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Diameter, Decomposability, and Minkowski Sums of Polytopes

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Abstract. We investigate how the Minkowski sum of two polytopes affects their graph and, in particular, their diameter. We show that the diameter of the Minkowski sum is bounded below by the diameter of each summand and above by, roughly, the product between the diameter of one summand and the number of vertices of the other. We also prove that both bounds are sharp. In addition, we obtain a result on polytope decomposability. More precisely, given two polytopes P and Q, we show that P can be written as a Minkowski sum with a summand homothetic to Q if and only if P has the same number of vertices as its Minkowski sum with Q.

1 Introduction

The Minkowski sum of two subsets of an Euclidean space is obtained by summing each element of one subset with each element of the other. The Minkowski sum of P and Q is denoted by $P + Q$. This operation turns up in a large number of different contexts ranging from the Brunn–Minkowski theorem to applications in civil engineering or motion planning. The special case when P and Q are polytopes is of particular interest. It is a model for the combinatorics of prismatoids used by Santos to disprove the Hirsch conjecture [10]. The face lattice of $P + Q$, and in particular its vertex set, has been studied by Fukuda and Weibel [5]. Recently, a sharp upper bound on the number of faces of $P+Q$ has been obtained by Adiprasito and Sanyal [1]. The question of the decomposability of a polytope, that is, whether it can be obtained as the Minkowski sum of two non-homothetic polytopes, has been considered in [7–9,11]. Among polytopes, the case of zonotopes is particularly interesting. These polytopes are the Minkowski sums of line segments. For any pair of positive integers d and k, zonotopes are conjectured to achieve the largest possible diameter over all the d-dimensional polytopes whose vertices have integer coordinates ranging from 0 to k [3]. Here, by the *diameter of a polytope*, we mean the diameter of the graph of a polytope, made up of its vertices and edges. We refer the reader to the textbooks by Fukuda [4], Grünbaum [6], and Ziegler [12] for comprehensive introductions to polytopes, Minkowski sums, and zonotopes.

Here, we focus on the possible diameter of (the graph of) the Minkowski sum of two polytopes. While this diameter is bounded below by the diameters of each

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summand, we will observe that it can grow arbitrarily large, even when the diameter of both summands is fixed. In fact, we will prove that this diameter cannot exceed, roughly, the product between the diameter of one summand and the number of vertices of the other. We will also show that this upper bound is sharp when the diameter and the number of vertices of both summands grow large. Along the way, we obtain a result on the decomposability of a polytope into a Minkowski sum. If P is the Minkowski sum of two polytopes Q and R, we say that Q and R are *summands* of P. A polytope that is not homothetic to at least one of its summands is called *decomposable* [11]. We will show that a polytope P has a summand homothetic to a polytope Q if and only if P and $P+Q$ have the same number of vertices. This allows for a convenient way to check polytope decomposability, especially in the case of lattice polytopes.

The article is based on a couple of propositions from [4], which we recall and extend in Section 2. Our result on polytope decomposability is given as a conclusion to Section 2. The question on the diameter of Minkowski sums is addressed in Sections 3 and 4. The bounds on that diameter are given in Section 3 and the proof that the upper bound is sharp in Section 4.

2 Some Properties of the Minkowski Sum of Polytopes

In the sequel, each time a Minkowski sum of two polytopes is considered, it is implicitly assumed that these polytopes are both contained in the same ambient Euclidean space. Note that we will make heavy use of linear maps of the form $x \mapsto c{\cdot}x$. In this notation, c and x are vectors in the considered ambient space and $c{\cdot}x$ denotes their scalar product. The following lemma is borrowed from [4]. It is in some sense our starting point. In particular, most of our results are based on it.

Lemma 2.1 ([4, Proposition 12.1]) *For any subset F of a polytope P and any subset G of a polytope Q, $F+G$ is a face of $P+Q$ if and only if*

(i) *F and G are faces of P and Q, respectively,*
(ii) *there exists a vector c such that the map $x \mapsto c{\cdot}x$ is minimized exactly at F in P and exactly at G in Q.*

By this lemma, given two polytopes P and Q, a face X of their Minkowski sum can always be written as the Minkowski sum of a unique face F of P and a unique face G of Q. In the sequel, the expression $F+G$ will be referred to as the *Minkowski decomposition* of X. Lemma 2.1 is illustrated on Figure 1 with the Minkowski sum of a triangle P and a line segment Q, where the Minkowski decomposition of each proper face of $P+Q$ is indicated by an arrow. Note, for instance, that when c is a vertical vector pointing down, the map $x \mapsto c{\cdot}x$ is minimized, in P, at the purple vertex placed at the top and, in Q, at Q itself. The sum of these two faces is the line segment at the top of $P+Q$. The following lemma, also borrowed from [4], tells how Minkowski sums affect vertex adjacency in the graph of a polytope.

