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PERMUTATION REPRESENTATIONS OF THE (2,4,r) TRIANGLE GROUPS

BRENT EVERITT

The abstract triangle groups $\Delta(2,4,r)$ can be defined for any positive integer r by $\Delta(2,4,r) = \langle x,y \mid x^2 = y^4 = (xy)^r = 1\rangle$. In this paper we show that for every $r \geqslant 6$, all but finitely many of the alternating groups A_n can be obtained as quotients of $\Delta(2,4,r)$.

1. INTRODUCTION

For positive integers p, q and r we can define the abstract triangle groups $\Delta(p,q,r)$ by

$$\Delta(p,q,r) = \langle x,y \mid x^p = y^q = (xy)^r = 1\rangle.$$

These arise most naturally in the study of the geometry of discrete groups (see Coxeter [5] or Lyndon [6]). For instance, it is well known that if

$$1/p + 1/q + 1/r < 1$$

then $\Delta(p,q,r)$ corresponds to the group of automorphisms of a tessellation of the hyperbolic plane by triangles with angles $\pi/p, \pi/q$ and π/r.

Some years ago, Graham Higman showed that for all but finitely many n, the alternating group A_n can be obtained as quotients of both the triangle groups $\Delta(2,3,7)$ and $\Delta(2,4,5)$. His method (which remained unpublished) involved the use of a small number of what were basically coset diagrams, together with a technique for combining diagrams to obtain new ones. His methods were subsequently refined and expanded by Conder in [1, 2] where he showed that in fact all the groups in the family $\Delta(2,3,r), r \geqslant 7$ shared this property. Recently, Mushtaq and Rota obtained a similar result for the groups $\Delta(2,k,l)$ for all even $k \geqslant 6$, [7]. Their project was continued by Mushtaq and Servatius in [8] where the result was established for $\Delta(2,p,q)$ with prime $p \geqslant 5$ and $q \geqslant 5p-3$.

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The author is indebted to his doctoral supervisor Marston Conder for suggesting the problem and method of proof of this paper.

The technique of all these authors was to draw diagrams in the manner discussed in the next section for the groups $\Delta^*(2,p,q)$ defined by

$$\Delta^*(2,p,q) = \langle x,y,t \mid x^2 = y^p = (xy)^q = t^2 = (xt)^2 = (yt)^2 = 1\rangle.$$

The result was obtained by showing that all but finitely many of the symmetric groups S_n could be obtained as quotients of the group $\Delta^*(2,p,q)$.

There has also been some interest in whether certain quotients of $\Delta^*(2,q,r)$ possess this property. In particular the groups $G^{k,l,m}$ can be defined by

$$G^{k,l,m} = \langle x,y,t \mid x^2 = y^k = (xy)^l = t^2 = (xt)^2 = (yt)^2 = (xyt)^m = 1\rangle.$$

In [**3, 4**] Conder showed that the groups $G^{6,6,6}$ and $G^{3,7,168}$ yield as quotients all but finitely many of the alternating groups A_n. An open question that remains is what is the smallest value of m such that $G^{3,7,m}$ has this property.

All these results used Higman's original method with coset diagrams. The same approach is used in this paper to show the following:

Theorem 1. *For every $r \geqslant 6$ all but finitely many of the alternating groups A_n can be obtained as quotients of the triangle groups $\Delta(2,4,r)$.*

2. Diagrams, Handles and Composition

We shall use Higman's method by drawing diagrams to illustrate transitive permutation representations of the groups $\Delta^*(2,4,r)$. A method for combining diagrams, called composition, will then create new representations for $\Delta^*(2,4,r)$.

Given a permutation representation of $\Delta^*(2,4,r)$ to S_n, we depict the action of the generators x,y and t on the set $\{1,2,\dots,n\}$ by drawing a diagram with n vertices in the following way:

- 4-cycles of y are represented by 4-gons permuted in an anticlockwise direction, transpositions of y by circular 2-gons whose endpoints are swapped, and fixed points of y by heavy dots,
- a transposition (a,b) of x is represented by an arc connecting the vertices a and b,
- the action of t is represented by a reflection in the vertical axis of symmetry.

As an example, the diagram in Figure 1 illustrates a transitive permutation representation of $\Delta^*(2,4,7)$ of degree 14, where

- x acts as the permutation $(3,7)(4,5)(6,10)(8,12)(11,13)$,
- y acts as $(1,2,3,4)(5,6)(7,8)(9,10,11,12)(13,14)$, and
- t acts as $(1,2)(3,4)(5,7)(6,8)(10,12)$.

