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COMPLETE LATTICE HOMOMORPHISM OF STRONGLY REGULAR CONGRUENCES ON E-INVERSIVE SEMIGROUPS

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Abstract

In this paper, we investigate strongly regular congruences on E-inversive semigroups S. We describe the complete lattice homomorphism of strongly regular congruences, which is a generalization of an open problem of Pastijn and Petrich for regular semigroups. An abstract characterization of left and right traces for strongly regular congruences is given. The strongly regular (sr) congruences on E-inversive semigroups S are described by means of certain strongly regular congruence triples (γ, K, δ) consisting of certain sr-normal equivalences γ and δ on $E(S)$ and a certain sr-normal subset K of S. Further, we prove that each strongly regular congruence on E-inversive semigroups S is uniquely determined by its associated strongly regular congruence triple.

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1. Introduction and preliminaries

We shall use the standard terminology and notation of semigroup theory, and the reader is referred to Higgins [5] and Howie [6]. As usual, $E(S)$ is the set of idempotents of a semigroup S, $\mathcal{L}$ $(\mathcal{R}, \mathcal{H})$ is a Green's relation on S, $V(a) = \{x \in S : axa = a, xax = x\}$ is the set of all inverses of a in S and $W(a) = \{x \in S : xax = x\}$ is the set of all weak inverses of a in S. The $\mathcal{L}$-class ($\mathcal{R}$-class, $\mathcal{H}$-class) containing the element a will be written L_a (R_a, H_a). A semigroup S is E-inversive if, for any $a \in S$, there exists $x \in S$ such that $ax \in E(S)$. From [12, Lemma 3.1], a semigroup S is E-inversive if and only if $W(a) \neq \emptyset$ for any $a \in S$. This is an extremely broad class of semigroups, certainly including all regular semigroups, but also containing all eventually regular

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semigroups (in which every element has a power that is regular: see [1]), all periodic semigroups and all semigroups with zero. This concept of E-inversive semigroups was introduced by Thierrin [15]. Some basic properties of E-inversive semigroups were given by Hayes [4], Mitsch [11] and Mitsch and Petrich [12]. Recently, congruences on E-inversive semigroups have been explored extensively. For the main results about congruences on E-inversive semigroups, the reader is referred to the references Luo *et al.* [10], Weipoltshammer [17] and Zheng [18].

It is possible to study congruences on E-inversive semigroups S by means of the notion of 'weak inverse', and the 'weak inverse' in E-inversive semigroups is an analogue of 'inverse' in regular semigroups. Luo and Li [7–9] described $\mathcal{R}$-unipotent congruences: the maximum idempotent separating congruence and orthodox congruences on eventually regular semigroups via replacing 'inverse' by 'weak inverse', which leads to a similar characterization of strongly regular congruences and strongly orthodox congruences on E-inversive semigroups as in regular semigroups (see [2, 3]). It is well known that the kernel-trace approach is an effective tool for handling congruences on regular semigroups. A systematic exposition of the achievements of this approach can be found in [13, 14]. Further, Pastijn and Petrich [13] gave an abstract characterization of congruences on arbitrary regular semigroups by means of kernel-trace pairs. Luo *et al.* [10] introduced concepts of regular congruences and characterized them in terms of kernels and traces, which also play a central role in this paper. They proved that each regular congruence on E-inversive semigroups S is uniquely determined by its regular congruence pair. This paper will continue this work and establish an analogue of the results in [13, 16] for regular semigroups.

For a set X, $\operatorname{Eq} X$ is the lattice of equivalence relations on X ordered by inclusion. For a congruence ρ on S, the relation $\operatorname{tr}\rho := \rho|_{E(S)}$ is called the trace of ρ and $\ker\rho = \{a \in S : a\rho a^2\}$ is called the kernel of ρ. Let ρ be a congruence on regular semigroups S. It is obvious that the mapping $\rho \longrightarrow \operatorname{tr}\rho$ is a complete $\cap$-homomorphism. An open problem was raised by Pastijn and Petrich in [13], who asked whether this mapping is a complete lattice homomorphism, or, equivalently, whether the normal equivalences on $E(S)$ form a complete sublattice of $\operatorname{Eq} E(S)$. Pastijn and Petrich [13] only proved it to be true for the special cases where S is locally inverse, group bound or orthodox. Trotter [16] answered the open question and proved that this property is satisfied in all classes of regular semigroups.

The aim of this paper is to establish E-inversive semigroup analogues of results obtained from the congruence theory of regular semigroups. After introducing some definitions and results in this section, we describe, in Section 2, the complete lattice homomorphism of strongly regular congruences, which is a generalization of the open problem of Pastijn and Petrich for regular semigroups. In Section 3, we introduce the left and the right traces of strongly regular congruences and prove that the mapping $\rho \longrightarrow \operatorname{tr}\rho$ is a complete lattice homomorphism for the strongly regular congruence ρ on E-inversive semigroups. In the last section, strongly regular congruence triples are introduced which are then used to describe an arbitrary strongly regular congruence on

E-inversive semigroups. These results generalize the corresponding results for regular semigroups (see [13]).

In this paper, S always denotes an E-inversive semigroup, unless otherwise stated. Let $e, f \in E(S)$. Recall that $S(e, f) = \{g \in E(S) : ge = g = fg, egf = ef\}$ is the sandwich set of e and f. It is known that $S(e, f) \neq \emptyset$ for all $e, f \in E(S)$ in a regular semigroup S (see [5]). Obviously, if $e, f \in E(S)$ and $e\mathcal{L}f$ ($e\mathcal{R}f$), then $S(e, f) = \{f\}$ ($S(e, f) = \{e\}$). Let γ be an equivalence relation on S. The greatest congruence contained in γ is denoted by γ^0 on S. Let $\mathcal{F}$ be a family of equivalence relations on S. Then $\bigcap_{\gamma \in \mathcal{F}} \gamma^0 = (\cap\mathcal{F})^0$ (see [13]). If τ is an equivalence on S, the congruence generated by τ is denoted by τ^*, which is the intersection of all congruences on S that contain τ.

The following definition gives a central concept of this paper.

Definition 1.1 [3]. A congruence ρ on E-inversive semigroups S is called a strongly regular congruence if, for each $a \in S$, there exists $a' \in W(a)$ such that $a\rho aa'a$.

Recall that a congruence ρ on a semigroup S is called regular if S/ρ is regular. Following from [17, Lemma 5.4], a congruence ρ on an eventually regular semigroup S is regular if and only if, for each $a \in S$, there exists $a' \in W(a)$ such that $a\rho aa'a$. Indeed, a congruence ρ on an eventually regular semigroup is regular if and only if it is strongly regular. But the regular congruences on an E-inversive semigroup do not always satisfy the above property. An example in [10] illustrates that there exists a regular congruence on E-inversive semigroups S, which is generally not strongly regular. Notice that all elements have a weak inverse in E-inversive semigroups. It follows that the class of E-inversive semigroups is the largest possible class on which strongly regular congruences exist. The set of all strongly regular congruences on E-inversive semigroups S is denoted by SRC(S). An equivalence τ on $E(S)$ is called normal if $\tau = \operatorname{tr} \tau^*$. In particular, an equivalence τ on $E(S)$ is called a sr-normal (strongly regular normal) if there exists a strongly regular congruence ρ on S such that $\tau = \operatorname{tr} \rho$.

We list some known results which will be used in the subsequent work.

Lemma 1.2 [10]. *If ρ is a strongly regular congruence on S and $a\rho$ is an idempotent of S/ρ, then an idempotent e can be found in $a\rho$ such that $H_e \leqslant H_a$.*

If ρ is a strongly regular congruence on an E-inversive semigroup S, then, according to Lemma 1.2,

$$\ker \rho = \{a \in S : (\exists e \in E(S))\, a\rho e\}.$$

Lemma 1.3. *Let S be E-inversive semigroups and θ be a strongly regular congruence on S. If $e, f, g \in E(S)$ such that $e\theta f\theta = f\theta = f\theta g\theta$, then, for each $x \in e\theta \cap E(S)$, $y \in g\theta \cap E(S)$, there exists $z \in f\theta \cap E(S)$ such that $xz = z = zy$.*

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Proof. Let $e, f, g \in E(S)$ such that $e\theta f\theta = f\theta = f\theta g\theta$. Since θ is a strongly regular congruence on S, for each $x \in e\theta \cap E(S)$, $y \in g\theta \cap E(S)$, there exists $a \in W((xfy)^2)$ such that $(xfy)^2\theta(xfy)^2a(xfy)^2$. Let $z = xfyaxfy$, then $z \in E(S)$ and

$$\begin{aligned} z &= (xfy)\, a(xfy)\theta(efg)\, a(efg)\theta faf \\ &= f^2af^2\theta(efg)^2a(efg)^2\theta(xfy)^2a(xfy)^2\theta(xfy)^2\theta efg\theta f. \end{aligned}$$

Meanwhile, $xz = z = zy$, as required. □

The following relationships, $\mathcal{R}^*_\tau, \mathcal{L}^*_\tau, \mathcal{H}_\tau, \mathcal{R}^*$ and $\mathcal{L}^*$ were introduced in [10], which generalize Green's relations.