Lemma 2.2 ([4, Proposition 12.4]) *Let P and Q be two polytopes. If u and v are adjacent vertices of $P+Q$ with Minkowski decompositions u_P+u_Q and v_P+v_Q, respectively, then u_P and v_P either coincide or are adjacent vertices of P. Similarly, u_Q and v_Q coincide or are adjacent vertices of Q.*

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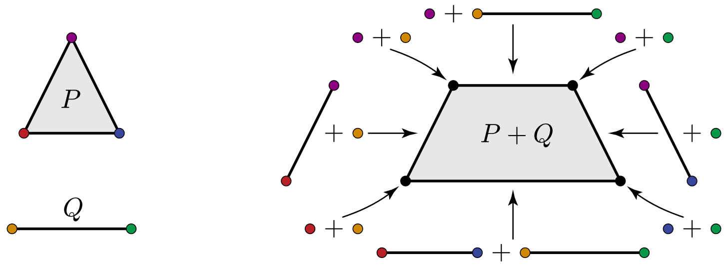

Figure 1: (Color online.) The Minkowski sum of a triangle and a line segment.

Observe that, for any vertex u of a polytope P, and any polytope Q, there exists a vertex v of Q such that $u + v$ is a vertex of $P + Q$. Indeed, consider a vector c such that the map $x \mapsto c{\cdot}x$ is uniquely minimized at u in P. This map is also minimized at a face F in Q. According to Lemma 2.1, $u + F$ is a face of $P + Q$, and the vertices of this face are precisely the Minkowski sums $u + v$ where v is a vertex of F. Since the Minkowski decomposition of a vertex of $P + Q$ is unique, we immediately obtain Lemma 2.3.

Lemma 2.3 *Let P and Q be two polytopes. There exists an injection ϕ from the vertex set of P into the vertex set of P + Q such that, for every vertex u of P, $\phi(u) = u + v$, where v is a vertex of Q.*

Consider a face F of a polytope P. Recall that the normal cone of P at F is the set of all the vectors c such that the map $x \mapsto c{\cdot}x$ is minimized, in P, exactly at a face that contains F. The normal fan of P is the complete polyhedral fan made up of the normal cones of P at all of its faces.

The following result is proved in [7].

Lemma 2.4 ([7, Theorem 4]) *If the normal fans of two polytopes P and Q coincide, then P has a summand homothetic to Q.*

Note that [7, Theorem 4] actually provides four statements equivalent to the normal fans of two polytopes coinciding. Lemma 2.4 only borrows the part of this theorem that we will make use of here.

Theorem 2.5 *A polytope P has a summand homothetic to a polytope Q if and only if P and P + Q have the same number of vertices.*

Proof Assume that P has a summand homothetic to Q; that is, P is equal to $\alpha Q + R$ for some positive number α and some polytope R. In this case, Lemma 2.1 provides a bijection between the vertex set of P and the vertex set of $P + Q$. Indeed, let u and v be two points in Q and R, respectively. By Lemma 2.1, $\alpha u + v$ is a vertex of P if and only if there exists a vector c such that the map $x \mapsto c{\cdot}x$ is uniquely minimized at αu in αQ and at v in R. This is equivalent to the map $x \mapsto c{\cdot}x$ being uniquely minimized at $(1 + \alpha)u$ in $(1 + \alpha)Q$ and at v in R. Since $P + Q$ is equal to $(1 + \alpha)Q + R$, it follows

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from Lemma 2.1 that the map $\alpha u + v \mapsto (1+\alpha)u + v$ is a bijection between the vertex set of P and the vertex set of $P+Q$.

Now assume that P and $P+Q$ have the same number of vertices. According to [12, Proposition 7.12], the normal fan of $P+Q$ refines the normal fan of P. In other words, the normal cones of $P+Q$ form polyhedral subdivisions of each of the normal cones of P. Since P and $P+Q$ have the same number of vertices, their normal fans contain the same number of full-dimensional cones and, therefore, the two fans must coincide. According to Lemma 2.4, P has a summand homothetic to $P+Q$ and, in turn, a summand homothetic to Q. ∎

The proof we give here for Theorem 2.5 illustrates how Lemma 2.1 can be used. Note, however, that this theorem can be given a shorter proof by making use of a result from [8] instead of Lemmas 2.1 and 2.4. Indeed, it follows from Theorem 2 therein that a polytope Q is homothetic to a summand of a polytope P if and only if the normal fan of P refines the normal fan of Q. According to [12, Proposition 7.12], this is equivalent to the equality between the normal fans of P and $P+Q$ and to the equality between their number of vertices.

