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The Operator Biprojectivity of the Fourier Algebra

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Abstract. In this paper, we investigate projectivity in the category of operator spaces. In particular, we show that the Fourier algebra of a locally compact group G is operator biprojective if and only if G is discrete.

1 Introduction

In [13], [14] Khelemskii investigated various homological properties in the category of Banach and topological algebras. (See also Taylor's work in [22].) He was able to show that the group algebra $\mathbf{L}^1(G)$ is biprojective in the category of Banach spaces if and only if G is compact. When G is abelian, by using the Fourier transform we can recognize Khelemskii's theorem in terms of the Fourier algebra. Recall that the Fourier transform allows us to naturally identify $\mathbf{L}^1(G)$ with $\mathbf{A}(\hat{G})$. Using the Pontryagin duality theorem, we easily conclude that for abelian groups, the Fourier algebra $\mathbf{A}(G)$ is biprojective (as a Banach algebra) if and only if G is discrete.

The goal of this paper is to study the biprojectivity of $\mathbf{A}(G)$ for all locally compact groups. The previous remarks suggest the possibility that all discrete groups are good candidates for $\mathbf{A}(G)$ to be biprojective, however in the category of Banach spaces this is not the case. Indeed, Khelemskii has shown that if $\mathcal{A}$ is a Banach algebra with a bounded approximate identity which is biprojective as a Banach algebra, then $\mathcal{A}$ is amenable as a Banach algebra. Although it is still an open problem, it seems likely that $\mathbf{A}(G)$ is amenable as a Banach algebra if and only if G has an abelian subgroup of finite index. Thus we are immediately restricted to a relatively small class of groups.

In fact, for the class of groups possessing an abelian subgroup of finite index, H. Steiniger (see [21]) showed that if G is a discrete group, then $\mathbf{A}(G)$ is biprojective as a Banach algebra. An alternative proof of this fact as well as its converse is provided in Theorem 4.7.

However, in [19] Ruan was able to show that when we view $\mathbf{A}(G)$ as a natural operator space under the structure it inherits from being a predual of a von Neumann algebra, $\mathbf{A}(G)$ is operator amenable if and only if G is amenable. This is the analogue of B. Johnson's famous theorem for the group algebra (see [12]). In this paper, we shall show that under this operator space structure, $\mathbf{A}(G)$ is biprojective if and only if G is discrete.

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In addition to his result, Ruan's work suggests that the category of operator spaces is the correct place to work when studying the Fourier algebra. In the author's opinion, this paper provides further evidence for this.

2 Notation and Preliminaries

Let G be a locally compact group, and let $\mathbf{L}^1(G)$ denote the group algebra. We let $\mathbf{A}(G)$ denote the subspace of $\mathcal{C}_0(G)$ consisting of functions of the form

$$u(x) = \sum_{i=1}^{\infty} \widehat{(f_i * \tilde{g}_i)}(x)$$

where $f_i, g_i \in \mathbf{L}^2(G)$, $\sum_{i=1}^{\infty} \|f_i\|_2 \, \|g_i\|_2 < \infty$, $\hat{f}(x) = f(x^{-1})$ and $\tilde{f}(x) = \overline{f(x^{-1})}$. $\mathbf{A}(G)$ is a commutative Banach algebra with respect to pointwise operations and the norm

$$\|u\|_{\mathbf{A}(G)} = \inf\Big\{ \sum_{i=1}^{\infty} \|f_i\|_2 \, \|g_i\|_2 \;\Big|\; u = \sum_{i=1}^{\infty} \widehat{(f_i * \tilde{g}_i)} \Big\},$$

called the *Fourier algebra* of G [9]. If G is abelian, then $\mathbf{A}(G)$ is simply the Fourier transform of the group algebra of $\hat{G}$. $\mathbf{A}(G)$ was introduced for non-commutative groups by Eymard in [9]. In any case, $\mathbf{A}(G)$ is a subspace of $C_0(G)$, the space of continuous functions vanishing at infinity. We highlight the very important fact that if G is not abelian, then $\hat{G}$ is not a group and thus there is no way for us to have an isomorphism between $\mathbf{A}(G)$ and the group algebra of $\hat{G}$. For each closed set $E \subset G$ we define the subspace $I(E)$ as follows:

$$I(E) = \{u \in \mathbf{A}(G) : u(x) = 0 \text{ for all } x \in E\}.$$

It is easy to see that $I(E)$ is a closed ideal in $\mathbf{A}(G)$. Let $\mathbf{VN}(G)$ denote the closure of $\mathbf{L}^1(G)$, considered as an algebra of convolution operators on $\mathbf{L}^2(G)$, with respect to the weak operator topology on $B\big(\mathbf{L}^2(G)\big)$. The von Neumann algebra $\mathbf{VN}(G)$ can be identified with the Banach space dual of $\mathbf{A}(G)$ [9]. An *operator space* is a vector space V together with a family $\| \quad \|_n$ of Banach space norms (called *operator space norms*) on $\mathbb{M}_n(V)$, the space of $n \times n$ matrices with entries in V such that

(i)
$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{n+m} = \max\{\|A\|_n, \|B\|_m\}$$

for each $A \in \mathbb{M}_n(V)$, $B \in \mathbb{M}_m(V)$, and

(ii)
$$\|([a_{ij}])A([b_{ij}])\|_n \le \|[a_{ij}]\| \, \|A\|_n \, \|[b_{ij}]\|$$

for each $[a_{ij}], [b_{ij}] \in \mathbb{M}_n(\mathbb{C})$ and $A \in \mathbb{M}_n(V)$.

Let X and Y be operator spaces and let $T\colon X \mapsto Y$. For each $n \in \mathbb{N}$ define

$$T^{(n)} \colon \mathbb{M}_n(X) \mapsto \mathbb{M}_n(Y)$$

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by

$$T^{(n)}[x_{ij}] = [Tx_{ij}].$$

The map T is said to be *completely bounded* (or simply *c.b.* for short) if $\sup\{\|T^{(n)}\|\} < \infty$. In this case we let $\|T\|_{\mathrm{cb}} = \sup\{\|T^{(n)}\|\}$. We say that T is a *complete isometry* if each $T^{(n)}$ is an isometry and that T is a *complete contraction* if each $T^{(n)}$ is a contraction. We say that two operator spaces X and Y are *c.b. isomorphic* if there exists a c.b. map $T\colon X \mapsto Y$ such that T^{-1} is also completely bounded. Furthermore we shall say that X and Y are *c.b. isometrically isomorphic* (or *completely isometrically isomorphic*) if the map T can be chosen to be a complete isometry. For the Hilbert space H, we let

$$H^{(n)} = \underbrace{H \oplus \cdots \oplus H}_{n}.$$

Since there is a canonical identification between $\mathbb{M}_n\big(B(H)\big)$ and $B(H^{(n)})$, it is easy to show that $B(H)$ (and hence any closed subspace) is an operator space. A fundamental result in the theory is that every operator space is completely isometrically isomorphic to a norm closed subspace S of $B(H)$, the algebra of bounded operators on the Hilbert space H, where the operator space structure on S is the structure inherited from $B(H)$ [6]. If we let $\mathrm{CB}(X, Y)$ denote the space of all completely bounded maps from X to Y, then $\mathrm{CB}(X, Y)$ has a natural operator structure which can be obtained by identifying $\mathbb{M}_n\big(\mathrm{CB}(X, Y)\big)$ with $\mathrm{CB}\big(X, \mathbb{M}_n(Y)\big)$. It is important to note that continuous linear functionals are automatically completely bounded. In fact, since we can identify X^* with $\mathrm{CB}(X, \mathbb{C})$, X^* is also an operator space called the *standard dual* of X (see [2]). For operator spaces X, Y and Z, we call a bilinear map $T\colon X \times Y \mapsto Z$ *jointly completely bounded*, if for $[x_{ij}] \in \mathbb{M}_n(X)$ and $[y_{kl}] \in \mathbb{M}_m(Y)$ we have that

