda_W \uparrow \infty} \limsup_{v \downarrow 0} \mathbb{P}^*(s \text{ is open} \mid S) = 0$ almost surely with respect to S. In view of the definition of closed streets, let $C(v)$ denote the event that Condition (A) is satisfied for s, and similarly $D(\lambda_W)$ and $E(v, \lambda, \lambda_W)$ for the events associated with Conditions (B) and (C). Then,

$$\mathbb{P}^*(s \text{ is open} \mid S) \leq \mathbb{P}(C^c(v) \mid S) + \mathbb{P}(D^c(\lambda_W) \mid S) + \mathbb{P}(E^c(v, \lambda, \lambda_W) \mid S),$$

where the first and third summands tend to zero as v tends to zero, and then the second summand also tends to zero as λ_W tends to infinity. This completes the proof. $\qquad\square$

Proof of Theorem 3 *part (i).* For the proof we can follow the same line of argument as used for the proof of Theorem 2, replacing Lemma 3 with Lemma 5 and adjusting the subsequent arguments. We need to prove existence of $\lambda_{W,c} < \infty$ such that, for all $\lambda_W > \lambda_{W,c}$, there exists $v_c(\lambda, \lambda_W)$ such that $\lim_{\alpha \uparrow \infty} \mathbb{P}(X^{\lambda,\text{th}}(\mathcal{C}_o(\mathbb{R}^2)) > X^{\lambda,\text{th}}(B_{8\alpha})) = 0$ for all $v < v_c(\lambda, \lambda_W)$. Referencing [11][Lemmas III.2, III.4, and III.5], as well as Lemma 2 and the initial argument presented in Theorem 2, it suffices to note that, with the help of Lemma 5, there exists $\lambda_{W,c} < \infty$ such that, for all $\lambda_W > \lambda_{W,c}$, there exists $v_c(\lambda, \lambda_W)$ such that, for all $v < v_c(\lambda, \lambda_W)$,

$$\mathbb{P}^*\big(\text{there exists } y \in \text{supp}\big(\kappa^S(o, dy)\big) \text{ and } s \in \ell_S(o, y) \text{ such that } s \text{ is open}\big) < \tfrac{1}{2}(100c\alpha_c)^{-2},$$

where α_c is defined as in the proof of Theorem 2. Then, using the same definitions for f and g, the result follows by an application of Lemma 4. $\qquad\square$

4.5. Proof of Proposition 2

Proof of Proposition 2. First note that, by the assumption of supercriticality, there is a positive probability that the typical device is connected to infinity within $g_{\rho,r}(S, X^\lambda)$ via at least one path of susceptible devices, i.e. in the absence of white knights. However, in the case that $T_I \leq T_W$, any white knight that is connected to the path can only eliminate the infection after the infected device has already passed the infection to the subsequent susceptible device, leading to an unstoppable sequence of infections along the path, independent of the choice of λ_W.

On the other hand, in the case $T_I > T_W$, it suffices to consider the model conditioned on the event $\{X_o \longleftrightarrow \infty\}$ that X_o is in the infinite cluster of $g_{\rho,r}(S, X^\lambda)$. Let us denote by D the graph distance of X_o to the set of white knights, and note that $D < \infty$ almost surely as $\lambda_W > 0$. Then, the closest white knight Y_i experiences an infection attempt at time DT_I. Note that, at this time, any infected device has a distance to Y_i given by at most $2D$. This comes from the fact that the transmission times are deterministic and there is a one-to-one correspondence between the graph distance to the typical device and elapsed time. However, there exists $N \in \mathbb{N}$ such that $NT_I > (N + 2D)T_W$, and therefore the infection can reach at most a graph distance of $N + D$. Indeed, any device X_i at graph distance $N + D$ from X_o is in contact with Y_i via $N + 2D$ edges connecting infected or patched devices. But, since the patching is faster than the infection, for this N, on the joint path between X_o, Y_i, and X_i, the patch will have caught up with the infection before time $DT_I + (2D + N)T_W$, which is strictly smaller than $(N + D)T_I$. $\qquad\square$

4.6. Proof of Theorem 4

Proof of Theorem 4. As $\varrho_{W,\min} > 0$, we can choose b large enough that $\varrho_{I,\min}^b < \varrho_{W,\min}$. Furthermore, we can choose μ' with $v'_{\min}$ such that $v_{\min}\rho < v'_{\min}\varrho_{I,\min}^b < \min(a_c/2, r, c/2)$.