A weaker version of Theorem 2.5 where P is a lattice polytope and Q is lattice segment is used in [2] in order to enumerate lattice polytopes with given properties. In the case of lattice polytopes, the summand homothetic to Q in the statement of Theorem 2.5 is homothetic to Q by an integer coefficient, which allows for a convenient enumeration procedure. A consequence of Theorem 2.5 is that it makes it possible to check whether a Minkowski difference is possible between P and a polytope homothetic to Q by only computing the vertices $P+Q$ and comparing its number of vertices to that of P. Another consequence is that it provides an convenient way to tell whether a lattice polytope P is a zonotope: it suffices to compute the Minkowski sums of P with each of its edges (up to parallelism) and to compare the number of vertices of the resulting polytopes with that of P.

3 Bounds on the Diameter of Minkowski Sums

The purpose of this section is to investigate the possible range for the diameter of a Minkowski sum in terms of the diameter and the number of vertices of its summands. In the sequel, the diameter of a polytope P is denoted by $\delta(P)$. We begin with a general lower bound that only depends on the diameter of the summands.

Theorem 3.1 *For any two polytopes P and Q,*

$$\delta(P+Q) \geq \max\{\delta(P), \delta(Q)\}.$$

Proof By Lemma 2.3, there exists an injection ϕ from the vertex set of P into the vertex set of $P+Q$ such that, for every vertex v of P, the Minkowski decomposition of $\phi(v)$ contains v as one of its two summands. Consider two vertices u and v of P distant of $\delta(P)$ in the graph of P. By Lemma 2.2, for any path of length l between $\phi(u)$ and $\phi(v)$ in the graph of $P+Q$, there exists a path of length at most l between u and v in the graph of P. As a consequence, the distance between u and v in the graph of P

is at most the distance between $\phi(u)$ and $\phi(v)$ in the graph of $P+Q$. Therefore, $\delta(P)$ is not greater than $\delta(P+Q)$ and, by symmetry, the desired inequality holds. ∎

The inequality provided by Theorem 3.1 is sharp, since $\delta(2P) = \delta(P)$ for any polytope P. This inequality is used in [2] in the case when Q is a line segment, in order to evaluate the diameter of lattice polytopes.

It turns out that there is no upper bound on the diameter of a Minkowski sum only in terms of the diameter of the summands. More precisely, we provide a pair of polytopes, each of diameter 2, whose diameter of the Minkowski sum can grow arbitrarily large. The construction relies on the following proposition that provides polytopes of any dimension and any diameter.

Proposition 3.2 *For any two positive integers d and k, there exists a polytope of dimension d and diameter k.*

Proof We distinguish two cases. First, assume that $k \geq d-1$. Consider a polygon with $2(k-d)+5$ vertices (whose diameter is therefore $k-d+2$) and a $(d-2)$-dimensional cube. Let P be the cartesian product of the polygon and the cube. This cartesian product can be alternatively obtained by taking a prism over the polygon and then a prism over this prism, and so on until the resulting polytope is d-dimensional. Since the diameter of a prism is the diameter of its base plus 1, the diameter of P is equal to k. Now assume that $k < d-1$. Consider a $(d-k+1)$-dimensional simplex and a $(k-1)$-dimensional cube. As above, the Minkowski sum P of the simplex and the cube is a d-dimensional polytope obtained by taking successive prisms over the simplex. Hence, as the diameter of a prism is the diameter of its base plus 1 and as simplices have diameter 1, the diameter of P is equal to k. ∎

By Proposition 3.3, the diameter of a Minkowski sum of two polytopes can grow arbitrarily large, even if both polytopes have a fixed diameter.

Proposition 3.3 *For any $d \geq 3$ and $k \geq 4$, there exist two d-dimensional polytopes, both of diameter 2, whose Minkowski sum has diameter k*

Proof By Proposition 3.2, there exists a polytope B of dimension $d-1$ and diameter $k-2$. We will think of B as embedded in a hyperplane H of $\mathbb{R}^d$. Consider two points p and q placed in $\mathbb{R}^d \setminus H$ in such a way that the line segment between p and q goes through the relative interior of B. Let P and Q be the pyramids over B whose apices are p and q. By construction, P and Q both have diameter 2. Note that $p+B$ and $q+B$ are two translates of B placed in distinct hyperplanes parallel to H. The Minkowski sum of P and Q is the convex hull of these two translates of B, and of the polytope $2B$ (the Minkowski sum of B with itself) placed between them in a third hyperplane parallel to H. In particular, all the faces of $p+B$ and $q+B$ are also faces of $P+Q$. Moreover, since the line segment between p and q goes through the relative interior of B, all the proper faces of $2B$ are faces of $P+Q$, and all the remaining proper faces of $P+Q$ are precisely obtained as the convex hull of $x+F$ and $2F$, where F is proper a face of B, and x is equal to p or to q. Combinatorially, $P+Q$ can be thought of as a prism on both sides of B. Since the diameter of a prism is the diameter of its base plus 1, the diameter of $P+Q$ is equal to k. ∎