$$\|T\|_{\mathrm{jcb}} = \sup\{\|[T(x_{ij}, y_{kl})]\|_{mn} : \|[x_{ij}]\|_n \le 1, \|[y_{kl}]\|_m \le 1\}$$

is finite. There is an operator space analogue of the projective tensor product which we denote $X \hat{\otimes} Y$ such that

$$\mathrm{JCB}(X, Y; Z) = \mathrm{CB}(X \hat{\otimes} Y, Z).$$

That is to say each jointly completely bounded map extends to a unique map on this *operator space projective tensor product*. In particular, there is a complete isometry between $(X \hat{\otimes} Y)^*$ and $\mathrm{CB}(X, Y^*)$. We can define the norm of a typical element in the operator space projective tensor product with the following. Let $[x_{ij}] \in \mathbb{M}_p(X)$ and $[y_{kl}] \in \mathbb{M}_q(Y)$. We define the tensor product $x \otimes y$ to be the $pq \times pq$ matrix

$$x \otimes y = [x_{ij} \otimes y_{kl}] \in \mathbb{M}_{pq}(X \otimes Y).$$

Given any element $u \in \mathbb{M}_n(X \otimes Y)$, we can write

$$u = \alpha(x \otimes y)\beta$$

for some $\alpha \in \mathbb{M}_{n,pq}(\mathbb{C})$, $x \in \mathbb{M}_p(X)$, $y \in \mathbb{M}_q(Y)$, and $\beta \in \mathbb{M}_{pq,n}(\mathbb{C})$. Now we have that the operator space projective tensor norm is given by

$$\|u\|_n = \inf\{\|\alpha\| \, \|x\| \, \|y\| \, \|\beta\|\}$$

where the infimum is taken over all such representations of u.

An associative algebra $\mathcal{A}$ which is also an operator space and is such that the multiplication

$$m\colon \mathcal{A} \hat{\otimes} \mathcal{A} \mapsto \mathcal{A}$$

is completely contractive is called a *completely contractive Banach algebra*. We note that if X is the predual of a von Neumann algebra $\mathcal{A}$, it inherits a natural operator space structure as follows: for $[x_{ij}] \in \mathbb{M}_n(X)$ we set

$$\|[x_{ij}]\|_n = \sup\{\|[f_{kl}(x_{ij})]\|_{nm} : [f_{kl}] \in \mathbb{M}_m(\mathcal{A}), \|[f_{kl}]\|_m \leq 1\}.$$

Thus the Fourier algebra can be given a natural operator structure by virtue of it being the predual of a von Neumann algebra. In this case, this operator space structure results in a completely contractive Banach algebra (see [19] and [2]). Given two operator spaces X and Y, we can consider the direct sum $X \oplus Y$ to be an operator space where

$$\|[x_{ij} \oplus y_{ij}]\|_n = \max\{\|[x_{ij}]\|_n, \|[y_{ij}]\|_n\}.$$

Unless otherwise noted, whenever we are given the direct sum of two operator spaces, we shall consider it to be an operator space in this way.

A *left Banach $\mathcal{A}$-module* is a left $\mathcal{A}$-module X that is itself a Banach space and for which

$$\|ax\|_X \leq \|a\|_{\mathcal{A}} \, \|x\|_X$$

for each $a \in \mathcal{A}$ and each $x \in X$. A *right* and *two-sided* Banach $\mathcal{A}$ module is defined analogously. We call a two sided module a *bimodule*. If X is a left Banach $\mathcal{A}$-module, then X^* becomes a right Banach $\mathcal{A}$-module with respect to the action

$$(\phi a)(x) = \phi(ax).$$

We call X^* a *dual right Banach $\mathcal{A}$-module*. Naturally we can define dual left and bimodules analogously. In the category of operator spaces there are two ways to define an operator module. In this paper we shall call an operator space X which is a left Banach $\mathcal{A}$-module, a *left operator $\mathcal{A}$-module* (or simply *left $\mathcal{A}$ module* if no confusion arises) if the module map is completely contractive with respect to the operator projective tensor product, that is to say the module map

$$\pi_X\colon \mathcal{A} \hat{\otimes} X \mapsto X$$

is completely contractive. Clearly we may define operator right and bimodules analogously. Furthermore if X is an operator module, then X^* becomes a dual operator module with the dual actions defined above. (The alternative way to construct an operator module is to use the Haagerup tensor product. While this tensor product may in some ways be more natural when studying concrete operator spaces, the operator space projective tensor product is a far better analogue of the Banach space projective tensor product. Indeed, since it is the categorical properties and not the structural properties of operator spaces which are of most interest to us, we shall only make use of the projective tensor product). Suppose X is a left operator $\mathcal{A}$-module and Y an operator space. Then we may consider $X \hat{\otimes} Y$ as a left operator $\mathcal{A}$ module by

$$a \cdot (x \otimes y) = ax \otimes y$$

for $a \in \mathcal{A}$, $x \in X$ and $y \in Y$. It is clear that if Y is a right operator module, then $X \hat{\otimes} Y$ becomes a right operator module in the analogous way. By a *chain complex* we mean a sequence of objects X_n with $n \in \mathbb{Z}$ and morphisms $d_n \colon X_{n+1} \mapsto X_n$ such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. The objects could be Banach spaces, Banach algebras, Banach modules *etc.* and the maps naturally will be respectively Banach space maps, Banach algebra maps, Banach module maps *etc.* Typically, a chain complex is written as

$$(\Xi)\colon \ \cdots \longrightarrow X_n \xrightarrow{d_n} X_{n+1} \longrightarrow \cdots .$$

The condition that $d_{n+1} \circ d_n = 0$ is clearly equivalent to $\operatorname{im} d_{n+1} \subset \ker d_n$.

3 Projectivity in Operator Spaces

Our primary goal in this section is to extend the natural results of projectivity in Banach spaces to the category of completely contractive Banach algebras. In some cases, the proofs of certain results are almost identical to the Banach algebra case which in turn are almost identical to the pure algebra case. In this instance we shall omit the proof. However, in other cases differences emerge between the categories or technical facts relying on the important properties of the objects are required to translate the old results to the new category. We will try to highlight these situations.

One of the basic concepts and tools in homological algebra is that of a short exact sequence. Recall that a short exact sequence of objects in an abelian category is a complex of the form

$$0 \longrightarrow X \xrightarrow{f} A \xrightarrow{g} Y \longrightarrow 0$$

where f is injective, g is surjective and $\ker g = \operatorname{im} f$. We note the expected fact that $A/f(X)$ is isomorphic in the category to Y, and naturally we write $A/f(X) \cong Y$. In this case we say that A is an extension of X by Y. Indeed, this principle holds in both the category of Banach spaces and the category of Banach algebras.

Unfortunately, this concept breaks down in the case of operator spaces. Consider the following short exact sequence:

$$0 \longrightarrow 0 \longrightarrow \mathrm{MAX}(X) \xrightarrow{\mathrm{id}} \mathrm{MIN}(X) \longrightarrow 0$$

where id represents the identity map. If X is any infinite dimensional Banach space, then $\mathrm{MAX}(X)/0 = \mathrm{MAX}(X)$ is not isomorphic in the category of operator spaces to $\mathrm{MIN}(X)$. In this sense, one of the basic objects of homology fails to "do what we

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want" in our new category. The basic way to repair this is to restrict the sequences under consideration. In [20], the authors considered cases where f was a "complete isometry" and g was a "complete quotient" map. In [24] the present author considered sequences where both f and g had inverses which were completely bounded. The first thing we shall now do is establish in a natural way, a broad class of short exact sequences which avoids this isomorphism dilemma, and we will show that in some sense this class is as broad as possible.