Definition 1.4 [10]. Let S be a semigroup and τ be an equivalence relation on $E(S)$. Define the binary relations on S, for $a, b \in S$, by

$$a\mathcal{R}^*_\tau b \Leftrightarrow \begin{array}{l}(\forall a' \in W(a))(\exists b' \in W(b))(aa'\tau bb')\& \\ (\forall b' \in W(b))(\exists a' \in W(a))(aa'\tau bb'),\end{array}$$

$$a\mathcal{L}^*_\tau b \Leftrightarrow \begin{array}{l}(\forall a' \in W(a))(\exists b' \in W(b))(a'a\tau b'b)\& \\ (\forall b' \in W(b))(\exists a' \in W(a))(a'a\tau b'b),\end{array}$$

$$a\mathcal{H}_\tau b \Leftrightarrow \begin{array}{l}(\forall a' \in W(a))(\exists b' \in W(b))(aa'\tau bb', a'a\tau b'b)\& \\ (\forall b' \in W(b))(\exists a' \in W(a))\ (aa'\tau bb', a'a\tau b'b).\end{array}$$

If τ is an equivalence relation on $E(S)$, we use the symbols $\mathcal{R}^*, \mathcal{L}^*$ instead of $\mathcal{R}^*_\tau, \mathcal{L}^*_\tau$, respectively. It is obvious that if a, b are regular elements of a semigroup S, then $a\mathcal{R}b$ $(a\mathcal{L}b, a\mathcal{H}b)$ if and only if $a\mathcal{R}^*b$ $(a\mathcal{L}^*b, a\ (\mathcal{R}^* \cap \mathcal{L}^*)b)$.

The next result will be used several times.

Lemma 1.5 [10]. *Let ρ be a strongly regular congruence on S with $\tau = \operatorname{tr}\rho$. Then:*

(i) $(e\rho)\ \mathcal{R}^*\ (f\rho) \Leftrightarrow e(\tau\mathcal{R}^*\tau)f \Leftrightarrow e\mathcal{R}^*_\tau f\ (e, f \in E(S))$;
(ii) $(a\rho)\ \mathcal{R}^*\ (b\rho) \Leftrightarrow a\ \mathcal{R}^*_\tau\ b\ (a, b \in S)$;
(iii) $\mathcal{R}^*_\tau = \rho\mathcal{R}^*\rho = \rho \vee \mathcal{R}^*$;
(iv) $\mathcal{R}^*_\tau|_{E(S)} = \tau\mathcal{R}^*\tau = (\rho \vee \mathcal{R}^*)|_{E(S)} = \tau \vee (\mathcal{R}^*|_{E(S)})$; and
(v) $\tau = \operatorname{tr}(\mathcal{R}^*_\tau \cap \mathcal{L}^*_\tau)^0 = \tau\mathcal{L}^*\tau \cap \tau\mathcal{R}^*\tau$.

2. Complete lattice homomorphism of strongly regular congruences

Trotter [16] answered an open question by Pastijn and Petrich in [13] by proving that the mapping $\rho \longrightarrow \operatorname{tr}\rho$ is a complete lattice homomorphism for a general congruence ρ in all classes of regular semigroups. In this section, we described the complete lattice homomorphism of strongly regular congruences, which is a generalization of the open problem of Pastijn and Petrich for regular semigroups.

Lemma 2.1. *Let S be an E-inversive semigroup and θ be a strongly regular congruence on S. For any $a \in S$, then $(a, a^2) \in \theta \Leftrightarrow S(a\theta, a\theta) = \{a\theta\}$.*

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Proof. '⇒' Since $(a, a^2) \in \theta$, $a\theta \in E(S/\theta)$. By Lemma 1.2, there exists $e \in E(S)$ such that $a\theta = e\theta$. Let $h \in E(S)$ and $h\theta \in S(a\theta, a\theta) = S(e\theta, e\theta)$. It follows that

$$(he)\theta = h\theta = (eh)\theta \quad \text{and} \quad (ehe)\theta = e\theta.$$

Then $h\theta = (ehe)\theta = e\theta = a\theta$.

'⇐' It is clear. □

Lemma 2.2. *Let* $e, f \in E(S)$, $c_1, c_2, \ldots, c_n \in S$, $\rho_1, \rho_2, \ldots, \rho_n \in \mathrm{SRC}(S)$ *and* $e = c_0, c_{i-1}\rho_i c_i, c_n = f$, $0 < i \leqslant n$. *Then there exist* $g_1, g_2, \ldots, g_n \in E(S)$ *such that* $e = g_0, g_{i-1} \operatorname{tr} \rho_i g_i, g_{i-1}g_i = g_i, g_n = g_n f$, $0 < i \leqslant n$.

Proof. Let $c_{i-1}\rho_i c_i, 0 < i \leqslant n$. Since ρ_i is a strongly regular congruence on S, there exist $c'_{i-1} \in W(c_{i-1})$ and $c'_i \in W(c_i)$ such that

$$c_i\rho_i = (c_i c'_i c_i)\rho_i \quad \text{and} \quad c_{i-1}\rho_i = (c_{i-1}c'_{i-1}c_{i-1})\rho_i, \quad 0 < i \leqslant n.$$

Let $c'_0 = e, c'_n = f$. Put $g_0 = e$ and select inductively $g'_i\rho_i \in S((g_{i-1}c'_i c_i)\rho_i, (g_{i-1}c'_i c_i)\rho_i)$, where $g'_i \in E(S)$. Then

$$(g_{i-1}g'_i)\rho_i = g'_i\rho_i = (g'_i c'_i c_i)\rho_i.$$

By Lemma 1.3, there exists $g_i \in E(S)$ such that

$$g_i\rho_i g'_i \quad \text{and} \quad g_{i-1}g_i = g_i = g_i c'_i c_i.$$

Since $c_{i-1}\rho_i c_i$, then $(c'_{i-1}c_{i-1})\rho_i \mathcal{L} (c'_i c_i)\rho_i$ in S/ρ_i. Thus $(c'_{i-1}c_{i-1})\rho_i = (c'_{i-1}c_{i-1})(c'_i c_i)\rho_i$. By the induction hypothesis, $g_{i-1}c'_{i-1}c_{i-1} = g_{i-1}$. It follows that

$$g_{i-1} = g_{i-1}c'_{i-1}c_{i-1}\rho_i g_{i-1}c'_{i-1}c_{i-1}c'_i c_i = g_{i-1}c'_i c_i,$$

and so $(g_{i-1}c'_i c_i)\rho_i \in E(S/\rho_i)$. By Lemma 2.1, $g'_i\rho_i g_{i-1}c'_i c_i$. Therefore

$$g_i\rho_i g'_i\rho_i g_{i-1}c'_i c_i\rho_i g_{i-1},$$

and so $g_{i-1} \operatorname{tr} \rho_i g_i$, as required. □

Note that there is a dual construction based on the alternative inductive selection of $g'_i\rho_i \in S((c_i c'_i g_{i-1})\rho_i, (c_i c'_i g_{i-1})\rho_i)$ giving $g_1, g_2, \ldots, g_n \in E(S)$ such that

$$e = g_0, \quad g_{i-1} \operatorname{tr} \rho_i g_i, g_i g_{i-1} = g_i, \quad g_n = f g_n, \quad 0 < i \leqslant n.$$

Lemma 2.3. *Let* $e, f \in E(S), \rho_i (0 < i \leqslant n)$ *and* g_n *satisfy the conclusion of Lemma 2.2 and* $fe = f$. *Then* $e\,(\bigvee_{i=1}^n \operatorname{tr} \rho_i) g_n \mathcal{L} f g_n (\bigvee_{i=1}^n \operatorname{tr} \rho_i) f$.