Now $v'_{\min}$ and $\varrho^b_{\mathrm{I},\min}$ satisfy the conditions of Theorem 1. A close inspection of the proof of Theorem 1 reveals that, under the conditions in this theorem, there exists an infinite sequence of crossing devices, boundary devices, and devices connecting boundary devices $\{X_{n_i}\}_{i\in\mathbb{N}}$ that connect the typical device to infinity in the dynamical model considered in Theorem 1. Let us argue that this given sequence is also connected in $g_{\rho,r}(S, X^\lambda)$. Indeed, as the chain of devices between boundary devices is strongly localized, they are connected for all $t \geq 0$, especially for a continuous time window of length ρ. Now, the crossing devices are localized in such a way that they travel at least a distance of $v'_{\min}\varrho^b_{\mathrm{I},\min}$ on their respective streets. As $v_{\min}\rho < v'_{\min}\varrho^b_{\mathrm{I},\min}$ the bridging devices can connect to their respective boundary devices. Therefore, the infection can propagate in $g_{\rho,r}(S, X^\lambda)$ along the given sequence to infinity, which completes the proof. $\qquad\square$

4.7. Proof of Theorem 5

Proof of Theorem 5. Using similar arguments to the analysis of chase–escape dynamics on the Gilbert graph in [23], we call a device $X_i \in V^{M,p}$ *good* if, in the case of an infection, the infection is transmitted to all neighbors of X_i before any neighbor of X_i is able to patch it. If there is an infinite path of *good* devices, every device on this path is guaranteed to become infected and hence, if there exists an infinite component of good devices with positive probability, this implies survival of the infection also with positive probability. Now, if a device X_i has at most M neighbors, the probability of being good is bounded by

$$\mathbb{P}\left(\min_{X_j\in N(X_i)} \rho_{\mathrm{W}}(X_j, X_i) > \max_{X_j\in X^\lambda\cap N(X_i)} \rho_{\mathrm{I}}(X_i, X_j)\right)$$

$$\geq \mathbb{P}\left(\min_{i\in\{1,\ldots,M\}} \rho_{\mathrm{W},i} > \max_{i\in\{1,\ldots,M\}} \rho^b_{\mathrm{I},i}\right) = \mathbb{P}\left(\min_{i\in\{1,\ldots,M\}} \rho_{\mathrm{W},i} > b^{-1}\max_{i\in\{1,\ldots,M\}} \rho_{\mathrm{I},i}\right), \quad (4)$$

independently of X_i, where the $\rho^b_{\mathrm{I},i}$, and respectively the $\rho_{\mathrm{W},i}$, are i.i.d. random variables with distribution ϱ^b_{I} and ϱ_{W}.

Now, by assumption, we can choose M and p such that $g^{M,p}_{\rho,r}(S, X^\lambda)$ percolates with positive probability. Hence, as ϱ_{W} had no atom at zero we can choose $b = b(M)$ sufficiently large such that the bound in (4) is larger than p.

Note that the bound is uniform over the devices, since the infection and patch times are independent by definition, and therefore we can couple the connectivity graph to an independent thinning with parameter p that dominates the process of good devices. As a consequence, $g^{M,p}_{\rho,r}(S, X^\lambda)$ is a subgraph of the thinned graph of good devices with at most M neighbors. Hence, we have established percolation of good devices with positive probability as desired. $\qquad\square$