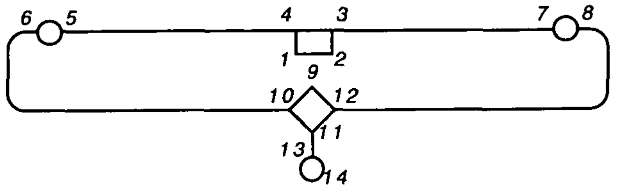

Figure 1

Note how the diagram is planar and consists of two 'faces' (one in the interior and one on the outside). It may be useful in the future to note that the sum of the number of polygonal sides and fixed points of y included in each face divides the order of xy. The reader should note that in figure 1 this sum is equal to seven for each face.

In future when there is no ambiguity we shall drop the numbering from our diagrams.

To obtain transitive permutation representations for all but finitely many n, we shall use a small collection of basic diagrams together with a technique for 'pasting' diagrams together to obtain new ones. To this end, all of our basic diagrams will possess the property that they contain two points, a and b, such that a and b are fixed by the action of x, and both y and t map a to b. We shall call such a structure a *handle* and denote it by $[a,b]$. Diagrammatically, a handle appears on the vertical axis of symmetry. In figure 1 for instance, $[1,2]$ forms a handle.

Let us suppose then, that we are given two diagrams for $\Delta^*(2,4,r)$ of degree n and m, say U and V, containing handles $[a,b]_U$ and $[a',b']_V$ respectively. We form the *composition*, UV, of U and V by placing U and V one above the other on the vertical axis of symmetry and adding arcs connecting a to a' and b to b'.

It is not too difficult to see that the resulting diagram depicts a transitive permutation representation for $\Delta^*(2,4,r)$ of degree $n+m$. Firstly, the relations $x^2 = y^4 = t^2 = (xt)^2 = (yt)^2 = 1$ are clearly still satisfied. Also if $(abc_1 \dots c_{r_1})$ and $(a'b'd_1 \dots d_{r_2})$ are the cycles of xy in U and V respectively that pass through the handles of each diagram, then in UV these become $(ab'c_1 \dots c_{r_1})(bd_1 \dots d_{r_2}a')$ with all other cycles of xy unaffected. In particular, xy still has order r in UV.

Later on we shall need to know the cycle structure of the action of the element xyt. In particular, we shall need to know the effect of composition on this cycle structure. In a manner similar to the action of xy, the only cycles of xyt that are effected by composition are those that pass through the handles of the respective diagrams. The two cycles passing through b and b' in U and V are juxtaposed while the fixed points a and a' are joined to form a transposition.

3. The Diagrams and Cycle Structures

For each value of $r \geqslant 6$ we shall use *three* basic diagrams $P(r), Q(r)$ and $R(r)$ each

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of which will depict a permutation representation of $\Delta^*(2,4,r)$. In the diagrams $P(r)$ and $Q(r)$, x,y and t will act as even permutations while in the diagram $R(r)$, x and y will be even and t will be odd. These will then be joined together by composition in a way that will give transitive permutation representations for $\Delta^*(2,4,r)$ onto S_n for all but finitely many values of n. Of course, we cannot present infinitely many diagrams, one for each value of $r \geqslant 6$. Instead we shall use a special technique first used by Conder in [**2**] and will adopt the notation used by him there.

Our first task will be to consider the positive integers *modulo eight* and give diagrams $P(k+8d), Q(k+8d)$ and $R(k+8d)$ for $k \in \{6,7,8,\dots,12,13\}$ and d a positive integer.

In the diagrams Figures $2-10$, the 'cross' symbol $\otimes$ stands for h 2-cycles (that is: h circles), for those values of h stipulated below the diagram. For instance, Figure 2 shows the diagram $P(6+2h+8d)$ for $h = 0,1,2,3$. Adding $0,1,2$ or 3 circles at each crossed position yields the four diagrams $P(6+8d), P(8+8d), P(10+8d)$ and $P(12+8d)$.

Once this has been done where appropriate we get 24 diagrams in total, a $P(k+8d), Q(k+8d)$ and $R(k+8d)$ for each value of $k \in \{6,7,\dots,12,13\}$.

The 'pod' symbol  in Figures $2-10$ stands for a string of d 4-cycles of y (that is: d squares). Since each 'face' in the basic diagrams has four such configurations adjacent to it (two contained in the face and two contained in a bordering face), the effect will be to add $8d$ polygonal sides on to each face. This means that the new diagram will represent a permutation representation for $\Delta^*(2,4,k+8d)$, since xy will now have order $k+8d$.

Hence we obtain our three diagrams $P(r), Q(r)$ and $R(r)$ for all $r \geqslant 6$.