When d is equal to 3, the construction in the proof of Proposition 3.3 consists in considering a convex polygon B with $2k-3$ vertices and two pyramids P and Q over this polygon whose apices are joined by a line segment going through the relative interior of B. A property of this construction is that both P and Q have diameter 2. It would be interesting to know whether a statement similar to that of Proposition 3.3 is true with polytopes of smaller diameter. More precisely, we ask the following.

Question 3.4 Do there exist a polytope of diameter 1 and a polytope of diameter 1 or 2 whose Minkowski sum has an arbitrarily large diameter?

On the one hand, Proposition 3.3 shows that there is no finite upper bound on the diameter of a Minkowski sum of polytopes only in terms of the diameter of the summands. In other words, the ratio

$$\frac{\delta(P+Q)}{\delta(P)\delta(Q)}$$

can grow arbitrarily large. On the other hand, there is a coarse upper bound for the diameter of $P+Q$ in terms of the number of vertices of P and Q, which we denote by $f_0(P)$ and $f_0(Q)$, respectively. Since a geodesic in the graph of $P+Q$ cannot visit a vertex twice, the diameter of $P+Q$ is at most the number of vertices of $P+Q$, which is in turn bounded above by $f_0(P)f_0(Q)$.

The main result of this section is the following refined bound, which combines the diameters of P and Q and the number of their vertices.

Theorem 3.5 *For any two polytopes P and Q,*

$$\delta(P+Q) < \min\big\{(\delta(P)+1)f_0(Q), f_0(P)(\delta(Q)+1)\big\}.$$

As will be shown in Section 4, this bound is sharp when the diameter of one summand grows large and the other summand is a line segment or a polygon with an arbitrarily large number of vertices. In order to prove Theorem 3.5, we will introduce the following family of graphs, whose vertex sets form a partition of the vertices of the Minkowski sum of two polytopes.

Definition 3.6 For any vertex u of a polytope P and any polytope Q, call $\Gamma_{P,Q}(u)$ the subgraph induced in the graph of $P+Q$ by the vertices whose Minkowski decomposition is of the form $u+v$, where v is a vertex of Q.

Note that the injection ϕ provided by Lemma 2.3 is precisely a map that sends each vertex u of P to a vertex of $\Gamma_{P,Q}(u)$. Let us illustrate the graphs $\Gamma_{P,Q}(u)$ using the Minkowski sum of a triangle P and a line segment Q depicted in Figure 1. When u is the vertex at the top of P, $\Gamma_{P,Q}(u)$ is the graph made up of the line segment at the top of $P+Q$ and its two vertices. When u is one of the vertices at the bottom of P, $\Gamma_{P,Q}(u)$ is made up of a single vertex and no edge; this vertex is the one bottom left of $P+Q$ if u is the vertex bottom left of P, and bottom right of $P+Q$ if u is the vertex bottom right of P. Further observe that $\Gamma_{Q,P}(u)$ is the oblique edge on the left of $P+Q$ together with its vertices when u is the vertex at the left of Q and the other oblique edge of $P+Q$ together with its vertices when u is the vertex at the right of Q.

Lemma 3.7 *Consider two polytopes P and Q. For any vertex u of P, the graph $\Gamma_{P,Q}(u)$ is connected.*

Proof Denote by N the interior of the normal cone of P at u and recall that N is precisely the set of the vectors c such that the map $x \mapsto c{\cdot}x$ is minimized exactly at u in P. According to Lemma 2.1, the Minkowski sum of u with a face F of Q is a face of $P+Q$ if and only if N is non-disjoint from the relative interior of the normal cone of Q at F. As a consequence, it follows from Definition 3.6 that for every vertex v of Q, $u+v$ is a vertex of $\Gamma_{P,Q}(u)$ if and only if some point belongs both to N and to the interior of the normal cone of Q at v.