4.8. Proof of Theorem 6

Proof of Theorem 6. Our strategy is again to employ the arguments from Sections 4.3 and 4.4. That is, we realize that it suffices to show absence of percolation in the geostatistical Cox–Boolean model $\mathcal{C} = \bigcup_{X_i\in X^{\lambda,\mathrm{th}}} B_{\ell_S(X_i)}(X_i)$, where

$$X^{\lambda,\mathrm{th}} = \left\{X_i \in X^\lambda : \min_{X_j\in N_\lambda(X_i)} \rho_{\mathrm{I}}(X_i, X_j) < \min_{X_j\in N_{\lambda_{\mathrm{W}}}(X_i)} \rho_{\mathrm{W}}(X_j, X_i)\right\},$$

with $N_\lambda(X_i)$ the set of neighbors of X_i in $g_{\rho,r}(S, X^\lambda)$ and $N_{\lambda_W}(X_i)$ the set of neighbors of X_i in $g_{\rho,r}(S, X_i \cup Y^{\lambda_W})$. Indeed, devices in $X^\lambda \setminus X^{\lambda,\mathrm{th}}$ are patched before they can transmit the infection and can therefore not contribute to the dissipation of the infection.

Again, the thinning procedure depends on the entire point cloud, but we can still employ Lemmas 2 and 4 and only need to replace Lemma 3 by the following, which is presented in terms of $\mathbb{P}^*$, the Palm version of $\mathbb{P}$.

Lemma 6. *There exists $c > 0$ such that*

$$\mathbb{P}(G(o, \alpha)) \le c\alpha^2 \mathbb{P}^*\left(\min_{X_j \in N_\lambda(o)} \rho_I(o, X_j) < \min_{X_j \in N_{\lambda_W}(o)} \rho_W(X_j, o)\right)$$

and

$$\limsup_{\lambda_W \uparrow \infty} \mathbb{P}^*\left(\min_{X_j \in N_\lambda(o)} \rho_I(o, X_j) < \min_{X_j \in N_{\lambda_W}(o)} \rho_W(X_j, o)\right) = 0.$$

Proof of Lemma 6. Again, by the Slivnyak–Mecke formula and translation invariance we can estimate

$$\mathbb{P}(G(o, \alpha)) \le \mathbb{P}(X^{\lambda,\mathrm{th}}(B_{10\alpha}) > 0) \le \mathbb{E}[X^{\lambda,\mathrm{th}}(B_{10\alpha})]$$

$$\le \lambda 10^2 \alpha^2 \mathbb{P}^*\left(\min_{X_j \in N_\lambda(o)} \rho_I(o, X_j) < \min_{X_j \in N_{\lambda_W}(o)} \rho_W(X_j, o)\right).$$

Now,

$$\mathbb{P}^*\left(\min_{X_j \in N_\lambda(o)} \rho_I(o, X_j) < \min_{X_j \in N_{\lambda_W}(o)} \rho_W(X_j, o)\right) \le \mathbb{E}^*\left[\mathbb{P}(\rho_W > \varrho_{I,\min})^{N_{\lambda_W}(o)}\right], \quad (5)$$

where ρ_W is a random variable distributed according to ϱ_W. Note that $\mathbb{P}(\rho_W > \varrho_{I,\min}) < 1$ by our assumptions on $\varrho_{W,\min}$. Further, the expression on the right-hand side of (5) can be further bounded from above by considering only those white knights that mimic the behavior of the typical device at the origin in the sense that they start close to the origin and have a target location close to the target location of the typical device. Then, by dominated convergence, the right-hand side of (5) tends to zero as λ_W tends to infinity, and this completes the proof. □

As the lemma is now proved, this concludes the proof of Theorem 6. □

Proof of Theorem 3 part (i). We can follow the same line of argument as used for the proof of Theorem 2, replacing Lemma 3 by Lemma 5 and adjusting the subsequent arguments. We need to prove the existence of $\lambda_{W,c} < \infty$ such that, for all $\lambda_W > \lambda_{W,c}$, there exists $v_c(\lambda, \lambda_W)$ such that $\lim_{\alpha \uparrow \infty} \mathbb{P}(X^{\lambda,\mathrm{th}}(C_o(\mathbb{R}^2)) > X^{\lambda,\mathrm{th}}(B_{8\alpha})) = 0$ for all $v < v_c(\lambda, \lambda_W)$. Referencing [11][Lemmas III.2, III.4, and III.5], as well as Lemma 2 and the initial argument presented in Theorem 2, it suffices to note that, with the help of Lemma 5, there exists $\lambda_{W,c} < \infty$ such that, for all $\lambda_W > \lambda_{W,c}$, there exists $v_c(\lambda, \lambda_W)$ such that, for all $v < v_c(\lambda, \lambda_W)$,