Proof. By Lemma 2.2, $g_n = g_n f$, so $g_n \mathcal{L} f g_n$. For any $0 < i \leqslant n$,

$$f = fe = fg_0, \quad fg_{i-1}\rho_i f g_i.$$

Let $p_0 = f$, and select inductively $p'_i\rho_i \in S(f\rho_i, g_i\rho_i)$. Since $g_n = g_n f$, we may choose $p_n = g_n$. Then

$$(p'_i\rho_i)(f\rho_i) = p'_i\rho_i = (g_i\rho_i)(p'_i\rho_i) \quad \text{and} \quad (f\rho_i)(p'_i\rho_i)(g_i\rho_i) = (f\rho_i)(g_i\rho_i).$$

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By Lemma 1.3, there exists $p_i \in E(S)$ such that

$$p_i \rho_i p_i' \quad \text{and} \quad p_i f = p_i = g_i p_i.$$

Then $(f\rho_i)(p_i\rho_i)(g_i\rho_i) = (f\rho_i)(g_i\rho_i)$. Notice that $g_j g_i = g_i$ for $0 < j < i \leqslant n$. So

$$\begin{aligned}(fp_j)(fp_i)\rho_i &= (fp_jp_i)\rho_i = (fp_jg_ip_i)\rho_i \quad (\text{since } p_i = g_ip_i)\\ &= (fp_jg_jg_ip_i)\rho_i = (fp_jg_j)\rho_i(g_ip_i)\rho_i \quad (\text{since } g_jg_i = g_i)\\ &= (fg_jg_ip_i)\rho_i \quad (\text{since } (fp_jg_j)\rho_i = (fg_j)\rho_i)\\ &= (fp_i)\rho_i \in E(S/\rho_i).\end{aligned}$$

It follows that

$$\begin{aligned}(fp_{i-1})\rho_i &= (fp_{i-1}fg_{i-1}p_{i-1})\rho_i = (fp_{i-1}fg_ip_{i-1})\rho_i \quad (\text{since } (fg_{i-1})\rho_i = (fg_i)\rho_i)\\ &= (fp_{i-1})(fp_ig_i)p_{i-1}\rho_i \quad (\text{since } (fp_ig_i)\rho_i = (fg_i)\rho_i)\\ &= (fp_{i-1})(fp_i)(fg_ip_{i-1})\rho_i \quad (\text{since } p_i = p_if)\\ &= (fp_{i-1})(fp_i)(fg_{i-1}p_{i-1})\rho_i \quad (\text{since } (fg_{i-1})\rho_i = (fg_i)\rho_i)\\ &= (fp_{i-1})(fp_i)(fp_{i-1})\rho_i \quad (\text{since } g_{i-1}p_{i-1} = p_{i-1})\\ &= (fp_i)(fp_{i-1})\rho_i.\end{aligned}$$

Then $(fp_{i-1})\rho_i\, \mathcal{R}\, (fp_i)\rho_i$.

If $h_1\rho_i \in S((fp_{i-1})\rho_i, (fp_i)\rho_i) = \{(fp_{i-1})\rho_i\}$, then

$$(fp_i)\rho_i \cdot h_1\rho_i = h_1\rho_i.$$

By Lemma 1.3, there exists $h \in E(S)$ such that $h\rho_i h_1$ and $(fp_i)h = h$. So $h\rho_i fp_{i-1}$ and therefore

$$(hfp_i)\rho_i = h\rho_i(fp_i)\rho_i = (fp_{i-1})\rho_i(fp_i)\rho_i = (fp_i)\rho_i.$$

Consequently,

$$(fp_{i-1})\,\mathrm{tr}\,\rho_i h\mathcal{R}(hfp_i)\,\mathrm{tr}\,\rho_i\,(fp_i).$$

Then $(h, hfp_i) \in \mathrm{tr}(\bigvee_{\rho\in\mathcal{F}}\rho)$. Let $a' = h, b' = hfp_i$. Notice that $a', b' \in E(S)$. By Lemma 2.2, there exist $d_1, d_2, \ldots, d_m \in E(S)$ and $\rho_1', \rho_2', \ldots, \rho_m' \in \mathcal{F}$ such that

$$a' = d_0, \quad d_{i-1}\,\mathrm{tr}\,\rho_i' d_i, \quad d_{i-1}d_i = d_i, \quad d_m = d_m b', \quad 0 < i \leqslant m.$$

Since $a'\mathcal{R}b'$, then

$$b' = a'b' = d_0b', \quad d_{i-1}b'\rho_i'd_ib', \quad d_mb' = d_m, \quad 0 < i \leqslant m.$$

Furthermore, since $d_i = d_{i-1}d_i = a'd_i$ and $a' = b'a'$, then

$$(d_ib')^2 = d_ib'd_ib' = d_ib'a'd_ib' = d_ia'd_ib' = d_ib',$$

and so $d_ib' \in E(S)$. Thus

$$a'\Big(\bigvee_{\rho\in\mathcal{F}}\mathrm{tr}\,\rho\Big)d_m\Big(\bigvee_{\rho\in\mathcal{F}}\mathrm{tr}\,\rho\Big)b'.$$

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It follows that

$$f = fp_0, \quad (fp_{i-1})\,\mathrm{tr}\,\rho_i h = a'\Big(\bigvee_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho\Big) d_m \Big(\bigvee_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho\Big)(hfp_i)\,\mathrm{tr}\,\rho_i(fp_i), \quad fp_n = fg_n.$$

Therefore $fg_n(\bigvee_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho)f$, as required. □

THEOREM 2.4. *Let S be an E-inversive semigroup and ρ be a strongly regular congruence on S. Then*

$$\rho \longrightarrow \mathrm{tr}\,\rho, \quad \rho \in \mathrm{SRC}(S)$$

is a complete lattice homomorphism from SRC(S) *into* Eq $E(S)$.

PROOF. Let $\mathcal{F}$ be a family of strongly regular congruences on S. It is easy to check that $\bigcap_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho = \mathrm{tr}(\bigcap_{\rho\in\mathcal{F}} \rho)$. Since $\bigvee_{\rho\in\mathcal{F}}(\mathrm{tr}\,\rho) \subseteq \mathrm{tr}(\bigvee_{\rho\in\mathcal{F}} \rho)$ always holds, it suffices to show that $\mathrm{tr}(\bigvee_{\rho\in\mathcal{F}} \rho) \subseteq \bigvee_{\rho\in\mathcal{F}}(\mathrm{tr}\,\rho)$.

For $e, f \in E(S)$, now let $e\ (\mathrm{tr} \bigvee_{\rho\in\mathcal{F}} \rho) f$. Then there exist $x_1, x_2, \ldots, x_n \in S$ and $\rho_1, \rho_2, \ldots, \rho_n \in \mathcal{F}$ such that

$$e = x_0, \quad x_{i-1}\rho_i x_i, \quad x_n = f, \quad 0 < i \leqslant n. \tag{1}$$

By Lemma 2.2 there exist $y_1, y_2, \ldots, y_n \in E(S)$ such that

$$e = y_0, \quad y_{i-1}\,\mathrm{tr}\,\rho_i y_i, \quad 0 < i \leqslant n. \tag{2}$$

Apply Lemma 2.2 again to

$$f = x_n\rho_n x_{n-1}\rho_{n-1}\cdots\rho_1 e\rho_1 y_1\rho_2\, y_2\cdots\rho_n y_n$$

(that is, the reverse of the chain of relations (1) followed by chain (2)) to get $z_1, z_2, \ldots, z_{2n} \in E(S)$, where

$$f = z_0, \quad z_{i-1}\,\mathrm{tr}\,\rho_{n-i+1} z_i, \quad z_{n+i-1}\,\mathrm{tr}\,\rho_i z_{n+i}, \quad z_{2n} = z_{2n}y_n, \quad 0 < i \leqslant n. \tag{3}$$

Now apply Lemma 2.2 to

$$e\rho_1 x_1\rho_2\cdots\rho_n f\rho_n z_1\rho_{n-1}z_2\rho_{n-2}\cdots\rho_n z_{2n}$$

((1) followed by (3)) to get $y_{n+1}, y_{n+2}, \ldots, y_{3n} \in E(S)$, where

$$e = y_0, \quad y_{i-1}\,\mathrm{tr}\,\rho_i\, y_i, \quad y_{n+i-1}\,\mathrm{tr}\,\rho_{n-i+1} y_{n+i}, \quad y_{2n+i-1}\,\mathrm{tr}\,\rho_i\, y_{2n+i},$$
$$y_{i-1}y_i = y_i, \quad y_{n+i-1}y_{n+i} = y_{n+i}, \quad y_{2n+i-1}y_{2n+i} = y_{2n+i}, \quad y_{3n}z_{2n} = y_{3n}.$$

The subchain

$$y_n\,\mathrm{tr}\,\rho_n y_{n+1}\cdots\mathrm{tr}\,\rho_1 y_{2n}\,\mathrm{tr}\,\rho_1 y_{2n+1}\cdots\mathrm{tr}\,\rho_n y_{3n}$$

satisfies the conclusion of Lemma 2.2 and, by (3), $z_{2n} = z_{2n}y_n$. So, by Lemma 2.3, there exist $a, b \in E(S)$ (in fact $a = y_{3n},\ b = z_{2n}y_{3n}$) such that (with (2) and (3))

$$e\Big(\bigvee_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho\Big) y_n \Big(\bigvee_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho\Big) a\mathcal{L}b \Big(\bigvee_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho\Big) z_{2n} \Big(\bigvee_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho\Big) f.$$

Since $(e, f) \in \mathrm{tr}(\bigvee_{\rho\in\mathcal{F}} \rho) \supseteq (\bigvee_{\rho\in\mathcal{F}} \mathrm{tr}\,\rho)$, then $(a, b) \in \mathrm{tr}(\bigvee_{\rho\in\mathcal{F}} \rho)$.