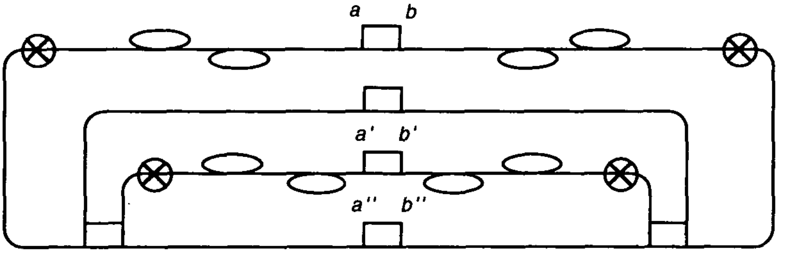

Figure 2. Diagram $P(6+2h+8d)$ $h = 0,1,2,3$ $24+8h+32d$ vertices

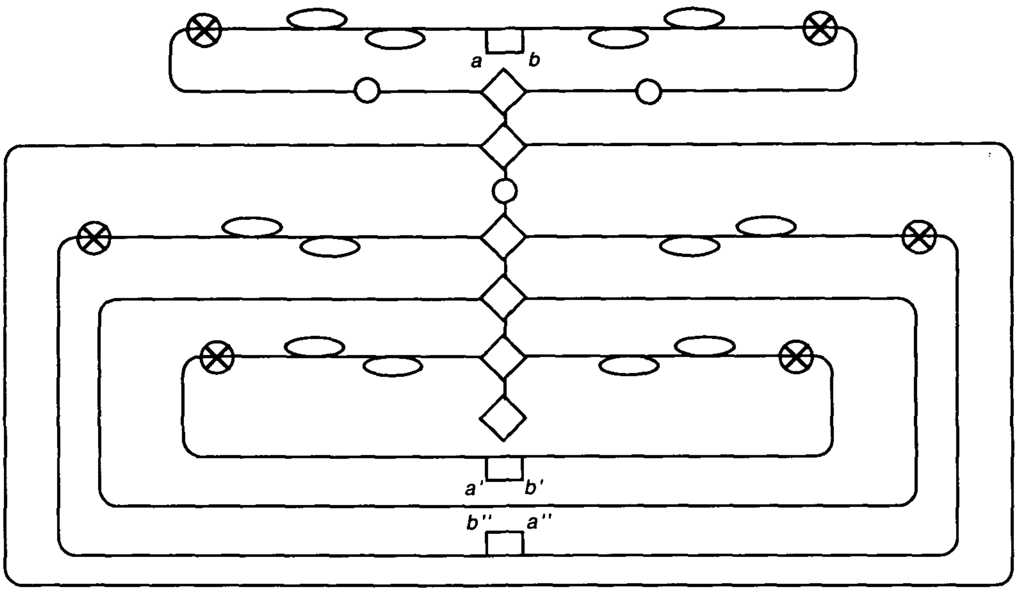

Figure 3. Diagram $P(7+2h+8d)$ $h=0,1,2,3$ $42+12h+48d$ vertices

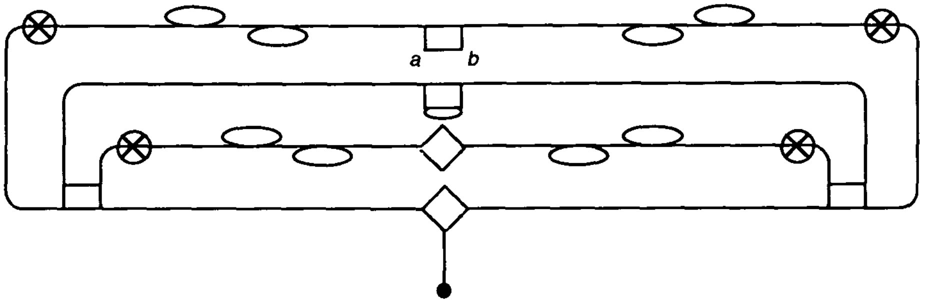

Figure 4. Diagram $Q(6+2h+8d)$ $h=0,1,2,3$ $25+8h+32d$ vertices

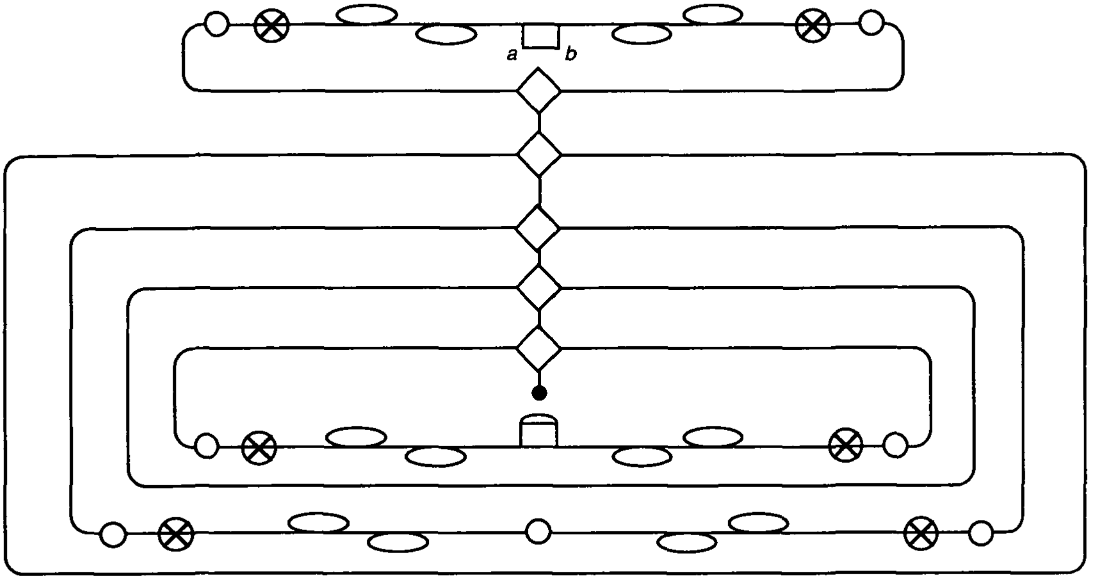

Figure 5. Diagram $Q(7+2h+8d)$ $h=0,1,2,3$ $43+12h+48d$ vertices

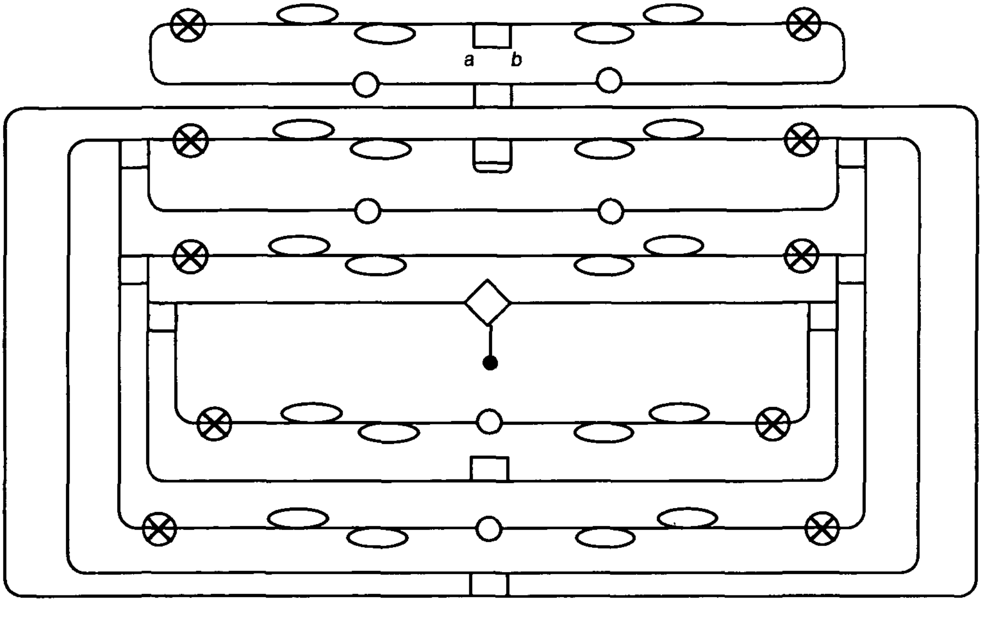

Figure 6. Diagram $R(6+2h+8d)$ $h=0,2$ $61+20h+80d$ vertices

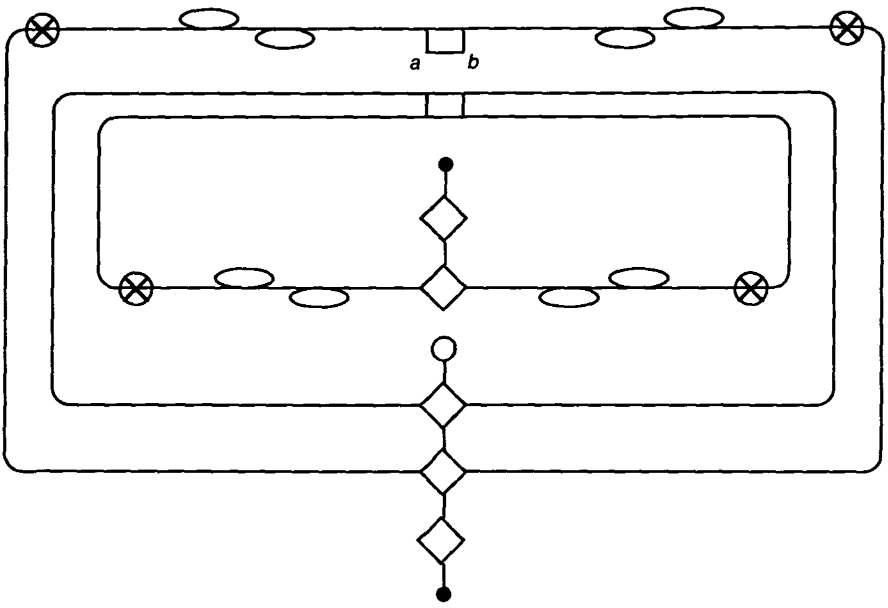

Figure 7. Diagram $R(8+2h+8d)$ $h=0,2$ $32+8h+32d$ vertices

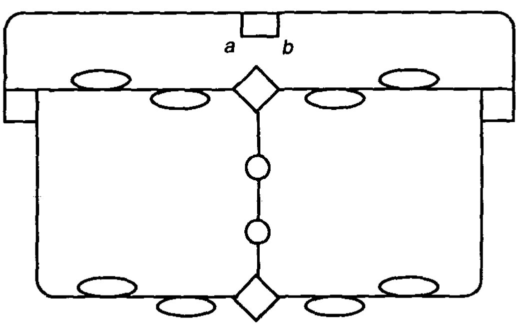

Figure 8. Diagram $R(7+8d)$ $28+32d$ vertices

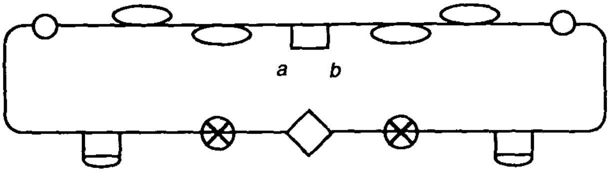

Figure 9. Diagram $R(9+2h+8d)$ $h=0,1$ $20+4h+16d$ vertices

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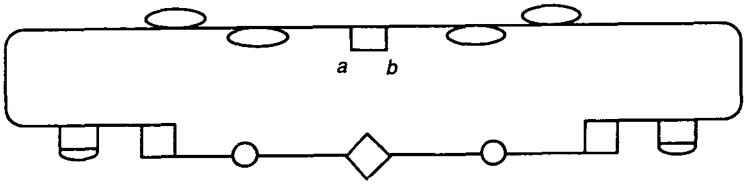

Figure 10. Diagram $R(13+8d)$ $28+16d$ vertices