$$\mathbb{P}^*\left(\text{there exists } y \in \mathrm{supp}(\kappa^S(o, \mathrm{d}y)) \text{ and } s \in \ell_S(o, y) \text{ such that } s \text{ is open}\right) < \tfrac{1}{2}(100c\alpha_c)^{-2},$$

where α_c is defined as in the proof of Theorem 2. Then, using the same definitions for f and g, the result follows by an application of Lemma 4. □

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References

[1] AUFFINGER, A., DAMRON, M. AND HANSON, J. (2017). *50 Years of First-Passage Percolation* (University Lect. Ser. **68**). American Mathematical Society, Providence, RI.

[2] BACCELLI, F. AND BŁASZCZYSZYN, B. (2010). *Stochastic Geometry and Wireless Networks: Volume I Theory*. Now Publishers Inc., Delft.

[3] BACCELLI, F. AND BŁASZCZYSZYN, B. (2010). *Stochastic Geometry and Wireless Networks: Volume II Application*. Now Publishers Inc., Delft.

[4] BECKMAN, E., COOK, K., EIKMEIER, N., HERNANDEZ-TORRES, S. AND JUNGE, M. (2021). Chase–escape with death on trees. *Ann. Prob.* **49**, 2530–2547.

[5] BENOMAR, Z., GHRIBI, C., CALI, E., HINSEN, A. AND JAHNEL, B. (2022). Agent-based modeling and simulation for malware spreading in D2D networks. In *Proc. 21st Int. Conf. Autonomous Agents and Multiagent Systems*. International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, pp. 91–99.

[6] BENOMAR, Z., GHRIBI, C., CALI, E., HINSEN, A., JAHNEL, B. AND WARY, J.-P. (2022). Multi-agent simulations for virus propagation in D2D 5G+ networks. WIAS preprint 2953.

[7] BERNSTEIN, E., HAMBLEN, C., JUNGE, M. AND REEVES, L. (2022). Chase–escape on the configuration model. *Electron. Commun. Prob.* **27**, 1–14.

[8] BETTSTETTER, C., HARTENSTEIN, H. AND PÉREZ-COSTA, X. (2004). Stochastic properties of the random waypoint mobility model. *Wireless Networks* **10**, 555–567.

[9] BHAMIDI, S., NAM, D., NGUYEN, O. AND SLY, A. (2021). Survival and extinction of epidemics on random graphs with general degree. *Ann. Prob.* **49**, 244–286.

[10] BŁASZCZYSZYN, B., HAENGGI, M., KEELER, P. AND MUKHERJEE, S. (2018). *Stochastic Geometry Analysis of Cellular Networks*. Cambridge University Press.

[11] CALI, E., HINSEN, A., JAHNEL, B. AND WARY, J.-P. (2022). Connectivity in mobile device-to-device networks in urban environments. *IEEE Trans. Inf. Theory*, doi: 10.1109/TIT.2023.3298278.

[12] COLETTI, C. F., DE LIMA, L. R., HINSEN, A., JAHNEL, B. AND VALESIN, D. (2021). Limiting shape for first-passage percolation models on random geometric graphs. *J. Appl. Prob.*, doi: 10.1017/jpr.2023.5.

[13] COURTAT, T. (2012). Promenade dans les cartes de villes-phénoménologie mathématique et physique de la ville-une approche géométrique. PhD thesis, Université Paris-Diderot.

[14] DOUSSE, O., FRANCESCHETTI, M., MACRIS, N., MEESTER, R. AND THIRAN, P. (2006). Percolation in the signal to interference ratio graph. *J. Appl. Prob.* **43**, 552–562.

[15] DURRETT, R., JUNGE, M. AND TANG, S. (2020). Coexistence in chase–escape. *Electron. Commun. Prob.* **25**, 1–14.

[16] FRANCESCHETTI, M. AND MEESTER, R. (2008). *Random Networks for Communication*. Cambridge University Press.