Let v and w be two vertices of Q such that $u+v$ and $u+w$ are vertices of $\Gamma_{P,Q}(u)$. Choose a point p_v that belongs to N and to the interior of the normal cone of Q at v. Similarly, let p_w be a point in the intersection of N with the interior of the normal cone of Q at w. Since p_v and p_w are picked from open sets, we can assume that the line segment between them does not meet a face of dimension less than $d-1$ in the normal fan of Q. This can be achieved by, if needed, perturbing p_v or p_w slightly. By construction, when going from p_v to p_w along the line segment that joins these points, one meets the interiors of a sequence of full-dimensional cones in the normal fan of Q, glued along cones of codimension 1. These cones are the normal cones of Q at the vertices and at the edges of a path in the graph of Q from v to w. By the convexity of N, the relative interiors of all these cones are non-disjoint from N. According to the above observation, the Minkowski sum of u with the vertices and the edges of the path we found in the graph of Q from v to w form a path from $u+v$ to $u+w$ in $\Gamma_{P,Q}(u)$. ∎

Lemma 3.8 tells how the subgraphs induced by the graphs $\Gamma_{P,Q}(u)$ relate to one another within the graph of $P+Q$.

Lemma 3.8 *Consider two polytopes P and Q. Two distinct vertices u and v of P are adjacent in the graph of P if and only if there exist a vertex of $\Gamma_{P,Q}(u)$ and a vertex of $\Gamma_{P,Q}(v)$ that are adjacent in the graph of $P+Q$.*

Proof First consider an edge of $P+Q$ between a vertex of $\Gamma_{P,Q}(u)$ and a vertex of $\Gamma_{P,Q}(v)$. This edge is the Minkowski sum of a face of P with a face of Q, both of dimension 0 or 1. The face of P is necessarily the line segment with vertices u and v because these vertices are distinct.

Now assume that u and v are adjacent in the graph of P. Consider a projection π on some linear hyperplane H of the ambient space that sends u and v to the same point. Observe that $\pi(u)$ is a vertex of $\pi(P)$ and consider a vector $c \in \mathbb{R}^d$ such that the map $x \mapsto c{\cdot}x$ is uniquely minimized at $\pi(u)$ in $\pi(P)$. This map is minimized at a face F in Q. According to Lemma 2.1, $\pi(u)+F$ is a face of $\pi(P)+\pi(Q)$, and the vertices of this face are precisely the Minkowski sums of u with the vertices of F. Hence, there exists a vertex of $\pi(P)+\pi(Q)$, obtained as the Minkowski sum of $\pi(u)$ with a vertex, say $\pi(w)$ of $\pi(Q)$. Since Minkowski sums commute with projections, $\pi(u+w)$ is a

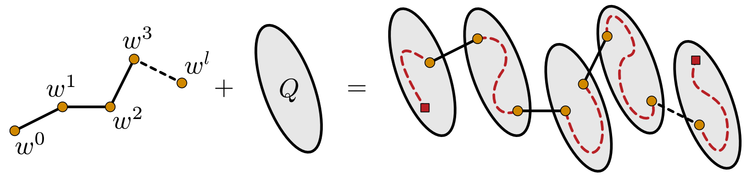

Figure 2: (Color online.) The Minkowski sum of a path with a polytope Q.

vertex of $\pi(P+Q)$. Now observe that the face of $P+Q$ whose image by π is $\pi(u+w)$ is either a vertex or an edge. Since u and v are distinct, this face is an edge between a vertex of $\Gamma_{P,Q}(u)$ and a vertex of $\Gamma_{P,Q}(v)$. ∎

We are now ready to prove Theorem 3.5. The strategy, illustrated in Figure 2, is to look at how a geodesic of length l in the graph of P is modified by the Minkowski sum with Q: informally, the Minkowski sum copies each of the $l+1$ vertices in the geodesic at most $f_0(Q)$ times. In particular, the term $\delta(P)+1$ in the bound provided by Theorem 3.5 can be thought of as the number of vertices of a path of length $\delta(P)$. Figure 2 shows how copies of the first and last vertices of the path (represented as red squares) can be connected via copies of the vertices and edges of the original path.

Proof of Theorem 3.5 Consider two vertices u and v of P such that in the graph of $P+Q$, the largest possible distance between a vertex of $\Gamma_{P,Q}(u)$ and a vertex of $\Gamma_{P,Q}(v)$ is exactly $\delta(P+Q)$. Denote by l the distance of u and v in the graph of P. We show that the distance, in the graph of $P+Q$, between a vertex of $\Gamma_{P,Q}(u)$ and a vertex of $\Gamma_{P,Q}(v)$ is at most $(l+1)f_0(Q)$. Consider a geodesic from u to v in the graph of P. Denote by w^0 to w^l the vertices along this geodesic in such a way that w^0 coincides with u, w^l coincides with v, and w^{i-1} is adjacent to w^i in the graph of P, for all i.