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By the dual of Lemma 2.2 there exist $g_1, g_2, \ldots, g_m \in E(S)$ and $\rho'_1, \rho'_2, \ldots, \rho'_m \in \mathcal{F}$ such that

$$a = g_0, \quad g_{i-1} \operatorname{tr} \rho'_i g_i, \quad g_i g_{i-1} = g_i, \quad g_m = b g_m, \quad 0 < i \leqslant m.$$

Since $a\mathcal{L}b$, then

$$b = ba = bg_0, \quad bg_{i-1} \rho'_i bg_i, \quad bg_m = g_m, \quad 0 < i \leqslant m.$$

Since $g_i = g_i g_0 = g_i a$ and $ab = a$, then

$$(bg_i)^2 = bg_i abg_i = bg_i ag_i = bg_i,$$

and so $bg_i \in E(S),\ 0 < i \leqslant m$. Thus

$$b\Big(\bigvee_{\rho\in\mathcal{F}} \operatorname{tr} \rho\Big) g_m \Big(\bigvee_{\rho\in\mathcal{F}} \operatorname{tr} \rho\Big) a.$$

But $e\ (\bigvee_{\rho\in\mathcal{F}} \operatorname{tr} \rho)\, a$ and $f\ (\bigvee_{\rho\in\mathcal{F}} \operatorname{tr} \rho) b$. So $e\ (\bigvee_{\rho\in\mathcal{F}} \operatorname{tr} \rho) f$. Thus

$$\operatorname{tr}\Big(\bigvee_{\rho\in\mathcal{F}} \rho\Big) \subseteq \bigvee_{\rho\in\mathcal{F}} (\operatorname{tr} \rho).$$

The theorem is therefore proved. □

3. The left and the right traces

In this section, we shall introduce some relationships about strongly regular congruences on S. They will turn out to be complete congruences induced by certain complete homomorphisms of $\mathrm{SRC}(S)$ into $\mathrm{Eq}\, E(S)$.

Definition 3.1. For a congruence ρ on E-inversive semigroups S, $\operatorname{ltr} \rho = \operatorname{tr}(\rho \vee \mathcal{L}^*)^0$ is called the left trace of ρ and $\operatorname{rtr} \rho = \operatorname{tr}(\rho \vee \mathcal{R}^*)^0$ is called the right trace of ρ.

An abstract characterization of left and right traces will be given by means of the following concepts.

Definition 3.2. An equivalence τ on $E(S)$ is:

(i) left normal if $\mathcal{L}^*_\tau|_{E(S)} = \tau\mathcal{L}^*\tau$ and $\tau = \operatorname{tr}(\mathcal{L}^*_\tau)^0$; and
(ii) right normal if $\mathcal{R}^*_\tau|_{E(S)} = \tau\mathcal{R}^*\tau$ and $\tau = \operatorname{tr}(\mathcal{R}^*_\tau)^0$.

An equivalence τ on $E(S)$ is right (left) sr-normal if and only if τ is both right (left) normal and sr-normal.

The following lemma will be used many times.

Lemma 3.3. *Let ρ be a strongly regular congruence on S with $\tau = \operatorname{tr} \rho$, $\tau_l = \operatorname{ltr} \rho$ and $\tau_r = \operatorname{rtr} \rho$. Then:*

(i) $\rho\mathcal{R}^*\rho = \rho \vee \mathcal{R}^* = (\rho \vee \mathcal{R}^*)^0 \vee \mathcal{R}^* = \mathcal{R}^*_{\tau_r}$;
(ii) $\tau\mathcal{R}^*\tau = \tau_r\mathcal{R}^*\tau_r$;

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(iii) $\tau_r = \mathrm{tr}(\rho \vee \mathcal{R}^*)^0 = \mathrm{tr}(\mathcal{R}^*_{\tau_r})^0 = \mathrm{rtr}(\rho \vee \mathcal{R}^*)^0 = \mathrm{rtr}(\mathcal{R}^*_{\tau_r})^0$; *and*
(iv) $\tau = \tau_l \cap \tau_r$.

Proof. (i) By Lemma 1.5(iii), it suffices to show that $\rho \vee \mathcal{R}^* = (\rho \vee \mathcal{R}^*)^0 \vee \mathcal{R}^*$. Obviously, $\rho \subseteq (\rho \vee \mathcal{R}^*)^0$. Then we may conclude that $\rho \vee \mathcal{R}^* \subseteq (\rho \vee \mathcal{R}^*)^0 \vee \mathcal{R}^*$. On the other hand, $(\rho \vee \mathcal{R}^*)^0 \subseteq \rho \vee \mathcal{R}^*$ and $\mathcal{R}^* \subseteq \rho \vee \mathcal{R}^*$ give $(\rho \vee \mathcal{R}^*)^0 \vee \mathcal{R}^* \subseteq \rho \vee \mathcal{R}^*$.

(ii) It follows from Lemma 1.5(iv) and part (i) that

$$\tau\mathcal{R}^*\tau = (\rho \vee \mathcal{R}^*)|_{E(S)} = ((\rho \vee \mathcal{R}^*)^0 \vee \mathcal{R}^*)|_{E(S)} = \tau_r \vee (\mathcal{R}^*|_{E(S)}) = \tau_r\mathcal{R}^*\tau_r.$$

(iii) This is immediate from part (i).

(iv) From [13, Result 1.4 and Lemma 2.5], together with Lemma 1.5, we find that

$$\tau = \mathrm{tr}(\mathcal{R}^*_\tau \cap \mathcal{L}^*_\tau)^0 = \mathrm{tr}(\mathcal{R}^*_\tau)^0 \cap \mathrm{tr}(\mathcal{L}^*_\tau)^0 = \tau_r \cap \tau_l.$$

Therefore, the desired equalities hold. □

Lemma 3.4. *An equivalence τ_r on $E(S)$ is right sr-normal if and only if it is the right trace of a strongly regular congruence on S. In this case, $\mathcal{R}^*_{\tau_r}$ is the greatest strongly regular congruence on S with right trace τ_r.*

Proof. Let τ_r be the right trace of a strongly regular congruence ρ. Then $\tau_r = \mathrm{tr}(\rho \vee \mathcal{R}^*)^0$. It follows from Lemma 1.5(iv) that $\mathcal{R}^*_{\tau_r}|_{E(S)} = \tau_r\mathcal{R}^*\tau_r$. By Lemma 3.3, we obtain $\tau_r = \mathrm{tr}(\rho \vee \mathcal{R}^*)^0 = \mathrm{tr}(\mathcal{R}^*_{\tau_r})^0$, and so τ_r is a right sr-normal equivalence.

Conversely, if τ_r is a right sr-normal equivalence, then $\mathcal{R}^*_{\tau_r}|_{E(S)} = \tau_r\mathcal{R}^*\tau_r$, and so $\mathcal{R}^*_{\tau_r}$ is an equivalence on S and $\tau_r = \mathrm{tr}(\mathcal{R}^*_{\tau_r})^0$. Using Lemma 1.5(iii), we may conclude that $(\mathcal{R}^*_{\tau_r})^0 \vee \mathcal{R}^* = \mathcal{R}^*_{\tau_r}$. Thus

$$\tau_r = \mathrm{tr}((\mathcal{R}^*_{\tau_r})^0 \vee \mathcal{R}^*)^0 = \mathrm{rtr}(\mathcal{R}^*_{\tau_r})^0.$$

Let ρ be a strongly regular congruence with $\mathrm{rtr}\,\rho = \tau_r$. Then $\rho \subseteq (\rho \vee \mathcal{R}^*)^0 = (\mathcal{R}^*_{\tau_r})^0$. By Lemma 3.3, we obtain $\mathrm{tr}(\mathcal{R}^*_{\tau_r})^0 = \tau_r$. Thus $\mathrm{tr}(\mathcal{R}^*_{\tau_r})^0$ is the greatest strongly regular congruence on S with right trace τ_r. □

Corollary 3.5. *An equivalence τ on $E(S)$ is sr-normal if and only if it is the intersection of a left sr-normal equivalence and a right sr-normal equivalence.*

Proof. Let τ be a sr-normal equivalence. Then there exists a strongly regular congruence ρ on S such that $\mathrm{tr}\,\rho = \tau$. By Lemma 3.3 and its dual, we may conclude that $\tau_l = \mathrm{ltr}\,\rho$ is a left sr-normal equivalence and $\tau_r = \mathrm{rtr}\,\rho$ is a right sr-normal equivalence. By Lemma 3.3, $\tau = \tau_r \cap \tau_l$.