Definition 3.1 Given two operator spaces X and Y, we say a c.b. map $T\colon X \mapsto Y$ has the *complete isomorphism property (c.i.p.)* if the image $T(X)$ is closed and the induced map $\tilde{T}\colon X/\ker T \mapsto T(X)$ is a c.b. isomorphism.

We note that any bounded map between Banach spaces satisfies the analogous property. This leads to the following:

Definition 3.2 A chain complex of operator spaces is called an *operator complex* if each of the differential maps has the complete isomorphism property. As discussed earlier, an important special case of a chain complex is of course the short exact sequences. In this paper, we call any short exact operator chain complex an *extension sequence* or 1-*extension sequence*. This leads to the following proposition which suggests that operator complexes are the correct tool for our category:

Proposition 3.3 *Suppose X, Y and Z are operator spaces such that*

$$(\Xi)\colon 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

forms an extension sequence. Then $Y/f(X)$ is c.b. isomorphic to Z.

Proof Since (Ξ) is a short exact sequence, we have $\operatorname{im} f = \ker g$ thus $Y/f(X) = Y/\ker g$. Since (Ξ) is an operator complex, the map g has the complete isomorphism property and hence there is a c.b. isomorphism between $Y/\ker g$ and $\operatorname{im} g = Z$ (by exactness). Thus $Y/f(X)$ is c.b. isomorphic to Z. ∎

Conversely, we have the following:

Proposition 3.4 *Suppose X, Y and Z are operator spaces with $X \subset Y$ such that Y/X is c.b. isomorphic to Z. Then there is an extension sequence (Ξ) of the form*

$$(\Xi)\colon 0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{f} Z \longrightarrow 0$$

where i represents the inclusion map $i\colon X \hookrightarrow Y$.

Proof Consider the canonical quotient map $q\colon Y \mapsto Y/X$. By construction the short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Y/X \longrightarrow 0$$

is an extension sequence. Let $T\colon Y/X \mapsto Z$ be a c.b. isomorphism. Then it is easy to see that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, equal] & Y \arrow[r, "q"] \arrow[d, equal] & Y/X \arrow[d, bend left, "T"] \\
X \arrow[r, "i"] & Y \arrow[r, "f"] & Z \arrow[u, bend left, "T^{-1}"]
\end{tikzcd}$$

Hence the bottom sequence is an extension sequence. ■

Unlike in algebra, it is usually necessary to consider further topological conditions on our extension sequences. An additional condition is that we will require im π_k to be complemented. The importance of this latter condition will become apparent a little later.

Definition 3.5 An exact operator complex of $\mathcal{A}$-modules

$$\cdots \longrightarrow X_{k-1} \xrightarrow{\pi_{k-1}} X_k \xrightarrow{\pi_k} X_{k+1} \xrightarrow{\pi_{k+1}} \cdots$$

is called *admissible* if there exist completely bounded maps (not necessarily $\mathcal{A}$-module maps) $\theta_k\colon X_{k+1} \mapsto X_k$ such that $\pi_k \circ \theta_k = \mathrm{id}_{\ker \pi_{k+1}}$. An admissible complex is said to *split* if the maps θ_k can be chosen to be module maps. Thus a complex is admissible exactly when it splits as a complex of $\mathbb{C}$-modules. Finally, we call a map $\phi\colon X \mapsto Y$ *admissible* if there exists a map $\theta\colon Y \mapsto X$ such that $\phi \circ \theta = \mathrm{id}_{\mathrm{im}\,\phi}$. Furthermore, we call this map θ a *right inverse* for ϕ. We now have an analogue of Proposition 1.1 from [5]. See also the special case of this in [20].

Lemma 3.6 *Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an extension sequence of $\mathcal{A}$-bimodules. Then there exists a completely bounded map $F\colon Y \mapsto X$ such that $Ff = \mathrm{id}_X$ if and only if there exists a map $G\colon Z \mapsto Y$ such that $gG = \mathrm{id}_Z$. Furthermore F is a module map if and only if G is.

Proof Suppose F exists. Then clearly the map fF is a completely bounded projection onto im $f \subset Y$. Thus the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[r, "g"] \arrow[d, "fF\oplus(1-fF)"] & Z \arrow[d, equal] \\
f(X) \arrow[r, hook, "i"] & f(X) \oplus Q \arrow[r, "g"] & Z
\end{tikzcd}$$

commutes, where Q is the complement of $f(X)$ in Y and the map $i\colon f(X) \mapsto Y$ is given by $i(x) = x \oplus 0$. Since $\operatorname{im} f = f(X) = \ker g$ and since g has the c.i.p., it follows that the induced map

$$\tilde{g}\colon Y/f(X) \mapsto Z$$

is a c.b. isomorphism. Since $Y/f(X) \cong Q$ it follows that $g|_Q$ is a c.b. isomorphism. Now let $G(z) = (g|_Q)^{-1}$. The fact that $gG = \mathrm{id}_Z$ is now trivial. Now assume that G exists. Similar to above, we see that Gg is a completely bounded projection onto a subspace P of Y which is c.b. isomorphic to Z. Let Q be the complement of P. Thus $1 - Gg$ is a completely bounded projection onto Q. Note that

$$g(1 - gG) = g - gGg = g - g = 0$$

thus $Q \subset \ker g = \operatorname{im} f$. Since $g|_P$ is an isomorphism, the reverse inclusion is obvious. Hence $Q = \ker g = \operatorname{im} f$. Since f has the c.i.p., the map $f^{-1}\colon Q \mapsto X$ is completely bounded. Thus we define

$$F\colon Y \mapsto Z$$

by

$$F(y) = f^{-1}\big(y - Gg(y)\big).$$

Clearly F satisfies the desired properties. The fact that F is a module map if and only if G is a module map is strictly algebraic. Suppose that G is a module map. Note that

$$F(ay) = f^{-1}\big(ay - Gg(ay)\big) = af^{-1}\big(y - Gg(y)\big) = aF(y).$$

The right module action is similar. Conversely if F is a module map, then the subspace Q is a submodule, hence $G|_Q$ is a module map. Thus G is a module map also. ■

We note that the above lemma fails for general short exact sequences. Consider the MAX / MIN example at the beginning of this section, see also [24]. It will arise that we will be given a completely contractive Banach algebra $\mathcal{A}$ and a left operator $\mathcal{A}$-module X, and we will wish to know when the module map

$$\pi\colon \mathcal{A} \hat{\otimes} X \mapsto X$$

has an inverse. Obviously this is impossible immediately whenever the map π is not onto. Recall that a module is called *neounital* if

$$\mathcal{A} \cdot X = \{a \cdot x : a \in \mathcal{A}, x \in X\} = X,$$

in which case π is clearly onto. If our completely contractive Banach algebra has a bounded approximate identity $\{e_\alpha\}$, and if $x = \lim e_\alpha x$ for all $x \in X$, then we can use Cohen's Factorization theorem to guarantee X is neounital. Indeed when studying $\mathbf{L}^1(G)$ this can be a useful approach. Once again, in our setting this method will fail us. In our primary example, $\mathbf{A}(G)$ does not have a bounded approximate identity when G is not amenable, and indeed it is known (see [17]) that for non-amenable groups G, $\mathbf{A}(G)$ is not a neounital $\mathbf{A}(G)$ module. Thus we make the following definition:

Definition 3.7 A left operator $\mathcal{A}$-module is called *semi-neounital* whenever the multiplication map

$$\pi\colon \mathcal{A} \hat{\otimes} X \longrightarrow X$$

is onto. We have the analogous definition for right and bimodules.