[17] GALL, Q., BŁASZCZYSZYN, B., CALI, E. AND EN-NAJJARY, T. (2021). Continuum line-of-sight percolation on Poisson–Voronoi tessellations. *Adv. Appl. Prob.* **53**, 510–536.

[18] GILBERT, E. N. (1961). Random plane networks. *J. Soc. Indust. Appl. Math.* **9**, 533–543.

[19] GOUÉRÉ, J.-B. (2008). Subcritical regimes in the Poisson Boolean model of continuum percolation. *Ann. Prob.* **36**, 1209–1220.

[20] HAENGGI, M. (2012). *Stochastic Geometry for Wireless Networks*. Cambridge University Press.

[21] HERNANDEZ-TORRES, S., JUNGE, M., RAY, N. AND RAY, N. (2022). Distance-dependent chase–escape on trees. Preprint, arXiv:2209.09876.

[22] HINSEN, A., JAHNEL, B., CALI, E. AND WARY, J.-P. (2020). Malware propagation in urban D2D networks. In *Proc. 18th Int. Symp. Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOPT)*. IEEE, Piscataway, NY.

[23] HINSEN, A., JAHNEL, B., CALI, E. AND WARY, J.-P. (2020). Phase transitions for chase–escape models on Poisson–Gilbert graphs. *Electron. Commun. Prob.* **25**, 1–14.

[24] HIRSCH, C., JAHNEL, B. AND CALI, E. (2019). Continuum percolation for Cox point processes. *Stoch. Process. Appl.* **129**, 3941–3966.

[25] HIRSCH, C., JAHNEL, B. AND MUIRHEAD, S. (2022). Sharp phase transition for Cox percolation. *Electron. Commun. Prob.* **27**, 1–13.

[26] JAHNEL, B. AND KÖNIG, W. (2020). *Probabilistic Methods in Telecommunications*. Birkhäuser, Basel.

[27] JAHNEL, B. AND TÓBIÁS, A. (2020). Exponential moments for planar tessellations. *J. Statist. Phys.* **179**, 90–109.

[28] JAHNEL, B. AND TÓBIÁS, A. (2022). SINR percolation for Cox point processes with random powers. *Adv. Appl. Prob.* **54**, 227–253.

[29] JAHNEL, B., TÓBIÁS, A. AND CALI, E. (2022). Phase transitions for the Boolean model of continuum percolation for Cox point processes. *Brazilian J. Prob. Statist.* **36**, 20–44.

[30] KESTEN, H. (2003). First-passage percolation. In *From Classical to Modern Probability*, eds P. Picco and J. San Martin. Springer, Berlin, pp. 93–143.

[31] LEE, S. (1997). The central limit theorem for Euclidean minimal spanning trees I. *Ann. Appl. Prob.* **7**, 996–1020.

[32] LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, Berlin.

[33] LIGGETT, T. M. (2013). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer, Berlin.

[34] LIGGETT, T. M., SCHONMANN, R. H. AND STACEY, A. M. (1997). Domination by product measures. *Ann. Prob.* **25**, 71–95.

[35] MÉNARD, L. AND SINGH, A. (2016). Percolation by cumulative merging and phase transition for the contact process on random graphs. *Annales Scientifiques de l'École Normale Supérieure* **49**, 1189–1238.

[36] NGUYEN, O. AND SLY, A. (2022). Subcritical epidemics on random graphs. Preprint, arXiv:2205.03551.

[37] PENROSE, M. AND YUKICH, J. (2002). Limit theory for random sequential packing and deposition. *Ann. Appl. Prob.* **12**, 272–301.

[38] PENROSE, M. AND YUKICH, J. (2003). Weak laws of large numbers in geometric probability. *Ann. Appl. Prob.* **13**, 277–303.

[39] TANG, S., KORDZAKHIA, G. AND LALLEY, S. P. (2018). Phase transition for the chase–escape model on 2D lattices. Preprint, arXiv:1807.08387.

[40] TÓBIÁS, A. (2020). Signal to interference ratio percolation for Cox point processes. *Lat. Amer. J. Prob. Math. Statist.* **17**, 273–308.