Suppose, conversely, that $\tau = \tau_r \cap \tau_l$, where τ_l is a left sr-normal and τ_r a right sr-normal equivalence on S and

$$\tau = \mathrm{tr}(\mathcal{L}^*_{\tau_l})^0 \cap \mathrm{tr}(\mathcal{R}^*_{\tau_r})^0 = \mathrm{tr}(\mathcal{L}^*_{\tau_l} \cap \mathcal{R}^*_{\tau_r})^0.$$

It follows that τ is a sr-normal equivalence. □

Now we introduce several relationships concerning strongly regular congruences SRC(S).

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NOTATION 3.6. For any $\rho, \theta \in \mathrm{SRC}(S)$, let

$$\rho T_l \theta \Leftrightarrow \operatorname{ltr} \rho = \operatorname{ltr} \theta,$$
$$\rho T_r \theta \Leftrightarrow \operatorname{rtr} \rho = \operatorname{rtr} \theta,$$
$$\rho T \theta \Leftrightarrow \operatorname{tr} \rho = \operatorname{tr} \theta,$$
$$\rho K \theta \Leftrightarrow \ker \rho = \ker \theta.$$

THEOREM 3.7. *The mapping* $\rho \longrightarrow \rho \vee \mathcal{R}^*$ $(\rho \in \mathrm{SRC}(S))$ *is a complete homomorphism of* $\mathrm{SRC}(S)$ *into* $\mathrm{Eq}\, S$ *which induces* T_r.

PROOF. Let $\mathcal{F}$ be a family of strongly regular congruences on S. Let $a, b \in S$ be such that $a\ (\bigcap_{\rho\in\mathcal{F}}(\rho \vee \mathcal{R}^*))b$. It follows that $a\ (\rho \vee \mathcal{R}^*)b$ for any $\rho \in \mathcal{F}$. By Lemma 1.5, $a\rho\mathcal{R}^* b\rho$. Since ρ is a strongly regular congruence on S, there exist $a' \in W(a), b' \in W(b)$ such that $a\rho aa'a, b\rho bb'b$. Let $e = aa'$, $f = bb'$. Then $e\rho\mathcal{R}^* a\rho\mathcal{R}^* b\rho\mathcal{R}^* f\rho$. It follows that $a(\rho \vee \mathcal{R}^*)e$ and $f(\rho \vee \mathcal{R}^*)b$. If $g_1\rho \in S(e\rho, f\rho) = \{e\rho\}$, then $g_1\rho e\rho = g_1\rho = f\rho g_1\rho$. By Lemma 1.3, there exists $g \in E(S)$ such that $ge = g = fg$. Thus $g\rho e$. Since $(e\rho)\mathcal{R}\,(f\rho)$, $(gf)\rho = g\rho f\rho = e\rho f\rho = f\rho$. Therefore, $e\tau g\mathcal{R}\,(gf)\tau f$. Consequently,

$$e\Big(\bigcap_{\rho\in\mathcal{F}}\rho\Big)g\mathcal{R}\,(gf)\Big(\bigcap_{\rho\in\mathcal{F}}\rho\Big)f,$$

that is, $e\ ((\bigcap_{\rho\in\mathcal{F}}\rho) \vee \mathcal{R}^*)f$. Therefore $a\ ((\bigcap_{\rho\in\mathcal{F}}\rho) \vee \mathcal{R}^*)b$, and so

$$\bigcap_{\rho\in\mathcal{F}}(\rho \vee \mathcal{R}^*) \subseteq \Big(\bigcap_{\rho\in\mathcal{F}}\rho\Big) \vee \mathcal{R}^*.$$

The reverse inclusion is obvious and so the equality holds. It follows that the mapping $\rho \longrightarrow \rho \vee \mathcal{R}^*$ $(\rho \in \mathrm{SRC}(S))$ is a complete homomorphism of $\mathrm{SRC}(S)$ into $\mathrm{Eq}\, S$.

If $\rho, \theta \in \mathrm{SRC}(S)$ are such that $\rho T_r \theta$, then $\operatorname{rtr}\rho = \operatorname{rtr}\theta$. By Lemma 3.3, we obtain

$$\rho \vee \mathcal{R}^* = (\operatorname{rtr}\rho)\mathcal{R}^*(\operatorname{rtr}\rho) = (\operatorname{rtr}\theta)\mathcal{R}^*(\operatorname{rtr}\theta) = \theta \vee \mathcal{R}^*.$$

Conversely, if $\rho \vee \mathcal{R}^* = \theta \vee \mathcal{R}^*$, then

$$\operatorname{rtr}\rho = \operatorname{tr}(\rho \vee \mathcal{R}^*)^0 = \operatorname{tr}(\theta \vee \mathcal{R}^*)^0 = \operatorname{rtr}\theta.$$

Therefore, the desired results hold. □

THEOREM 3.8. *The mapping* $\rho \longrightarrow (\rho \vee \mathcal{R}^*)|_{E(S)}$ $(\rho \in \mathrm{SRC}(S))$ *is a complete lattice homomorphism of* $\mathrm{SRC}(S)$ *into* $\mathrm{Eq}\, S$ *which induces* T_r.

PROOF. Let $\mathcal{F}$ be a family of strongly regular congruences on S. Let $e, f \in E(S)$ be such that $e\ ((\bigvee_{\rho\in\mathcal{F}}\rho) \vee \mathcal{R}^*)f$. Then $e\ (\bigvee_{\rho\in\mathcal{F}}(\rho \vee \mathcal{R}^*))f$ and there exist $x_0, x_1, \ldots, x_n$ and $\rho_1, \rho_2, \ldots, \rho_n \in \mathcal{F}$ such that

$$e = x_0(\rho_1 \vee \mathcal{R}^*)x_1(\rho_2 \vee \mathcal{R}^*)x_2 \cdots x_{n-1}(\rho_n \vee \mathcal{R}^*)x_n = f.$$

For each $0 < i \leqslant n$, by Lemma 3.3, we may conclude that $\rho_i \vee \mathcal{R}^* = \mathcal{R}^*_{\tau_r}$. It follows that

$$e = x_0\mathcal{R}^*_{\tau_r}x_1\mathcal{R}^*_{\tau_r}x_2 \cdots x_{n-1}\mathcal{R}^*_{\tau_r}\,x_n = f,$$

and so $e\mathcal{R}^*_{\tau_r} f$. By following exactly the same argument of the corresponding part of Theorem 3.7, we may conclude that there exist $g', h' \in E(S)$ such that $e\tau_r g'\mathcal{R}h'\tau_r f$. Therefore,

$$\Big(\Big(\bigvee_{\rho\in\mathcal{F}} \rho\Big) \vee \mathcal{R}^*\Big)\Big|_{E(S)} \subseteq \bigvee_{\rho\in\mathcal{F}} (\rho \vee \mathcal{R}^*)|_{E(S)}.$$

Let $e, f \in E(S)$ be such that $e\ (\bigcap_{\rho\in\mathcal{F}}(\rho \vee \mathcal{R}^*))f$. It follows that $e\ (\rho \vee \mathcal{R}^*)f$ for any $\rho \in \mathcal{F}$. By Lemma 1.5, we obtain $e\rho\mathcal{R}\ f\rho$. As in the proof of Theorem 3.7, there exist $g, h \in E(S)$ such that $e\tau g\mathcal{R}h\tau f$. Consequently,

$$e\Big(\bigcap_{\rho\in\mathcal{F}} \rho\Big) g\mathcal{R}h\Big(\bigcap_{\rho\in\mathcal{F}} \rho\Big) f,$$

that is, $e\ ((\bigcap_{\rho\in\mathcal{F}} \rho) \vee \mathcal{R}^*)|_{E(S)} f$. Hence,

$$\bigcap_{\rho\in\mathcal{F}} (\rho \vee \mathcal{R}^*)|_{E(S)} \subseteq \Big(\Big(\bigcap_{\rho\in\mathcal{F}} \rho\Big) \vee \mathcal{R}^*\Big)\Big|_{E(S)}.$$

The reverse inclusion obviously holds.