According to Lemma 3.7, $\Gamma_{P,Q}(w^i)$ is a connected graph. We will denote the diameter of this graph by $\delta(\Gamma_{P,Q}(w^i))$. By Lemma 3.8, some vertex of $\Gamma_{P,Q}(w^{i-1})$ is adjacent to a vertex of $\Gamma_{P,Q}(w^i)$ in the graph of $P+Q$. Therefore, the largest distance in the graph of $P+Q$ between any vertex of $\Gamma_{P,Q}(u)$ and any vertex of $\Gamma_{P,Q}(v)$, that is the diameter of $P+Q$, is bounded as

$$\delta(P+Q) \le l + \sum_{i=0}^{l} \delta(\Gamma_{P,Q}(w^i)). \tag{3.1}$$

Now observe that $\Gamma_{P,Q}(w^i)$ has at most $f_0(Q)$ vertices. As a direct consequence, its diameter is at most $f_0(Q)-1$, and (3.1) yields

$$\delta(P+Q) < (l+1)f_0(Q).$$

Since l is the distance between two vertices in the graph of P, it is bounded above by $\delta(P)$, and we obtain the desired inequality. ∎

4 The Polytopes $\Xi(k,l)$ and $\widetilde{\Xi}(k,l,m)$

In this section, we describe two families of 3-dimensional polytopes. The first family, which we will denote by $\Xi(k,l)$, shows that Theorem 3.5 is sharp for the Minkowski

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sum with a line segment, even when the diameter of the other summand is large. In other words, one can nearly double the diameter of a polytope by taking the Minkowski sum with a line segment. The other family of polytopes, which will be denoted by $\widetilde{\Xi}(k,l,m)$, will show that Theorem 3.5 is also sharp for the Minkowski sum with a polygon, even when both the number of vertices of the polygon and the diameter of the other summand are arbitrarily large.

Consider the 3-dimensional polytope $\Xi(5,4)$ sketched in Figure 3. The left of the figure shows $\Xi(5,4)$ from above, and the right of the figure shows it from below. The vertices represented as blue squares are the vertices of a regular decagon A. In particular they all belong to $\mathbb{R}^2$, which we think of as a *horizontal* plane. On the view of $\Xi(5,4)$ from above, the vertices marked with red disks are only slightly above $\mathbb{R}^2$ and their orthogonal projection on $\mathbb{R}^2$ belongs to every other edge of the decagon. Similarly, the vertices marked with green disks on the view of $\Xi(5,4)$ from below are just slightly below $\mathbb{R}^2$ and their orthogonal projection on $\mathbb{R}^2$ also belongs to every other edge of A, but with the requirement that a red and a green vertex never project on the same edge of A. It follows that $\Xi(5,4)$ has vertical facets, sketched in the center of the figure, each with two blue vertices and three other vertices, either all red or all green. The way the vertical facets are glued to the other facets of $\Xi(5,4)$ is indicated by arrows in the figure. The polytope $\Xi(5,4)$ also has two congruent horizontal facets colored grey in Figure 3, each with 20 vertices. All the other facets of $\Xi(5,4)$ are quadrilaterals or isosceles triangles. Each quadrilateral shares an edge with a horizontal facet and an edge with a vertical facet. Each triangle shares a vertex with a horizontal facet and an edge with a vertical facet. Observe that $\Xi(5,4)$ admits a natural generalization. One can define a similar 3-dimensional polytope whose projection on $\mathbb{R}^2$ is a regular polygon with $2k$ vertices (which we shall also denote by A) instead of a decagon, and such that there are $l-1$ red (or green) vertices between two blue vertices, instead of just 3. The resulting 3-dimensional polytope, which we will denote by $\Xi(k,l)$, still has two horizontal facets, each with kl vertices. It also has $2k$ vertical facets, each with two blue vertices and $l-1$ red or green vertices. The other facets of $\Xi(k,l)$ are $2k$ isosceles triangles and $2kl$ quadrilaterals.

Proposition 4.1 *The diameter of $\Xi(k,l)$ is at most $k+l+2$.*

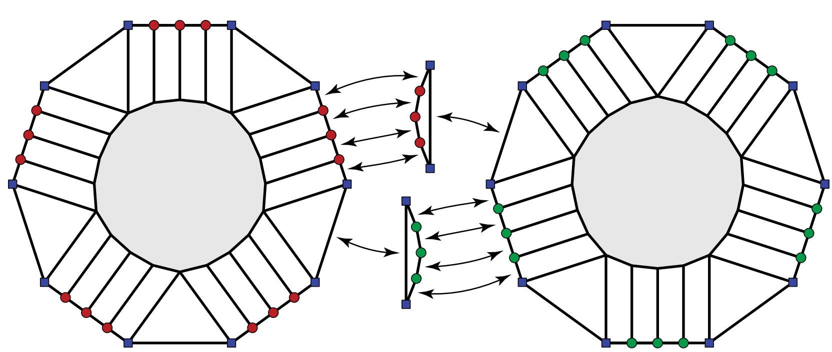

Figure 3: (Color online.) The 3-dimensional polytope $\Xi(5,4)$.