Given a Banach algebra $\mathcal{A}$ we can construct its unitization $\mathcal{A}_+$ as follows: Let $\mathcal{A}_+ = \mathcal{A} \oplus \mathbb{C}$, and now define multiplication by

$$(a, \alpha) \cdot (b, \beta) = (ab + \alpha b + \beta a, \alpha\beta).$$

Now note that if X is a left (right, bi-) $\mathcal{A}$-module, then X becomes a unital left (right, bi-) $\mathcal{A}_+$-module with respect to the action

$$(a, \alpha) \cdot x = a \cdot x + \alpha x$$

(a right and bimodule structure is defined analogously). In [8] Effros and Ruan showed that there is an operator space structure such that $\mathcal{A}_+$ is indeed a completely contractive Banach algebra. Using the same techniques, we can show that any operator $\mathcal{A}$ module X becomes a unital operator $\mathcal{A}_+$ module. For completeness, we recall their construction for $\mathcal{A}_+$, then we extend this in the obvious way to show X is an operator $\mathcal{A}_+$ module.

Suppose we are given two operator spaces V and W. We construct the operator space $V^* \oplus W^*$. The induced operator space structure on the predual will be denoted $V \oplus_1 W$. We have the following important fact concerning this structure:

Proposition 3.8 *Suppose X is an operator space and suppose that*

$$\phi\colon V \longrightarrow X \quad \text{and} \quad \psi\colon W \longrightarrow X$$

are completely contractive. Then the map

$$\phi \oplus_1 \psi\colon V \oplus_1 W \longrightarrow X$$

given by

$$(\phi \oplus_1 \psi)(v \oplus_1 w) = \phi(v) + \psi(w)$$

is completely contractive.

Now we give $\mathcal{A}_+$ the operator space structure $\mathcal{A} \oplus_1 \mathbb{C}$. Using the previous proposition, Effros and Ruan have shown that if $\mathcal{A}$ is a completely contractive Banach algebra then so is $\mathcal{A}_+$. Using the same technique, we have:

Lemma 3.9 *If X is a left (right, bi-) operator $\mathcal{A}$ module, then X is a neounital left (right, bi-) operator $\mathcal{A}_+$ module.*

Proof The fact that X is neounital is obvious. Now we simply note that the module map $(a, \alpha) \cdot x \mapsto a \cdot x + \alpha x$ is the sum of two completely contractive maps which, by Proposition 3.8 is clearly completely contractive. Thus X is a left operator $\mathcal{A}_+$ module (see [8]). The right and bimodule cases follow analogously. ■

Our basic goal in this section is to study projectivity in the category of operator spaces. Thus we introduce the following natural notation:

$$\mathrm{CB}_{\mathcal{A},\mathbb{C}}(X,Z) = \{T \in \mathrm{CB}(X,Z) \mid T(ax) = aT(x)\ \forall x \in X, a \in \mathcal{A}\}$$
$$\mathrm{CB}_{\mathbb{C},\mathcal{A}}(X,Z) = \{T \in \mathrm{CB}(X,Z) \mid T(xa) = T(x)a\ \forall x \in X, a \in \mathcal{A}\}$$
$$\mathrm{CB}_{\mathcal{A},\mathcal{A}}(X,Z) = \{T \in \mathrm{CB}(X,Z) \mid T(axb) = aT(x)b\ \forall x \in X, a, b \in \mathcal{A}\}.$$

Naturally, these sets define respectively the morphisms in the category of left, right and two-sided operator modules. Given X and Y, left operator $\mathcal{A}$ modules, we can define a contravariant functor denoted $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(?, Z)$ as follows: for any c.b. module map

$$\phi \colon X \mapsto Y$$

we define

$$\mathrm{CB}_{\mathcal{A},\mathbb{C}}(\phi, Z) = \phi_* \colon\ \mathrm{CB}_{\mathcal{A},\mathbb{C}}(Y,Z) \mapsto \mathrm{CB}_{\mathcal{A},\mathbb{C}}(X,Z)$$

given by

$$\phi_*(T)(x) = T\big(\phi(x)\big)\,.$$

Clearly we can define the contravariant functors $\mathrm{CB}_{\mathbb{C},\mathcal{A}}(?, Z)$ and $\mathrm{CB}_{\mathcal{A},\mathcal{A}}(?, Z)$ analogously. Furthermore, using the obvious changes, we can define covariant functors $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(Z, ?)$, $\mathrm{CB}_{\mathbb{C},\mathcal{A}}(Z, ?)$ and $\mathrm{CB}_{\mathcal{A},\mathcal{A}}(Z, ?)$. To see that ϕ_* is completely bounded, we have the following: for $[x_{kl}] \in \mathbb{M}_m(X)$ and $[T_{ij}] \in \mathbb{M}_n\big(\mathrm{CB}_{\mathcal{A},\mathbb{C}}(X,Z)\big)$

$$\begin{aligned}
\|\phi_*^{(n)}\|_n &= \sup\{\|\phi_*^{(n)}([T_{ij}])([x_{kl}])\| : \|[T_{ij}]\|_n \le 1, \|[x_{kl}]\|_m \le 1\} \\
&= \sup\Big\{ \Big\| \big[T_{ij}\big(\phi(x_{kl})\big) \big] \Big\|_{nm} : \|[T_{ij}]\|_n \le 1, \|[x_{kl}]\|_m \le 1 \Big\} \\
&\le \sup\{\|[T_{ij}]\|_n \, \|[\phi(x_{kl})]\|_m : \|[T_{ij}]\|_n \le 1, \|[x_{kl}]\|_m \le 1\} \\
&\le \sup\{\|[T_{ij}]\|_n \, \|\phi\|_{\mathrm{cb}} \, \|[x_{kl}]\|_m : \|[T_{ij}]\|_n \le 1, \|[x_{kl}]\|_m \le 1\} \\
&\le \|\phi\|_{\mathrm{cb}}\,.
\end{aligned}$$

The reader may recall that there is a second functor of interest in homology theory, namely the tensor product functor which we shall now introduce in our category. Suppose we are given two operator $\mathcal{A}$-bimodules X and Y. We define the tensor product $X \otimes_{\mathcal{A}} Y$ as follows: Consider the operator subspace N of $X \hat{\otimes} Y$ given by the closed linear span of elements of the form

$$xa \otimes y - x \otimes ay.$$

Now define $X \otimes_{\mathcal{A}} Y$ by

$$X \otimes_{\mathcal{A}} Y = X \hat{\otimes} Y / N.$$

Although we shall not utilize the functorial properties in this paper, it is worthwhile noting that we can recognize $? \otimes_{\mathcal{A}} Z$ as a covariant functor as follows: for any c.b. module map

$$\phi\colon X \mapsto Y$$

we have

$$\phi \otimes_{\mathcal{A}} Z = \phi_*\colon X \otimes_{\mathcal{A}} Z \mapsto Y \otimes_{\mathcal{A}} Z$$

given by

$$\phi_*(x \otimes_{\mathcal{A}} z) = \phi(x) \otimes_{\mathcal{A}} z.$$

To see that ϕ_* is completely bounded we first note that the map

$$\phi \hat{\otimes} \mathrm{id}_Z\colon X \hat{\otimes} Z \mapsto Y \hat{\otimes} Z$$

is completely bounded with $\|\phi \hat{\otimes} \mathrm{id}_Z\|_{\mathrm{cb}} \leq \|\phi\|_{\mathrm{cb}}$ (see [3]). Then since the following diagram is commutative

$$\begin{tikzcd}
X \hat{\otimes} Z \arrow[r, "\phi\hat{\otimes}\mathrm{id}_Z"] \arrow[d, "q_1"'] & Y \hat{\otimes} Z \arrow[d, "q_2"] \\
X \otimes_{\mathcal{A}} Z \arrow[r, "\phi_*"'] & Y \otimes_{\mathcal{A}} Z
\end{tikzcd}$$

where q_i are the canonical quotients, it follows that ϕ_* is completely bounded. Using identical arguments, it is now easy to see how to construct a covariant functor $X \otimes_{\mathcal{A}} ?$.