It follows that the mapping $\rho \longrightarrow (\rho \vee \mathcal{R}^*)|_{E(S)}$ $(\rho \in \mathrm{SRC}(S))$ is a complete lattice homomorphism of SRC(S) into Eq S. For $\rho, \theta \in \mathrm{SRC}(S)$, we may conclude that

$$\begin{aligned}
(\rho \vee \mathcal{R}^*)|_{E(S)} &= (\theta \vee \mathcal{R}^*)|_{E(S)} \\
&\Leftrightarrow (\mathrm{tr}\,\rho)\mathcal{R}^*(\mathrm{tr}\,\rho) = (\mathrm{tr}\,\theta)\mathcal{R}^*(\mathrm{tr}\,\theta) \\
&\Leftrightarrow \mathrm{tr}(\rho \vee \mathcal{R}^*)^0 = \mathrm{tr}(\theta \vee \mathcal{R}^*)^0 \\
&\Rightarrow \rho\ T_r\ \theta \\
&\Rightarrow (\mathrm{tr}\,\rho)\mathcal{R}^*(\mathrm{tr}\,\rho) = (\mathrm{tr}\,\theta)\mathcal{R}^*(\mathrm{tr}\,\theta) \\
&\Leftrightarrow (\rho \vee \mathcal{R}^*)|_{E(S)} = (\theta \vee \mathcal{R}^*)|_{E(S)}.
\end{aligned}$$

Hence $(\rho \vee \mathcal{R}^*)|_{E(S)} = (\theta \vee \mathcal{R}^*)|_{E(S)} \Leftrightarrow \rho\ T_r \theta$. □

COROLLARY 3.9. *Let ρ be a strongly regular congruence on S. Then $T = T_l \cap T_r$ and*

$$\rho \longrightarrow ((\rho \vee \mathcal{L}^*)|_{E(S)}, (\rho \vee \mathcal{R}^*)|_{E(S)}) \quad (\rho \in \mathrm{SRC}(S))$$

is a complete lattice homomorphism of SRC(S) *into* $(\mathrm{Eq}\,E(S))^2$ *which induces T.*

PROOF. For $\rho, \theta \in \mathrm{SRC}(S)$,

$$\begin{aligned}
\rho T\theta &\Leftrightarrow \mathrm{tr}\,\rho = \mathrm{tr}\,\theta \\
&\Rightarrow \mathrm{rtr}\,\rho = \mathrm{tr}((\mathrm{tr}\,\rho)\mathcal{R}^*(\mathrm{tr}\,\rho))^0 \\
&\qquad\qquad = \mathrm{tr}((\mathrm{tr}\,\theta)\mathcal{R}^*(\mathrm{tr}\,\theta))^0 = \mathrm{rtr}\,\theta, \\
&\qquad \mathrm{ltr}\,\rho = \mathrm{tr}((\mathrm{tr}\,\rho)\mathcal{L}^*(\mathrm{tr}\,\rho))^0 \\
&\qquad\qquad = \mathrm{tr}((\mathrm{tr}\,\theta)\mathcal{L}^*(\mathrm{tr}\,\theta))^0 = \mathrm{ltr}\,\theta, \\
&\Rightarrow \rho(T_l \cap T_r)\theta \\
&\Rightarrow \mathrm{tr}\,\rho = \mathrm{rtr}\,\rho \cap \mathrm{ltr}\,\rho = \mathrm{rtr}\,\theta \cap \mathrm{ltr}\,\theta = \mathrm{tr}\,\theta \\
&\Rightarrow \rho T\theta.
\end{aligned}$$

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Thus $T = T_l \cap T_r$. It is easy to show the remaining part in the statement of the corollary. □

4. Strongly regular congruences triples

Pastijn and Petrich [13] introduced the concept of congruence triples in arbitrary regular semigroups. These triples represent a close analogue of admissible triples which are used for describing congruences on a Rees matrix semigroup. In this section, we shall establish an analogue for strongly regular congruences on E-inversive semigroups. To this end, we first introduce the following concepts.

NOTATION 4.1. Let S be an E-inversive semigroup, $\gamma \in \mathrm{Eq}(S/\mathcal{L}^*)$ and $\delta \in \mathrm{Eq}(S/\mathcal{R}^*)$. For any $a, b \in S$, let

$$a\bar{\gamma}b \Leftrightarrow \mathcal{L}^*_a \gamma \mathcal{L}^*_b, \quad a\bar{\delta}b \Leftrightarrow \mathcal{R}^*_a \delta \mathcal{R}^*_b.$$

It is clear that $\mathcal{L}^* \subseteq \bar{\gamma}$, $\mathcal{R}^* \subseteq \bar{\delta}$ and $\bar{\gamma}, \bar{\delta} \in \mathrm{Eq}\, S$.

DEFINITION 4.2. An equivalence $\gamma \in \mathrm{Eq}(S/\mathcal{L}^*)$ is sr-normal if $\gamma = (\bar{\gamma}^0 \vee \mathcal{L}^*)/\mathcal{L}^*$, where $\bar{\gamma}^0 \in \mathrm{SRC}(S)$ and an equivalence $\delta \in \mathrm{Eq}(S/\mathcal{R}^*)$ is sr-normal if $\delta = (\bar{\delta}^0 \vee \mathcal{R}^*)/\mathcal{R}^*$, where $\bar{\delta}^0 \in \mathrm{SRC}(S)$.

Let ρ be a strongly regular congruence on S. Then $(\rho \vee \mathcal{L}^*)/\mathcal{L}^*$ $((\rho \vee \mathcal{R}^*)/\mathcal{R}^*)$ is said to be a strongly regular $\mathcal{L}^*$-part (strongly regular $\mathcal{R}^*$-part) of ρ.

LEMMA 4.3. *An equivalence $\delta \in \mathrm{Eq}(S/\mathcal{R}^*)$ is sr-normal if and only if δ is the strongly regular $\mathcal{R}^*$-part of a strongly regular congruence ρ on S.*

PROOF. If δ is sr-normal, then δ is the strongly regular $\mathcal{R}^*$-part of the strongly regular congruence $\bar{\delta}^0$. Conversely, let δ be the $\mathcal{R}^*$-part of the strongly regular congruence ρ (that is $\delta = (\rho \vee \mathcal{R}^*)/\mathcal{R}^*$) or, in other words, $\bar{\delta} = \rho \vee \mathcal{R}^*$. Clearly $\rho \subseteq \bar{\delta}^0$, so that $\rho \vee \mathcal{R}^* \subseteq \bar{\delta}^0 \vee \mathcal{R}^*$. On the other hand, $\bar{\delta}^0 \subseteq \rho \vee \mathcal{R}^*$, and so $\bar{\delta}^0 \vee \mathcal{R}^* \subseteq \rho \vee \mathcal{R}^*$. Therefore $\delta = (\rho \vee \mathcal{R}^*)/\mathcal{R}^* = (\bar{\delta}^0 \vee \mathcal{R}^*)/\mathcal{R}^*$. Thus δ is sr-normal, as required. □

Let S be a semigroup and K be a subset of S. A congruence ρ on S saturates K if $a \in K$ implies $a\rho \subseteq K$. The greatest congruence on S which saturates K is denoted by π_K. Recall from [13, Result 1.5] that for $a, b \in S$, $a\pi_K b$ if and only if

$$xay \in K \Leftrightarrow xby \in K, \quad (x, y \in S^1),$$

and $\pi_K = \theta_K^0$, where the equivalence relation θ_K on S is defined by $a, b \in S$ and

$$a\theta_K b \Leftrightarrow a, b \in K \quad \text{or} \quad a, b \in S - K.$$

A subset K of S is called sr-normal (strongly regular normal) if there exists a strongly regular congruence ρ on S such that $K = \ker \rho$. It follows from [10, Lemma 2.7] that K is sr-normal if and only if π_K is a strongly regular congruence on S. In this case, π_K is the greatest strongly regular congruence on S with kernel K.

We now give the concept of a strongly regular congruence triple for E-inversive semigroups.