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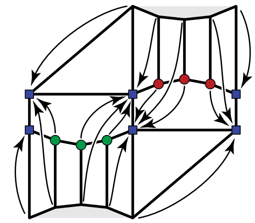

Figure 4: (Color online.) The map λ.

Proof Observe that the distance in the graph of $\Xi(k,l)$ from a red or green vertex to a blue vertex is at most $l/2$. Since the vertices of the horizontal facets are adjacent to a red or a green vertex, their distance to a blue vertex in the graph of $\Xi(k,l)$ is at most $l/2+1$. As two blue vertices are distant by at most k in the graph of A, we obtain the desired upper bound on the diameter of $\Xi(x,l)$. ∎

Proposition 4.2 *The Minkowski sum of* $\Xi(k,4)$ *with a vertical line segment has diameter at least* $2k$.

Proof First observe that taking the Minkowski sum of $\Xi(k,4)$ with a vertical line segment Σ does not modify the non-vertical facets of $\Xi(k,4)$, except for a possible translation. The only facets of $\Xi(k,4)$ whose geometry is modified by the Minkowski sum are the vertical ones. In these facets, the blue vertices are replaced by a translate of Σ. The two vertices of this edge can be understood as two copies of a blue vertex, and will also be referred to as blue vertices. In particular, the vertical facets of $\Xi(k,4)+\Sigma$ incident to a given blue vertex now share an edge, as shown in Figure 4.

Consider the map λ that sends each blue vertex of $\Xi(k,4)+\Sigma$ to itself and every other vertex of $\Xi(k,4)+\Sigma$ to a blue vertex, as indicated with arrows in Figure 4. While the figure only depicts λ next to a pair of vertical facets, the rest of the map can be recovered using the rotational symmetry of $\Xi(k,4)+\Sigma$. Observe that λ maps any two adjacent vertices of $\Xi(k,4)+\Sigma$ to adjacent or identical vertices. In particular, this map transforms a path between two blue vertices in the graph of $\Xi(k,4)+\Sigma$ into a path whose length has not increased between the same two blue vertices. Along the path resulting from the transformation, all the vertices are blue. As a consequence, the distance between two blue vertices can be measured within the cycle induced by blue vertices in the graph of $\Xi(k,4)+\Sigma$. Since this cycle has diameter $2k$, $\Xi(k,4)$ necessarily has diameter at least $2k$, as desired. ∎

Combining Propositions 4.1 and 4.2 shows that the upper bound provided by Theorem 3.5 is asymptotically sharp for the Minkowski sum with a line segment when the diameter of the other summand grows large.

Theorem 4.3 *If* Σ *is a vertical line segment, then*

$$\lim_{k\to\infty}\frac{\delta(\Xi(k,4)+\Sigma)}{\delta(\Xi(k,4))}=2.$$

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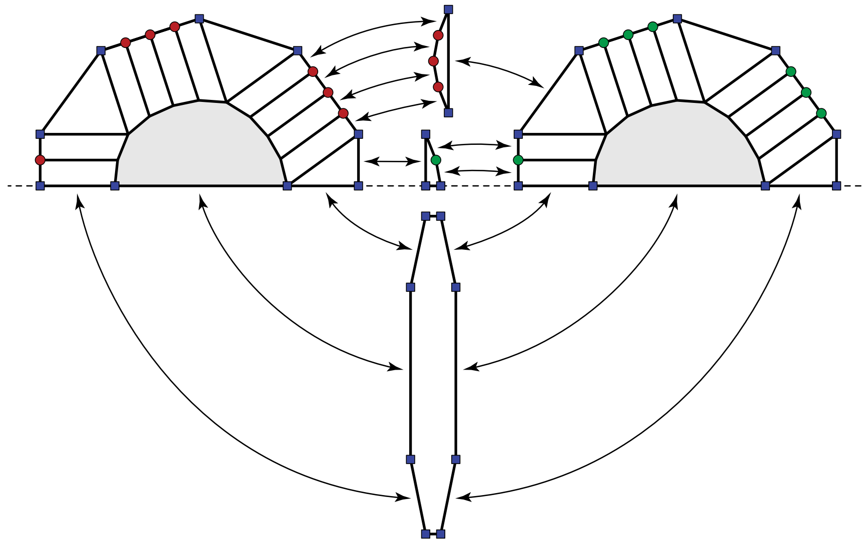

Figure 5: (Color online.) The polytope $\Theta(5,4)$.