Definition 3.10 A left operator $\mathcal{A}$-module Y is called *(left) projective* if, for any admissible complex Ξ, the complex $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(Y,\Xi)$ is exact. That is to say if

$$(\Xi)\colon \cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots$$

is admissible, then the complex

$$\begin{aligned}\big(\mathrm{CB}_{\mathcal{A},\mathbb{C}}(Y,\Xi)\big)\colon \cdots \longrightarrow \mathrm{CB}_{\mathcal{A},\mathbb{C}}(Y,X_{n-1}) &\longrightarrow \mathrm{CB}_{\mathcal{A},\mathbb{C}}(Y,X_n)\\ \longrightarrow \mathrm{CB}_{\mathcal{A},\mathbb{C}}(X_{n+1},Y) &\longrightarrow \cdots\end{aligned}$$

is exact. If Y is a right module, we call Y *(right) projective* if $\mathrm{CB}_{\mathbb{C},\mathcal{A}}(Y,\Xi)$ is exact. Finally if Y is a bimodule, we shall say Y is *biprojective* or *projective as a bimodule* if $\mathrm{CB}_{\mathcal{A},\mathcal{A}}(Y,\Xi)$ is exact. Of special note is that an object may be projective in one category, while not in another. The following theorem is well known in both the categories of linear spaces and Banach spaces and the proof is similar and thus omitted. (See for example [18] and [14]).

Theorem 3.11 *Let X be a left operator $\mathcal{A}$-module. Then the following are equivalent*

(1) *X is projective*
(2) *for any admissible extension sequence (Ξ), $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(X,\Xi)$ is exact*
(3) *for any c.b. admissible surjection $\phi\colon Y \mapsto Z$ and any c.b. module map $\theta\colon X \mapsto Z$, there is a c.b. module map $\psi\colon X \mapsto Y$ such that the following diagram commutes*

$$\begin{tikzcd} & X \arrow[d, "\theta"] \arrow[dl, dotted, "\psi"'] \\ Y \arrow[r, "\phi"'] & Z \end{tikzcd}$$

(4) *if Q is a submodule of Y, then every $\theta \in \mathrm{CB}_{\mathcal{A},\mathbb{C}}(X, Y/Q)$ has an extension to $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(X,Y)$.*

It is easy to see that any module is a projective $\mathbb{C}$ module. As a consequence we shall see that for any module E, the module of the form $\mathcal{A}_+ \hat{\otimes} E$, is projective. We require the following *reduction* formula. The idea of the proof is similar to the pure algebra case, except for the matrix calculations.

Proposition 3.12 *$\mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+, X)$ is c.b. isometrically isomorphic to X and $\mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+ \hat{\otimes} X, Y)$ is c.b. isometrically isomorphic to $\mathrm{CB}(X,Y)$ for all X and Y. Similarly we have complete isometries $\mathrm{CB}_{\mathbb{C},\mathcal{A}_+}(\mathcal{A}_+, X) \cong X$ and furthermore $\mathrm{CB}_{\mathbb{C},\mathcal{A}_+}(X \hat{\otimes} \mathcal{A}_+, Y) \cong \mathrm{CB}(X,Y)$.*

Proof Let $T \in \mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+, X)$. Then $T(a) = aT(e)$ for all $a \in \mathcal{A}_+$ where e is the identity element. Let $x_T = T(e)$. The map $T \mapsto x_T$ is clearly a bijection. Also for $[T_{ij}] \in \mathbb{M}_n\big(\mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+, X)\big)$ we have for $[a_{kl}] \in \mathbb{M}_m(\mathcal{A}_+)$

$$\begin{aligned} \|[T_{ij}]\|_n &= \sup\{\|[T_{ij}(a_{kl})]\| : \|[a_{kl}]_m\| \le 1\} \\ &= \sup\{[\|a_{kl} \cdot (x_T)_{ij}]\| : \|a_{kl}\|_m \le 1\} \\ &\le \|[(x_T)_{ij}]\|_n \end{aligned}$$

but if we consider the element $e \in \mathbb{M}_1(\mathcal{A}_+)$ we have

$$\|[T_{ij}]\|_n \ge \|[T_{ij}(e)]\| = \|[(x_T)_{ij}]\|_n.$$

Thus the natural map is a c.b. isometric isomorphism. For the second identification we proceed similarly. As before, it is easy to see that $T \in \mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+ \hat{\otimes} X, Y)$ is defined on elements of the form $e \otimes x$. Thus the map $T \mapsto \bar{T}$ is an isomorphism between $\mathrm{CB}(X,Y)$ and $\mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+ \hat{\otimes} X, Y)$, where $\bar{T}(x) = T(e \otimes x)$. Note that we can identify the space $\mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+ \hat{\otimes} X, Y)$ with $\mathrm{JCB}(\mathcal{A}_+, X; Y)$, the space of maps

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which are jointly completely bounded from $\mathcal{A}_+ \times X$ to Y, such that $T(a, x) = aT(e, x)$. Thus we have

$$\begin{aligned}\|[T_{pq}]\|_n &= \sup\{\|[T_{pq}(a_{ij}, x_{kl})]\| : \|[a_{ij}]\| \le 1, \|[x_{kl}]\| \le 1\} \\ &= \sup\{\|[a_{ij} \cdot T_{pq}(e, x_{kl})]\| : \|[a_{ij}]\| \le 1, \|[x_{kl}]\| \le 1\} \\ &\le \sup\{\|[a_{ij}]\|\, \|[\bar{T}_{pq}(x_{kl})]\| : \|[a_{ij}]\| \le 1, \|[x_{kl}]\| \le 1\} \\ &= \|[\bar{T}_{pq}]\|_n.\end{aligned}$$

The reverse equality follows by taking $a_{ij} = e$ as before. The assertions concerning $\mathrm{CB}_{\mathbb{C},\mathcal{A}_+}(\mathcal{A}_+, X)$ and $\mathrm{CB}_{\mathbb{C},\mathcal{A}_+}(X \hat{\otimes} \mathcal{A}_+, Y)$ are proved in a similar manner. ■

As a consequence of the above proposition, we have the following corollary.

Corollary 3.13 *We have complete isometries* $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+, X) \cong X$ *and* $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+ \hat{\otimes} X, Y) \cong \mathrm{CB}(X, Y)$.

Proof To prove the first equality, it suffices to show that $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+, X) = \mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+, X)$, where $\mathcal{A}_+$ and X are considered as $\mathcal{A}$ modules on the left and $\mathcal{A}_+$ modules on the right. Let $T \in \mathrm{CB}_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+, X)$ and let (a, α) and $(b, \beta) \in \mathcal{A}_+$. Now

$$\begin{aligned}T[(a, \alpha)(b, \beta)] &= T(ab + \beta a + \alpha b, \alpha\beta) \\ &= T[(ab + \beta a, 0) + (\alpha b, \alpha\beta)] \\ &= a \cdot T(b, \beta) + \alpha T(b, \beta) \\ &= (a, \alpha) \cdot T(b, \beta).\end{aligned}$$

A similar calculation shows $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+ \hat{\otimes} X, Y) = \mathrm{CB}_{\mathcal{A}_+,\mathbb{C}}(\mathcal{A}_+ \hat{\otimes} X, Y)$. Now we simply apply the previous Proposition. ■

Corollary 3.14 *Any module of the form* $\mathcal{A}_+ \hat{\otimes} E$ *for any* $\mathcal{A}$*-module* E *is projective as a left operator* $\mathcal{A}$*-module.*

Proof In view of the above proposition, the complex $\mathrm{CB}_{\mathcal{A},\mathbb{C}}(\mathcal{A}_+ \hat{\otimes} E, \Xi)$ reduces to $\mathrm{CB}(E, \Xi)$. Since any module is a projective $\mathbb{C}$ module,it follows that the complex is exact. Hence $\mathcal{A}_+ \hat{\otimes} E$ is projective. ■

It is easy to see that the last four theorems have the obvious generalizations to the category of right and bimodules. In particular we can conclude:

Corollary 3.15 *Any module of the form* $E \hat{\otimes} \mathcal{A}_+$ *(resp.* $\mathcal{A}_+ \hat{\otimes} E \hat{\otimes} \mathcal{A}_+$*) is a projective right (resp. bi-) module for any module E.*

A module of the above form is usually called a *free* module. The following three technical propositions will lead to another important reduction formula.