Definition 4.4. Let S be E-inversive semigroups and K a sr-normal subset of S. Let $\gamma \in \mathrm{Eq}(S/\mathcal{L}^*)$ and $\delta \in \mathrm{Eq}(S/\mathcal{R}^*)$ be two sr-normal equivalences on $E(S)$. The triple (γ, K, δ) is called a strongly regular congruence triple for S if:

(i) $\bar{\gamma} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{L}^*$, $\bar{\delta} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}^*$;
(ii) $K \subseteq \ker \bar{\gamma}^0$, $\bar{\gamma} \subseteq \theta_K^0 \vee \mathcal{L}^*$;
(iii) $K \subseteq \ker \bar{\delta}^0$, $\bar{\delta} \subseteq \theta_K^0 \vee \mathcal{R}^*$; and
(iv) $\pi_K \cap \mathcal{H}_{\mathrm{tr}(\bar{\gamma}\cap\bar{\delta})^0}$ is a strongly regular congruence on S.

In such a case we define

$$\rho_{(\gamma,K,\delta)} = (\bar{\gamma} \cap \theta_K \cap \bar{\delta})^0.$$

The following result describes strongly regular congruences on E-inversive semigroups in terms of strongly regular congruence triples, which generalizes [13, Theorem 6.6] for regular semigroups.

Theorem 4.5. *Let (γ, K, δ) be a strongly regular congruence triple for S. Then $\rho_{(\gamma,K,\delta)}$ is the unique strongly regular congruence ρ on S such that γ is the strongly regular $\mathcal{L}^*$-part of ρ, $K = \ker \rho$ and δ is the strongly regular $\mathcal{R}^*$-part of ρ.*

Conversely, let ρ be a strongly regular congruence on S. Then $(\gamma, K, \delta) = ((\rho \vee \mathcal{L}^*)/\mathcal{L}^*, \ker \rho, (\rho \vee \mathcal{R}^*)/\mathcal{R}^*)$ is a strongly regular congruence triple for S and $\rho = \rho_{(\gamma,K,\delta)}$.

Proof. If (γ, K, δ) is a strongly regular congruence triple, then

$$\begin{aligned}
\ker \rho_{(\gamma,K,\delta)} &= \ker(\bar{\gamma} \cap \theta_K \cap \bar{\delta})^0 \\
&= \ker(\bar{\gamma}^0 \cap \theta_K^0 \cap \bar{\delta}^0) \quad \text{(by [13, Result 1.4])} \\
&= \ker \bar{\gamma}^0 \cap \ker \theta_K^0 \cap \ker \bar{\delta}^0 \quad \text{(by [13, Lemma 2.5])} \\
&= \ker \bar{\gamma}^0 \cap K \cap \ker \bar{\delta}^0 \quad \text{(since } K \text{ is a sr-normal subset of } S) \\
&= K \quad \text{(since } K \subseteq \ker \bar{\gamma}^0 \text{ and } K \subseteq \ker \bar{\delta}^0).
\end{aligned}$$

Let $\tau = \mathrm{tr}(\bar{\gamma} \cap \bar{\delta})^0$. Then $\bar{\gamma} = \mathcal{L}_\tau^*$ and $\bar{\delta} = \mathcal{R}_\tau^*$. From Theorem 2.3 in [10], $\mathcal{H}_\tau = (\mathcal{L}_\tau^* \cap \mathcal{R}_\tau^*)^0 = \bar{\gamma}^0 \cap \bar{\delta}^0$. Notice that $\pi_K = \theta_K^0$. By Definition 4.4(iv), we may now conclude that $\rho_{(\gamma,K,\delta)}$ is a strongly regular congruence on S.

Since (γ, K, δ) is a strongly regular congruence triple,

$$\begin{aligned}
\rho_{(\gamma,K,\delta)} \vee \mathcal{R}^* &= (\bar{\gamma} \cap \theta_K \cap \bar{\delta})^0 \vee \mathcal{R}^* \\
&= ((\bar{\gamma} \cap \bar{\delta})^0 \cap \theta_K^0) \vee \mathcal{R}^* \quad \text{(by [13, Result 1.4])} \\
&= ((\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}^*) \cap (\theta_K^0 \vee \mathcal{R}^*) \quad \text{(by Theorem 3.7)} \\
&= \bar{\delta} \cap (\theta_K^0 \vee \mathcal{R}^*) \quad \text{(since } \bar{\delta} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}^*) \\
&= \bar{\delta} \quad \text{(since } \bar{\delta} \subseteq \theta_K^0 \vee \mathcal{R}^*),
\end{aligned}$$

and so $(\rho_{(\gamma,K,\delta)} \vee \mathcal{R}^*)/\mathcal{R}^* = \delta$ is the strongly regular $\mathcal{R}^*$-part of $\rho_{(\gamma,K,\delta)}$. Dually, γ is the strongly regular $\mathcal{L}^*$-part of $\rho_{(\gamma,K,\delta)}$.

Let (γ, K, δ) be a strongly regular congruence triple and $\theta \in \mathrm{SRC}(S)$ such that γ is the $\mathcal{L}^*$-part of θ, $K = \ker\theta$ and δ is the strongly regular $\mathcal{R}^*$-part of θ. Then $\ker\theta = K = \ker\rho_{(\gamma,K,\delta)}$. Furthermore,

$$\begin{aligned}\mathrm{tr}\,\theta &= \mathrm{ltr}\,\theta \cap \mathrm{rtr}\,\theta \quad \text{(by Lemma 3.3)}\\ &= \mathrm{tr}(\theta \vee \mathcal{L}^*)^0 \cap \mathrm{tr}(\theta \vee \mathcal{R}^*)^0\\ &= \mathrm{tr}\,\bar{\gamma}^0 \cap \mathrm{tr}\,\bar{\delta}^0.\end{aligned}$$

Similarly, $\mathrm{tr}\,\rho_{(\gamma,K,\delta)} = \mathrm{tr}\,\bar{\gamma}^0 \cap \mathrm{tr}\,\bar{\delta}^0$.

From [10, Corollary 2.5], we may now conclude that $\theta = \rho_{(\gamma,K,\delta)}$.

Conversely, let $\rho \in \mathrm{SRC}(S)$ and let $\gamma = (\rho \vee \mathcal{L}^*)/\mathcal{L}^*$, $K = \ker\rho$ and $\delta = (\rho \vee \mathcal{R}^*)/\mathcal{R}^*$. By Lemma 4.3 and its dual, γ and δ are sr-normal equivalences on $\mathrm{Eq}(S/\mathcal{L}^*)$ and $\mathrm{Eq}(S/\mathcal{R}^*)$, respectively. It is clear that K is a sr-normal subset of S. We note that $\bar{\gamma} = \rho \vee \mathcal{L}^*$ and $\bar{\delta} = \rho \vee \mathcal{R}^*$. Since $\mathcal{R}^* \subseteq \bar{\delta}$, $(\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}^* \subseteq \bar{\delta}$. Further, since $\rho \subseteq (\bar{\gamma} \cap \bar{\delta})^0$, it follows that $\mathrm{tr}\,\rho \subseteq (\bar{\gamma} \cap \bar{\delta})^0$. Together with Lemma 1.5(iii), we may conclude that

$$\bar{\delta} = \rho \vee \mathcal{R}^* = \mathcal{R}^*_\tau \subseteq (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}^*.$$

Therefore, $\bar{\delta} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}^*$. A dual reasoning gives $\bar{\gamma} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{L}^*$, so (i) is satisfied.

From $\rho \subseteq (\rho \vee \mathcal{R}^*)^0 = \bar{\delta}^0$, $K = \ker\rho \subseteq \ker\bar{\delta}^0$. Moreover, it follows from $\ker\rho = K$ that $\rho \subseteq \theta_K^0$. Therefore,

$$\bar{\delta} = \rho \vee \mathcal{R}^* \subseteq \theta_K^0 \vee \mathcal{R}^*.$$

Dually, we have $K \subseteq \bar{\gamma}^0$ and $\bar{\gamma} \subseteq \theta_K^0 \vee \mathcal{L}^*$, so (ii) and (iii) are satisfied.

Further, $\rho \subseteq (\bar{\gamma}^0 \cap \bar{\delta}^0)$ and $\rho \subseteq \pi_K$. Hence $\rho \subseteq (\pi_K \cap \mathcal{H}_{\mathrm{tr}(\bar{\gamma}\cap\bar{\delta})^0})$, and so $\pi_K \cap \mathcal{H}_{\mathrm{tr}(\bar{\gamma}\cap\bar{\delta})^0}$ is a strongly regular congruence on S. Thus (iv) is satisfied.

It follows that (γ, K, δ) is a strongly regular congruence triple. From the first part of the proof we may now conclude that $\rho = \rho_{(\gamma,K,\delta)}$. □

The following result gives a simple expression for $\rho_{(\gamma,K,\delta)}$.