Before moving on to the proof of the theorem, it will be useful to take note of the cycle structures of the element xyt in each of our diagrams. Although there are infinitely many we are helped by the fact that for $d \geqslant 2$, the addition of d squares at the appropriate positions in each diagram only results in the creation of 2-cycles and 6-cycles in the cycle structure of xyt. Specifically, each symmetric pairing of pods gives rise to a 2-cycle and a 6-cycle for each value of $d \geqslant 2$.

The cycle structures are listed in Table 1 at the end of the paper, where each number represents a cycle in xyt of that length and repetitions are denoted by exponents. The positions of the handles $[a, b]$ in each cycle structure have been shown, and certain prime cycles have been highlighted for future use. For instance, if we take the diagram in Figure 2 with $h = 0$ and $d = 0$ we get a cycle structure of xyt with the points a, a' and a'' fixed, b, b' and b'' contained in three 4-cycles, and the remaining cycle structure comprising a five cycle and a fixed point.

4. The proof of the theorem

The proof of Theorem 1 relies heavily on the following theorem of Jordan (see Wielandt [**9**]).

Theorem 2. *Let G be a transitive permutation group on a set of size n such that G is primitive and contains a p-cycle for some prime $p < n-2$. Then G is either A_n or S_n.*

Take l and m to be positive integers with $m \leqslant l$. We shall now utilise our diagrams by forming a 'chain' of $l+1$ copies of $P(k+8d)$, $m+1$ copies of $Q(k+8d)$ and one copy of $R(k+8d)$.

To this end, for $k = 6, 10$ and 11, form a chain of l copies of $P(k+8d)$ by joining each handle $[a, b]_P$ to the next handle $[a', b']_P$. Each $P(k+8d)$ in this chain will then have one free handle remaining, namely $[a'', b'']_P$ (except the first and last diagram in the chain which will have two free handles each—but more on that later). Then attach m copies of $Q(k+8d)$ to this chain by joining $[a, b]_Q$ to $[a'', b'']_P$.