Proposition 3.16 $(X \otimes_{\mathcal{A}} Y)^*$ *is c.b. isometrically isomorphic to* $\mathrm{CB}_{\mathbb{C},\mathcal{A}}(X, Y^*)$.

Proof We have that the map $\phi \mapsto T_\phi$ given by

$$\langle \phi, x \otimes y \rangle = \langle T_\phi(x), y \rangle$$

is a c.b. isometric isomorphism between $(X \hat{\otimes} Y)^*$ and $\mathrm{CB}(X, Y^*)$ [19]. Now

$$(X \otimes_{\mathcal{A}} Y)^* = \left(\frac{X \hat{\otimes} Y}{N} \right)^* \cong N^\perp$$

where $N = \overline{\mathrm{span}}\{xa \otimes y - x \otimes ay\}$ for $a \in \mathcal{A}$, $x \in X$ and $y \in Y$. Now $\phi \in N^\perp$ if and only if

$$\phi(xa \otimes y - x \otimes ay) = 0.$$

Thus for all $y \in Y$,

$$\langle T_\phi(xa), y \rangle = \langle T_\phi(x), ay \rangle = \langle T_\phi(x)a, y \rangle.$$

Thus $T_\phi \in \mathrm{CB}_{\mathbb{C},\mathcal{A}}(X, Y^*)$. The reverse inclusion is clear. ■

Proposition 3.17 *We have the complete isometric isomorphism $\mathcal{A}_+ \otimes_{\mathcal{A}_+} X \cong X$.*

Proof The map $\phi \colon X \mapsto \mathcal{A}_+ \otimes_{\mathcal{A}_+} X$ given by

$$\phi(x) = e \otimes x$$

is easily seen to be completely contractive. Furthermore the map

$$\phi^* \colon (\mathcal{A}_+ \otimes_{\mathcal{A}_+} X)^* \longrightarrow X^*$$

is exactly the composition of the c.b. isomorphisms of Proposition 3.16 and Proposition 3.12 from $(\mathcal{A}_+ \otimes_{\mathcal{A}_+} X)^*$ to X^*. Thus $(\phi^{-1})^*$ is completely contractive, hence ϕ^{-1} is completely contractive. ■

Now we have the reduction formula:

Corollary 3.18 *We have the complete isometric isomorphism $\mathcal{A}_+ \otimes_{\mathcal{A}} X \cong X$.*

Proof For $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$ and $x \in X$ we have

$$\begin{aligned}(a, \lambda) \otimes_{\mathcal{A}} x &= [(a, 0) + (0, \lambda)] \otimes_{\mathcal{A}} x \\ &= a \cdot e \otimes_{\mathcal{A}} x + \lambda e \otimes_{\mathcal{A}} x \\ &= e \otimes (ax + \lambda x) = e \otimes (a, \lambda)x.\end{aligned}$$

Hence in view of the previous proposition, it follows that we have a complete isometric isomorphism. ■

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Armed with all these technical facts, we are now able to equate projectivity with the splitting of certain exact sequences. For the Banach space versions of these results see [13] and [14].

Lemma 3.19 *Suppose P is a left operator $\mathcal{A}$-module, and let $\pi_L \colon \mathcal{A}_+ \hat{\otimes} P \longrightarrow P$ be the module map onto P and N its kernel. Then the admissible sequence*

$$(\mathfrak{M}_L)\colon 0 \longrightarrow N \longrightarrow \mathcal{A}_+ \hat{\otimes} P \xrightarrow{\pi_L} P \longrightarrow 0$$

splits if and only if P is projective.

Proof First note that the sequence is clearly short exact, and since the map $\tau \colon P \mapsto \mathcal{A}_+ \hat{\otimes} P$ given by $\tau(p) = e \otimes p$ is clearly a completely bounded inverse for π_L, by Lemma 3.6 $(\mathfrak{M}_L)$ is admissible. The remainder of the proof is essentially "diagram chasing" so we omit the details. ■

Clearly we can establish the following corollary with the same methods:

Corollary 3.20 *Suppose P is a right (resp. bi-) module. Then P is right (resp. bi-) projective if and only if the admissible sequence*

$$0 \longrightarrow N \longrightarrow P \hat{\otimes} \mathcal{A}_+ \xrightarrow{\pi_R} P \longrightarrow 0$$

$$(\textit{resp. } 0 \longrightarrow N \longrightarrow \mathcal{A}_+ \hat{\otimes} P \hat{\otimes} \mathcal{A}_+ \xrightarrow{\pi_B} P \longrightarrow 0)$$

splits, where π_R (resp. π_B) is the right (resp. bi-) module map and N its kernel.

A direct application of the previous corollary yields:

Theorem 3.21 *Suppose X is a projective left module and Y is a projective right module, then $X \hat{\otimes} Y$ is a biprojective $\mathcal{A}$-bimodule.*

It should not be hard to see how the techniques of the last two theorems can prove the following general result:

Lemma 3.22 *Suppose that P is a left (resp. right, bi-) projective $\mathcal{A}$-module and $\theta \colon P \to Q$ is a module map with a right inverse which is also a module map. Then Q is also left (right, bi-) projective.*

Now we can show the following:

Lemma 3.23 *Suppose X is biprojective and Y is any left module. Then $X \otimes_{\mathcal{A}} Y$ is left projective.*

Proof Since X is biprojective, there exists by Corollary 3.20 a bimodule map

$$\rho_X \colon X \longrightarrow \mathcal{A}_+ \hat{\otimes} X \hat{\otimes} \mathcal{A}_+$$

which is a right inverse for the module map π_X. Now consider the map

$$\rho_X \otimes_{\mathcal{A}} \mathrm{id}_Y \colon X \otimes_{\mathcal{A}} Y \longrightarrow \mathcal{A}_+ \hat{\otimes} X \hat{\otimes} \mathcal{A}_+ \otimes_{\mathcal{A}} Y \cong \mathcal{A}_+ \hat{\otimes} X \hat{\otimes} Y$$

where the last congruence follows from Proposition 3.18. It is easy to see that this map is a left module map and a right inverse for the map

$$\tau\colon \mathcal{A}_+ \hat{\otimes} X \hat{\otimes} Y \longrightarrow X \otimes_{\mathcal{A}} Y$$

given by

$$\tau(a \otimes x \otimes y) = ax \otimes_{\mathcal{A}} y.$$

Hence $X \otimes_{\mathcal{A}} Y$ is left projective by Lemma 3.22. ■

Let us now consider the sequence

$$(\mathfrak{M})\colon 0 \longrightarrow N \longrightarrow \mathcal{A} \hat{\otimes} P \longrightarrow P \longrightarrow 0.$$

Clearly the sequence $\mathfrak{M}$ is exact only when the module map is onto. Recall that the module P is called *semi-neounital* when the module map $\pi\colon \mathcal{A} \hat{\otimes} P \longrightarrow P$ is onto. Note that in this case we have $\overline{\mathcal{A} \cdot P} = P$. This leads to the following:

Proposition 3.24 *A semi-neounital left (resp. right) module P is projective if and only if the sequence*

$$0 \longrightarrow N \longrightarrow \mathcal{A} \hat{\otimes} P \xrightarrow{\pi} P \longrightarrow 0$$

$$(\textit{resp. } 0 \longrightarrow N \longrightarrow P \hat{\otimes} \mathcal{A} \xrightarrow{\pi} P \longrightarrow 0)$$

splits.