Proposition 4.6. *Let (γ, K, δ) be a strongly regular congruence triple for an E-inversive semigroup S. Then for any $a, b \in S$,*

$$a\rho_{(\gamma,K,\delta)}b \Leftrightarrow \mathcal{L}^*_a\gamma\mathcal{L}^*_b,\quad \mathcal{R}^*_a\delta\mathcal{R}^*_b \quad \text{and} \quad ab' \in K \quad \text{for all } b' \in W(b).$$

Proof. Let $\tau = \mathrm{tr}\,\rho_{(\gamma,K,\delta)}$. Then

$$\begin{aligned}\bar{\delta} &= \rho_{(\gamma,K,\delta)} \vee \mathcal{R}^* \quad \text{(by Theorem 4.5)}\\ &= \mathcal{R}^*_\tau \quad \text{(by Lemma 1.5(iii))}.\end{aligned}$$

Therefore, for $a, b \in S$, we may conclude that

$$\mathcal{R}^*_a\delta\mathcal{R}^*_b \Leftrightarrow a\mathcal{R}^*_\tau b$$

and a dual reasoning yields

$$\mathcal{L}^*_a\gamma\mathcal{L}^*_b \Leftrightarrow a\mathcal{L}^*_\tau b.$$

The result now follows immediately from [10, Corollary 2.10]. □

The following two results are very easily proved by Theorems 3.7 and 4.5.

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Proposition 4.7. *Let* SRCT(S) *be the poset of all strongly regular congruence triples for E-inversive semigroups under the partial order given by*

$$(\gamma, K, \delta) \subseteq (\gamma', K', \delta') \Leftrightarrow \gamma \subseteq \gamma', K \subseteq K', \delta \subseteq \delta'.$$

Then the mappings

$$\rho \longrightarrow ((\rho \vee \mathcal{L}^*)/\mathcal{L}^*, \ker\rho, (\rho \vee \mathcal{R}^*)/\mathcal{R}^*), \quad (\gamma, K, \delta) \longrightarrow \rho_{(\gamma,K,\delta)}$$

are mutually inverse isomorphisms of SRC(S) *and* SRCT(S).

Proposition 4.8. *Let* $(\gamma_i, K_i, \delta_i), i = 1, 2$ *be strongly regular congruence triples for* S. *Let* $\rho_i = \rho_{(\gamma_i,K_i,\delta_i)}, i = 1, 2$. *Then*

$$\rho_1 T_l \rho_2 \Leftrightarrow \gamma_1 = \gamma_2,$$
$$\rho_1 K \rho_2 \Leftrightarrow K_1 = K_2,$$
$$\rho_1 T_r \rho_2 \Leftrightarrow \delta_1 = \delta_2.$$

The following result gives the close relationship among sr-normal equivalences.

Proposition 4.9. *Let* $\gamma \in \text{Eq}(S/\mathcal{L}^*)$ *and* $\delta \in \text{Eq}(S/\mathcal{R}^*)$ *be sr-normal equivalences such that Definition 4.4(i) is satisfied. Then*

$$\tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)} = \text{tr}(\bar{\gamma} \cap \bar{\delta})^0$$

is a sr-normal equivalence on $E(S)$ *and*

$$\gamma = \mathcal{L}^*_\tau/\mathcal{L}^*, \quad \delta = \mathcal{R}^*_\tau/\mathcal{R}^*.$$

Conversely, if τ *is a sr-normal equivalence on* $E(S)$, *then* $\gamma = \mathcal{L}^*_\tau/\mathcal{L}^* \in \text{Eq}(S/\mathcal{L}^*)$ *and* $\delta = \mathcal{R}^*_\tau/\mathcal{R}^* \in \text{Eq}(S/\mathcal{R}^*)$ *are sr-normal equivalences such that Definition 4.4(i) is satisfied and* $\tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)}$.

Proof. Let γ and δ be sr-normal equivalences satisfying Definition 4.4(i). We put $\rho = (\bar{\gamma} \cap \bar{\delta})^0$ and $\tau = \text{tr}\,\rho$. Then

$$\begin{aligned}\tau &= \tau\mathcal{L}^*\tau \cap \tau\mathcal{R}^*\tau \quad \text{(by Lemma 1.5(v))}\\ &= (\rho \vee \mathcal{L}^*)|_{E(S)} \cap (\rho \vee \mathcal{R}^*)|_{E(S)} \quad \text{(by Lemma 1.5(iv))}\\ &= \bar{\gamma}|_{E(S)} \cap \bar{\delta}|_{E(S)} \quad \text{(since Definition 4.4(i) is satisfied)}\\ &= (\bar{\gamma} \cap \bar{\delta})|_{E(S)}.\end{aligned}$$

We conclude that $\text{tr}(\bar{\gamma} \cap \bar{\delta})^0 = \tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)}$ is a sr-normal equivalence on $E(S)$.

Conversely, let τ be a sr-normal equivalence on $E(S)$, then there exists $\rho \in \text{SRC}(S)$ such that $\tau = \text{tr}\,\rho$. From Theorem 4.5, we may conclude that $\gamma = (\rho \vee \mathcal{L}^*)/\mathcal{L}^*$ and $\delta = (\rho \vee \mathcal{R}^*)/\mathcal{R}^*$ are sr-normal equivalences on $S/\mathcal{L}^*$ and $S/\mathcal{R}^*$, respectively. By Lemma 1.5(iii) and its dual, $\gamma = \mathcal{L}^*_\tau/\mathcal{L}^*$ and $\delta = \mathcal{R}^*_\tau/\mathcal{R}^*$. Further,

$$\begin{aligned}(\bar{\gamma} \cap \bar{\delta})|_{E(S)} &= ((\rho \vee \mathcal{L}^*) \cap (\rho \vee \mathcal{R}^*))|_{E(S)}\\ &= (\rho \vee \mathcal{L}^*)|_{E(S)} \cap (\rho \vee \mathcal{R}^*)|_{E(S)}\\ &= \tau\mathcal{L}^*\tau \cap \tau\mathcal{R}^*\tau \quad \text{(by Lemma 1.5(iv))}\\ &= \tau \quad \text{(by Lemma 1.5(v))}.\end{aligned}$$

The proposition is therefore proved. □

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Recall from [10, Definition 2.8] that the concept of a regular congruence pair for E-inversive semigroups was introduced. The connection between a strongly regular congruence triple and a regular congruence pair is provided by the following proposition.

Proposition 4.10. *Let (γ, K, δ) be a strongly regular congruence triple. If $\tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)}$, then (K, τ) is a regular congruence pair and $\rho_{(\gamma,K,\delta)} = \rho_{(K,\tau)}$.*

Let (K, τ) be a regular congruence pair. If $\gamma = \mathcal{L}^*_\tau/\mathcal{L}^*$ and $\delta = \mathcal{R}^*_\tau/\mathcal{R}^*$, then (γ, K, δ) is a strongly regular congruence triple and $\rho_{(K,\tau)} = \rho_{(\gamma,K,\delta)}$.

Proof. Let (γ, K, δ) be a strongly regular congruence triple. As in the proof of Theorem 4.5, we may prove that

$$\operatorname{tr}\rho_{(\gamma,K,\delta)} = \operatorname{tr}\bar{\gamma}^0 \cap \operatorname{tr}\bar{\delta}^0.$$

From Proposition 4.9, we may now conclude that

$$\tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)} = \operatorname{tr}(\bar{\gamma} \cap \bar{\delta})^0 = \operatorname{tr}\rho_{(\gamma,K,\delta)}.$$

On the other hand, by Theorem 4.5, we have that $K = \ker\rho_{(\gamma,K,\delta)}$. It follows from [10, Theorem 2.9] that (K, τ) is a regular congruence pair and that $\rho_{(\gamma,K,\delta)} = \rho_{(K,\tau)}$.

Conversely, if (K, τ) is a regular congruence pair, then

$$\gamma = \mathcal{L}^*_\tau/\mathcal{L}^* = (\rho_{(K,\tau)} \vee \mathcal{L}^*)/\mathcal{L}^*$$

and similarly,

$$\delta = \mathcal{R}^*_\tau/\mathcal{R}^* = (\rho_{(K,\tau)} \vee \mathcal{R}^*)/\mathcal{R}^*.$$

Therefore,

$$K = \ker\rho_{(K,\tau)}.$$

By Theorem 4.5, we may conclude that (γ, K, δ) is a strongly regular congruence triple and that $\rho_{(\gamma,K,\delta)} = \rho_{(K,\tau)}$. □

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