For $k = 7, 9$ and 13 chain the $P(k+8d)$ together by joining $[a, b]_P$ to $[a'', b'']_P$ and attach the $Q(k+8d)$ by joining $[a, b]_Q$ to $[a', b']_P$. For $k = 8$ and 12, join $[a', b']_P$ to $[a'', b'']_P$ and $[a, b]_Q$ to $[a, b]_P$.

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In any case, once this has been done attach one more copy of $P(k+8d)$ to one end of the chain of P's and one more $Q(k+8d)$ to this P.

Finally, attach one copy of $R(k+8d)$ to the whole structure in the following way. If $k = 6, 10$ or 11 attach $R(k+8d)$ to the bottom $P(k+8d)$ in the chain by joining $[a,b]_R$ to the free handle $[a',b']_P$. For $k = 7, 8, 9, 12$ or 13 attach $R(k+8d)$ to the top $P(k+8d)$ in the chain by joining $[a,b]_R$ to $[a',b']_P$.

Let us now use P, Q and R to denote the number of vertices of the diagrams $P(k+8d), Q(k+8d)$ and $R(k+8d)$, for any choice of k and d.

By the above process we obtain a diagram $S(k+8d)$ for $\Delta^*(2,4,k+8d)$ that has $(l+1)P+(m+1)Q+R$ vertices, where l and m are positive integers with $m \leqslant l$. Observe that for $k = 6, 8, 10$ or 12, and fixed d, we get $P = 24+4(k-6)+32d$ and $Q = 25+4(k-6)+32d$. Similarly, for $k = 7, 9, 11$ or 13, we have $P = 42+6(k-7)+48d$ and $Q = 43+6(k-7)+48d$.

In either case (for any fixed k and d) P and Q differ by one and hence are relatively prime.

Since any sufficiently large integer can be expressed in the form $n = (l+1)P+(m+1)Q+R$ for some l and $m \leqslant l$, we now have transitive permutation representations of $\Delta^*(2,4,r)$ of degree n for all but finitely many n.

In order to take advantage of Theorem 2, the reader should note that for fixed $k \in \{6, 7, \dots, 12, 13\}$ and any fixed d, the structure of xyt in one of the diagrams $P(k+8d), Q(k+8d)$ or $R(k+8d)$ contains a cycle of prime length. Furthermore, this prime cycle has length not dividing the order of any other cycle in the resulting diagram $S(k+8d)$. Hence, there exists an integer s such that $(xyt)^s$ is just a power of this prime cycle. These prime cycles have been highlighted in the cycle structures given in Section 3. Note also that for some values of k and d, these prime cycles arise through cycles including points from handles being juxtaposed through composition in the construction of $S(k+8d)$.

We now have all the hypotheses for Theorem 2 except for primitivity. Suppose that the representation depicted by $S(r)$ is imprimitive. Then all the points of the prime cycle $(xyt)^s$ must lie in the same block B of imprimitivity. The reader can check that every prime cycle highlighted in Section 3 contains a point and its image under the action of x, a point and its image under the action of y, and a fixed point of t. (The last of these assertions is clear since the distrubution of points in any cycle of xyt must be symmetric about the t-axis.) Hence x, y and t all fix the block B, contradicting imprimitivity.

Finally, recall that the diagrams $R(r)$ depict representations with t acting as an odd permutation. By Theorem 2, we deduce that the permutation group depicted by $S(r)$ is the symmetric group S_n. However, x and y yield *even* permutations in S_n and since the subgroup $< x, y >$ has index two in $\Delta^*(2,4,r)$ we obtain the desired result.

This completes the proof of Theorem 1.

COROLLARY 3. *For any fixed $r \geqslant 6$, all but finitely many of the symmetric groups S_n can be generated by elements x, y and t that satisfy*

$$x^2 = y^4 = (xy)^r = t^2 = (xt)^2 = (yt)^2 = 1,$$

and all but finitely many of the alternating groups A_n can be generated by elements x and y that satisfy