Proof If the sequence splits, then there exists a c.b. module map $\rho\colon P \longrightarrow \mathcal{A} \hat{\otimes} P$ which is a right inverse for π. Clearly ρ is also a right inverse for π_L (as in Lemma 3.19). Thus by Lemma 3.19, P is projective. Conversely, if P is projective, then by Lemma 3.19, there exists $\tau\colon P \mapsto \mathcal{A}_+ \hat{\otimes} P$ which is a right inverse module map for π_L. However we note

$$\tau(\mathcal{A} \cdot P) = \mathcal{A} \cdot \tau(P) \subset \mathcal{A} \cdot (\mathcal{A}_+ \hat{\otimes} P) \subset \mathcal{A} \hat{\otimes} P.$$

Extending by continuity, it follows that $\tau(P) \subset \mathcal{A} \hat{\otimes} P$. Thus τ is an inverse for π. The right module case follows similarly. ■

Recall that a Banach algebra $\mathcal{B}$ is biprojective exactly when $\mathcal{B}$ is a projective (in the category of Banach spaces) $\mathcal{B}$-module. Thus we are led to the following analogous definition:

Definition 3.25 A completely contractive Banach algebra $\mathcal{A}$ is called *operator biprojective* if it is projective as an operator $\mathcal{A}$-bimodule.

This leads us to the main theorem of this section.

Theorem 3.26 *Suppose $\mathcal{A}$ is semi-neounital. Then $\mathcal{A}$ is operator biprojective if and only if the sequence*

$$0 \longrightarrow N \longrightarrow (\mathcal{A} \hat{\otimes} \mathcal{A}) \xrightarrow{\pi} \mathcal{A} \longrightarrow 0$$

splits as bimodules.

Proof Suppose $\mathcal{A}$ is semi-neounital and biprojective. Then from Corollary 3.20 we have that the sequence

$$0 \longrightarrow M \longrightarrow \mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+ \xrightarrow{\pi_B} \mathcal{A} \longrightarrow 0$$

splits, where M is the kernel of the module map $\pi_B \colon \mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+ \longrightarrow \mathcal{A}$. Now let $\pi_R \colon \mathcal{A} \hat{\otimes} \mathcal{A}_+ \mapsto \mathcal{A}$ be the "right" module map and let $\rho \colon \mathcal{A} \longrightarrow \mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+$ be a bimodule map which is a right inverse for π_B. Clearly the map

$$\mathrm{id} \hat{\otimes} \pi_R \colon \mathcal{A}_+ \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}_+ \mapsto \mathcal{A}_+ \hat{\otimes} \mathcal{A}$$

is also a bimodule map, and a simple calculation shows that

$$\rho' = (\mathrm{id} \hat{\otimes} \pi_R) \circ \rho \colon \mathcal{A} \mapsto \mathcal{A}_+ \hat{\otimes} \mathcal{A}$$

is a bimodule map which is an inverse for the left module map π_L. Following the idea in Proposition 3.24 we have that

$$\rho'(\mathcal{A} \cdot \mathcal{A}) \subset \mathcal{A} \cdot (\mathcal{A}_+ \hat{\otimes} \mathcal{A}) = \mathcal{A} \hat{\otimes} \mathcal{A}$$

hence $\rho'(\mathcal{A}) \subset \mathcal{A} \hat{\otimes} \mathcal{A}$ by continuity. Thus ρ' is a bimodule map which is a right inverse for π. Since ρ, π_R and id are all completely bounded, so is ρ'. Conversely, if the sequence splits, it follows that $\mathcal{A}$ is both left projective and right projective, by Proposition 3.24. Hence by Theorem 3.21 $\mathcal{A} \hat{\otimes} \mathcal{A}$ is operator biprojective. Since the sequence splits as bimodules, there exists a bimodule map $\rho \colon \mathcal{A} \mapsto \mathcal{A} \hat{\otimes} \mathcal{A}$. Thus $\mathcal{A}$ is biprojective by Lemma 3.22. ∎

4 Application to the Fourier Algebra

To apply the results of the previous section to the Fourier Algebra, we first must observe that $\mathbf{A}(G)$ is a semi-neounital $\mathbf{A}(G)$-module for all groups G. To realize this we begin with the following three technical propositions.

Proposition 4.1 *Let H be a closed subgroup of G. Then $\mathbf{A}(G)/I(H)$ is completely isometrically isomorphic with $\mathbf{A}(H)$.*

Proof Let $\tilde{u} \in \mathbf{A}(G)/I(H)$. Define $\Gamma \colon \mathbf{A}(G)/I(H) \mapsto \mathbf{A}(H)$ by $\Gamma(\tilde{u}) = v|_H$ where v is chosen so that $Q(v) = \tilde{u}$ and $Q \colon \mathbf{A}(G) \mapsto \mathbf{A}(G)/I(H)$ is the quotient map. It is known that Γ is an isometric isomorphism of $\mathbf{A}(G)/I(H)$ onto $\mathbf{A}(H)$ [10]. Let $\mathbf{VN}_H(G)$ be the weak closure in $\mathbf{VN}(G)$ of $\mathrm{span}\{\lambda_G(h) : h \in H\}$. To see that Γ is a complete isometry observe that $I(H)^{\perp} = \mathbf{VN}_H(G)$. Also $\mathbf{VN}_H(G)$ is a von Neumann subalgebra of $\mathbf{VN}(G)$ which is $*$-isomorphic with $\mathbf{VN}(H)$ [9]. It follows from [2] that $\big(\mathbf{A}(G)/I(H)\big)^*$ is completely isometrically isomorphic to $\mathbf{VN}_H(G)$ and hence to $\mathbf{VN}(H)$. Thus $\big(\mathbf{A}(G)/I(H)\big)$ is completely isometrically isomorphic to $\mathbf{A}(H)$. ∎

Proposition 4.2 *Let H be an open subgroup. Then $1_H\mathbf{A}(G)$ is completely isometrically isomorphic with $\mathbf{A}(H)$.*

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Proof Let $u \in \mathbf{A}(H)$. Let $\underline{u} \in \mathbf{A}(G)$ be such that $\underline{u}(x) = u(x)$ if $x \in H$ and $\underline{u}(x) = 0$ otherwise. It is well known that $\Gamma\colon \mathbf{A}(H) \mapsto \mathbf{A}(G)$ defined by $\Gamma(u) = \underline{u}$ is an isometric isomorphism of $\mathbf{A}(H)$ into $1_H\mathbf{A}(G)$ [9]. Let $[u_{ij}] \in \mathbb{M}_n\big(\mathbf{A}(H)\big)$ with $\|[u_{ij}]\|_n = 1$. It follows from Proposition 5.2.2 that $\|[\underline{\tilde{u}_{ij}}]\|_n = 1$. Let $\epsilon > 0$. We can find $[v_{ij}] \in \mathbb{M}_n\big(\mathbf{A}(G)\big)$ such that $[\tilde{v}_{ij}] = [\underline{\tilde{u}_{ij}}]$ and $\|[v_{ij}]\| \le 1 + \epsilon$. Now

$$\|\Gamma^{(n)}([u_{ij}])\| = \|[\underline{u_{ij}}]\|_n = \|P([v_{ij}])\|_n$$

where $P(v) = 1_H v$. However $1_H \in \mathbf{B}(G)$ and $\|1_H\| = 1$. It follows that $\|P\|_{\mathrm{cb}} = 1$. Hence

$$\|\Gamma^{(n)}([u_{ij}])\|_n \le 1 + \epsilon.$$

Therefore we can conclude that $\|\Gamma\|_{\mathrm{cb}} \le 1$. To complete the proof, observe that $\Gamma^{-1}\colon 1_H\mathbf{A}(G) \mapsto \mathbf{A}(H)$ is simply the restriction of the quotient map $Q\colon \mathbf{A}(G) \mapsto \mathbf{A}(G)/I(H)$ composed with the complete isometry of Proposition 4.1. It follows that $\|\Gamma^{-1}\|_{\mathrm{cb}} = 1$ and hence that Γ is a complete isometry. ■