$$x^2 = y^4 = (xy)^r = 1.$$

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TABLE 1

Diagram	*Cycle Structure*
$P(6+8d)$	
$d=0$	$a\,a'\,a''\,(b4)(b'4)(b''4)\,1\,5$
$d=1$	$a\,a'\,a''\,(b3)(b'3)(b''8)\,1\,2^4\,4^2\,5^2\,9$
$d\geq 2$	$a\,a'\,a''\,(b3)(b'3)(b''7)\,1\,2^{8+4(d-2)}\,4^2\,5^4\,6^{4(d-2)}\,8^3$
$P(7+8d)$	
$d=0$	$a\,a'\,a''\,(b5)(b'5)(b''14)\,2\,10$
$d=1$	$a\,a'\,a''\,(b8)(b'9)(b''3)\,1\,2^7\,3^2\,4^3\,15\,16$
$d\geq 16$	$a\,a'\,a''\,(b5)(b'5)(b''3)\,2^{13+6(d-2)}\,3^2\,4^3\,5\,6^{6(d-2)}\,8^5\,14\,16$
$P(8+8d)$	
$d=0$	$a\,a'\,a''\,(b3b''3)(b'7)\,1\,6^2$
$d=1$	$a\,a'\,a''\,(b3)(b'3)(b''7)\,1\,2^4\,4^2\,8\,10^2$
$d\geq 2$	$a\,a'\,a''\,(b3)(b'3)(b''7)\,1\,2^{8+4(d-2)}\,4^2\,5^2\,6^{4(d-2)}\,8^3\,9^2$
$P(9+8d)$	
$d=0$	$a\,a'\,a''\,(b18)(b'7b''7)\,2\,7^2$
$d=1$	$a\,a'\,a''\,(b21)(b'5)(b''17)\,2^7\,3\,4^3\,5\,8\,11$
$d\geq 2$	$a\,a'\,a''\,(b5)(b'5)(b''7)\,2^{13+6(d-2)}\,3\,4^3\,6^{6(d-2)}\,7\,8^4\,14\,19$
$P(10+8d)$	
$d=0$	$a\,a'\,a''\,(b8)(b'8)(b''8)\,1\,9$
$d=1$	$a\,a'\,a''\,(b3)(b'3)(b''12)\,1\,2^4\,4^2\,9^2\,13$
$d\geq 2$	$a\,a'\,a''\,(b3)(b'3)(b''11)\,1\,2^{8+4(d-2)}\,4^2\,5^2\,6^{4(d-2)}\,8^2\,9^2\,12$
$P(11+8d)$	
$d=0$	$a\,a'\,a''\,(b9)(b'9)(b''22)\,2\,18$
$d=1$	$a\,a'\,a''\,(b13)(b'13)(b''7)\,2^7\,3\,4^3\,7\,19\,20$
$d\geq 2$	$a\,a'\,a''\,(b5)(b'9)(b''7)\,2^{13+6(d-2)}\,3\,4^3\,6^{6(d-2)}\,7\,8^4\,9\,12\,14\,20$
$P(12+8d)$	
$d=0$	$a\,a'\,a''\,(b5b''5)(b'11)\,1\,10^2$
$d=1$	$a\,a'\,a''\,(b3)(b'3)(b''11)\,1\,2^4\,4^2\,12\,14^2$
$d\geq 2$	$a\,a'\,a''\,(b3)(b'3)(b''11)\,1\,2^{8+4(d-2)}\,4^2\,5^2\,6^{4(d-2)}\,8^2\,12\,13^2$
$P(13+8d)$	
$d=0$	$a\,a'\,a''\,(b26)(b'11b''11)\,2\,11^2$
$d=1$	$a\,a'\,a''\,(b25)(b'9)(b''21)\,2^7\,3\,4^3\,9\,12\,15$
$d\geq 2$	$a\,a'\,a''\,(b5)(b'9)(b''11)\,2^{13+6(d-2)}\,3\,4^3\,6^{6(d-2)}\,8^4\,9\,11\,12\,14\,24$
$Q(6+8d)$	
$d=0$	$a\,(b5)\,5\,6\,7$
$d=1$	$a\,(b9)\,2^4\,3\,4^3\,7^2\,9$
$d\geq 2$	$a\,(b5)\,2^{8+4(d-2)}\,3\,4^3\,6^{1+4(d-2)}\,7^3\,8^3$

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$Q(7+8d)$	
$d=0$	$a\,(b5)\,8\,14^2$
$d=1$	$a\,(b9)\,2^6\,3^2\,4^3\,5\,7\,19^2$
$d\geq 2$	$a\,(b5)\,2^{12+6(d-2)}\,3^2\,4^3\,5^3\,6^{6(d-2)}\,7\,8^4\,18^2$
$Q(8+8d)$	
$d=0$	$a\,(b8)\,7^2\,9$
$d=1$	$a\,(b12)\,2^4\,3\,4^3\,7\,8\,\mathbf{13}$
$d\geq 2$	$a\,(b5)\,2^{8+4(d-2)}\,3\,4^3\,6^{1+4(d-2)}\,7\,8^3\,11^2$
$Q(9+8d)$	
$d=0$	$a\,(b18)\,7\,8\,20$
$d=1$	$a\,(b23)\,2^6\,3^2\,4^3\,5\,8\,12\,\mathbf{23}$
$d\geq 2$	$a\,(b5)\,2^{12+6(d-2)}\,3^2\,4^3\,5^3\,6^{6(d-2)}\,8^4\,11\,22^2$
$Q(10+8d)$	
$d=0$	$a\,(b9)\,9\,10\,11$
$d=1$	$a\,(b13)\,2^4\,3\,4^3\,11^2\,13$
$d\geq 2$	$a\,(b5)\,2^{8+4(d-2)}\,3\,4^3\,6^{1+4(d-2)}\,8^2\,11^3\,12$
$Q(11+8d)$	
$d=0$	$a\,(b9)\,12\,22^2$
$d=1$	$a\,(b13)\,2^6\,3^2\,4^3\,5\,11\,27^2$
$d\geq 2$	$a\,(b5)\,2^{12+6(d-2)}\,3^2\,4^3\,5^3\,6^{6(d-2)}\,8^3\,\mathbf{11}\,12\,26^2$
$Q(12+8d)$	
$d=0$	$a\,(b12)\,11^2\,\mathbf{13}$
$d=1$	$a\,(b16)\,2^4\,3\,4^3\,11\,12\,\mathbf{17}$
$d\geq 2$	$a\,(b5)\,2^{8+4(d-2)}\,3\,4^3\,6^{1+4(d-2)}\,8^2\,\mathbf{11}\,12\,15^2$
$Q(13+8d)$	
$d=0$	$a\,(b26)\,11\,12\,28$
$d=1$	$a\,(b31)\,2^6\,3^2\,4^3\,5\,12\,16\,\mathbf{31}$
$d\geq 2$	$a\,(b5)\,2^{12+6(d-2)}\,3^2\,4^3\,5^3\,6^{6(d-2)}\,8^3\,12\,15\,30^2$
$R(6+8d)$	
$d=0$	$a\,(b4)\,2\,4\,5\,6\,8\,12\,\mathbf{17}$
$d=1$	$a\,(\mathbf{b8})\,1\,2^{11}\,3^4\,4^5\,5\,8\,9\,10\,12\,14\,18$
$d\geq 2$	$a\,(b5)\,1\,2^{21+10(d-2)}\,3^4\,4^5\,5^3\,6^{1+10(d-2)}\,7^3\,8^7\,\mathbf{11}\,14\,16$
$R(7+8d)$	
$d=0$	$a\,(b3)\,6^2\,\mathbf{11}$
$d=1$	$a\,(b3)\,2^4\,4^3\,5\,10^3$
$d\geq 2$	$a\,(b3)\,2^{8+4(d-2)}\,4^3\,5\,6^{2+4(d-2)}\,8^4\,10$

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$R(8+8d)$	
$d=0$	$a\,(b5)\,3^3\,5\,11$
$d=1$	$a\,(b9)\,2^4\,3^3\,4^2\,5\,7\,10$
$d\geq 2$	$a\,(b5)\,2^{8+4(d-2)}\,3^3\,4^2\,5^2\,6^{2+4(d-2)}\,8^3\,10$
$R(9+8d)$	
$d=0$	$a\,(b7)\,11$
$d=1$	$a\,(b11)\,2^2\,3\,4\,12$
$d\geq 2$	$a\,(b5)\,2^{4+2(d-2)}\,3\,4\,6^{2(d-2)}\,8\,10\,12$
$R(10+8d)$	
$d=0$	$a\,(b8)\,1\,2\,8\,9\,10\,12\,20\,29$
$d=1$	$a\,(\mathbf{b12})\,1\,2^{11}\,3^4\,4^5\,9\,12\,13\,14\,16\,22\,26$
$d\geq 2$	$a\,(b5)\,1\,2^{21+10(d-2)}\,3^4\,4^5\,5^2\,6^{1+10(d-2)}\,7\,8^5\,9\,11^2\,12^2\,15\,22\,24$
$R(11+8d)$	
$d=0$	$a\,(b9)\,13$
$d=1$	$a\,(b13)\,2^2\,3\,4\,14$
$d\geq 2$	$a\,(b5)\,2^{4+2(d-2)}\,3\,4\,6^{2(d-2)}\,8\,12\,14$
$R(12+8d)$	
$d=0$	$a\,(b8)\,3^2\,5\,7\,19$
$d=1$	$a\,(b13)\,2^4\,3^3\,4^2\,5\,6\,11\,18$
$d\geq 2$	$a\,(b5)\,2^{8+4(d-2)}\,3^3\,4^2\,5\,6^{2+4(d-2)}\,8^2\,9\,12\,18$
$R(13+8d)$	
$d=0$	$a\,(b9)\,5\,6^2$
$d=1$	$a\,(b13)\,2^2\,3\,4\,6^3$
$d\geq 2$	$a\,(b5)\,2^{4+2(d-2)}\,3\,6^{3+2(d-2)}\,8\,12$

Department of Mathematics and Statistics
University of Auckland
PO Box 92019
Auckland
New Zealand