Proposition 4.3 *Let H be a closed subgroup of G. Then the restriction map $R\colon \mathbf{A}(G) \mapsto \mathbf{A}(H)$ is a completely contractive homomorphism of $\mathbf{A}(G)$ onto $\mathbf{A}(H)$.*

Proof It is well known that R is a continuous homomorphism of $\mathbf{A}(G)$ onto $\mathbf{A}(H)$. Again we let $\mathbf{VN}_H(G)$ be the weak closure in $\mathbf{VN}(G)$ of $\operatorname{span}\{\lambda_G(h) : h \in H\}$. Then $\mathbf{VN}_H(G)$ is a von Neumann subalgebra of $\mathbf{VN}(G)$. Moreover, $R^*\colon \mathbf{VN}(H) \mapsto \mathbf{VN}(G)$ is a $*$-isomorphism of $\mathbf{VN}(H)$ onto $\mathbf{VN}_H(G)$ ([9] or [1]). It follows that R^* is completely contractive and hence that R is also completely contractive. ■

Corollary 4.4 *For all groups G, the Fourier algebra $\mathbf{A}(G)$ is semi-neounital.*

Proof Here we use the c.b. isomorphism $\mathbf{A}(G) \hat{\otimes} \mathbf{A}(G) \cong \mathbf{A}(G \times G)$ given by

$$\Phi(u \otimes v)(s,t) = u(s)v(t).$$

(See [7].) Let $G_D = \{(s,s) : s \in G\}$ be the diagonal subgroup. By the previous proposition we have that the restriction map 1_{G_D} maps $\mathbf{A}(G\times G)$ onto $\mathbf{A}(G_D) \cong \mathbf{A}(G)$. Thus the following diagram commutes:

$$\begin{tikzcd}
\mathbf{A}(G) \hat{\otimes} \mathbf{A}(G) \arrow[rr, "\Phi"] \arrow[dr, "m"'] & & \mathbf{A}(G \times G) \arrow[dl, "1_{G_D}"] \\
& \mathbf{A}(G) &
\end{tikzcd}$$

In particular, the multiplication map m is onto. ■

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See [15, VII.1.16] for the analogous result for $\mathbf{L}^1(G)$. It is perhaps important to note that the isomorphism $\mathbf{A}(G) \hat{\otimes} \mathbf{A}(G) \cong \mathbf{A}(G \times G)$ used in the above proposition no longer holds in the Banach space category. This failure is one of the driving forces behind our desire to recognize $\mathbf{A}(G)$ as an operator space, and not simply a Banach space. Since $\mathbf{A}(G)$ is semi-neounital for all groups G, we can use Proposition 3.26 to classify for which groups G, $\mathbf{A}(G)$ is operator biprojective. We note that $\mathbb{C}$ is a left operator $\mathbf{A}(G)$ module under the module action

$$u \cdot \alpha = \alpha u(e).$$

This takes us to the main theorem of this paper.

Theorem 4.5 *The following are equivalent:*

(1) $\mathbf{A}(G)$ *is operator biprojective;*
(2) G *is discrete;*
(3) *the left operator* $\mathbf{A}(G)$ *module* $\mathbb{C}$ *is projective.*

Proof (3) $\Rightarrow$ (2): Let $\mathcal{I}_0$ denote the ideal of functions $u \in \mathbf{A}(G)$ which are equal to zero at e. Since $\mathcal{I}_0$ is cofinite dimensional, there exists a bounded projection P onto $\mathcal{I}_0$. Now $1 - P\colon \mathbf{A}(G) \longrightarrow Q$ where Q is the complement of $\mathcal{I}_0$ is $\mathbf{A}(G)$. Clearly $Q \cong \mathbf{A}(G)/\mathcal{I}_0 \cong \mathbb{C}$. Let $\gamma\colon \mathbf{A}(G) \longrightarrow \mathbb{C}$ be given by $\gamma(u) = u(e)$. Certainly γ is completely bounded, and since there exists a c.b. map from $\mathbb{C}$ to $Q \subset \mathbf{A}(G)$, it follows that γ is admissible. Since $\mathbb{C}$ is projective, there is a right inverse module map for γ, call it τ. Now for all $u \in \mathbf{A}(G)$ we have that $u \cdot \tau(1) = \tau(u \cdot 1) = \tau\big(u(e)\big)$. Since for each $s \in G$ such that $s \neq e$ we can find an element $u \in \mathbf{A}(G)$ such that $u(s) = 0$ and $u(e) = 1$ it follows that $[\tau(1)](s) = 0$ for all $s \neq e$. Thus G is discrete.

(2) $\Rightarrow$ (1): Once again we can use the isomorphism $\mathbf{A}(G \times G) \cong \mathbf{A}(G) \hat{\otimes} \mathbf{A}(G)$ given by $(u \otimes v)(s,t) = u(s)v(t)$. The map $\tau\colon \mathbf{A}(G) \mapsto \mathbf{A}(G \times G)$ given by $\tau(u)(s,t) = u(s)\delta_s^t$, where δ is the Kronecker delta function, is a right inverse for the multiplication map

$$m\colon \mathbf{A}(G) \hat{\otimes} \mathbf{A}(G) \longrightarrow \mathbf{A}(G).$$

It now suffices to show that this map τ is completely bounded. Let $G_D = \{(s,s) : s \in G\}$. Clearly $\tau\big(\mathbf{A}(G)\big) \subset 1_{G_D}\big(\mathbf{A}(G \times G)\big)$. But we have

$$1_{G_D}\big(\mathbf{A}(G \times G)\big) \cong \mathbf{A}(G_D) \cong \mathbf{A}(G)$$

by Proposition 4.2. Thus τ is completely bounded. Now we may apply Theorem 3.26 to conclude that $\mathbf{A}(G)$ is biprojective.

(1) $\Rightarrow$ (3): Since $\mathbf{A}(G)$ is biprojective, it follows from Lemma 3.23 that $\mathbf{A}(G) \otimes_{\mathbf{A}(G)} \mathbb{C}$ is left projective. Clearly

$$\mathbf{A}(G) \otimes_{\mathbf{A}(G)} \mathbb{C} \subset \mathbf{A}(G)_+ \otimes_{\mathbf{A}(G)} \mathbb{C} \cong \mathbb{C}.$$

Thus $\mathbf{A}(G) \otimes_{\mathbf{A}(G)} \mathbb{C} \cong \mathbb{C}$. Hence $\mathbb{C}$ is left projective. ■

We complete this paper with some observations regarding the Banach space structure of the Fourier algebra. Suppose $\mathcal{A}$ is a completely contractive Banach algebra which possesses the MAX operator space structure, *i.e.*, $\mathcal{A} = \mathrm{MAX}(\mathcal{A})$. In this case $\mathcal{A} \hat{\otimes} \mathcal{A} = \mathcal{A} \otimes_\gamma \mathcal{A}$ where $\otimes_\gamma$ is the Banach space projective tensor product. Using Khelimskii's results (see [13] for example) it is easy to see that $\mathcal{A}$ is operator biprojective if and only if it is biprojective as a Banach algebra. First note that we have the following:

Theorem 4.6 *$\mathbf{A}(G)$ and $\mathrm{MAX}\big(\mathbf{A}(G)\big)$ are c.b. isomorphic if and only if G has an abelian subgroup of finite index.*

Proof See [11, Theorem 4.5]. ■

Now we have an alternate proof of [21, Proposition 23] and its converse.

Theorem 4.7 *Suppose G has an abelian subgroup of finite index. Then $\mathbf{A}(G)$ is biprojective as a Banach algebra if and only if G is discrete.*

Proof If G has an abelian subgroup of finite index, then the operator biprojectivity of $\mathbf{A}(G)$ and the biprojectivity of $\mathbf{A}(G)$ as a Banach algebra as equivalent by Theorem 4.6 and the discussion proceeding it. Thus by Theorem 4.5, the result follows immediately. ■

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