The polytope $\Xi(k,l)$ is now modified into another polytope whose diameter gets multiplied by the number of vertices (that can be arbitrary) of a well-chosen polygon under the Minkowski sum with this polygon. The first step of this modification, depicted in Figure 5 when $k = 5$ and $l = 4$, consists in cutting $\Xi(k,l)$ in half. The cut is performed along a vertical plane M that contains the center of two opposite edges of A. This plane is represented as a dashed line in the figure. If l is even, which we will assume from now on, then M contains two edges of $\Xi(k,l)$, and cuts in half its two horizontal grey facets, two of its triangular facets, and two of its vertical facets. In particular, the intersection of M and $\Xi(k,l)$ is an octagon, as shown in Figure 5. Now consider the polytope $\Theta(k,l)$ obtained as the intersection of $\Xi(k,l)$ with one of the closed half-spaces bounded by M. The octagon $M \cap \Xi(k,l)$ is a vertical facet of $\Theta(k,l)$ whose all eight vertices will be thought of as blue vertices and represented as blue squares. All the other vertices of $\Theta(k,l)$ will keep the color they have as vertices of $\Xi(k,l)$. The orthogonal projection on $\mathbb{R}^2$ of $\Theta(k,l)$ is now a polygon with $k+2$ vertices. The way the vertical facets of $\Theta(k,l)$ are glued to the other facets of $\Theta(k,l)$ is indicated by arrows in Figure 5.

We will further modify $\Theta(k,l)$ into a polytope $\widetilde{\Xi}(k,l,m)$ by gluing small polytopes to each of the vertical facets of $\Theta(k,l)$ that are disjoint from M. In order to build these polytopes, we will use homothetic translates of the vertical polygon Π with $m+1$ vertices depicted on the left of Figure 6 when $m = 4$. Let us first describe this polygon. The intersection of Π with M is the longest edge of Π, which will be referred to as e. In the figure, the half-space limited by M that does not contain $\Theta(k,l)$ is striped and one can see that the vertices of Π outside of M are on the same side of M than $\Theta(k,l)$. As also shown on the figure, the orthogonal projections on M of these vertices belong to the relative interior of e. The largest distance to M of a vertex

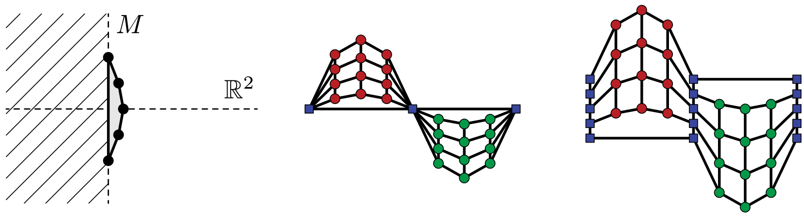

Figure 6: (Color online.) The polygon Π (left), the polytopes P_F glued to the vertices facets of $\Theta(k,l)$ in order to build $\widetilde{\Xi}(k,l,m)$ (center), and a sketch of the Minkowski sum between Π and these polytopes (right).

of Π will be denoted by ε. Observe that ε can be taken arbitrarily small, which will be instrumental for the construction of $\widetilde{\Xi}(k,l,m)$.

Now consider a vertical facet F of $\Theta(k,l)$ that is disjoint from M. The announced polytope P_F, which we will glue to F, will be the convex hull of F and of $l-1$ polygons homothetic to Π. Consider a red or a green vertex v of F and call e' the vertical line segment incident to v whose other vertex is in the horizontal edge of F. Denote by α the real number such that αe and e' have the same length. We can then translate $\alpha\Pi$ and glue it to F in such a way that e and e' coincide. The polytope P_F is the convex hull of F and of the $l-1$ homothetic translates of Π glued to F when v ranges over the red or green vertices of F. The projection of P_F back on F is depicted in the center of Figure 6 for two consecutive vertical facets of $\Theta(k,4)$ when $m=4$. Note that the projection is made along the direction orthogonal to M. Further note that, apart from two blue vertices, all the vertices of P_F will be colored red or green depending on whether F has red or green vertices. If ε is small enough, then gluing these polytopes to each of the vertical facets of $\Theta(k,l)$ that are disjoint from M results in a new polytope $\widetilde{\Xi}(k,l,m)$ whose vertex set contains all the vertices of $\Theta(k,l)$, together with $(k-1)(l-1)(m-1)$ new vertices. The diameter of this polytope is bounded as follows.

Proposition 4.4 *The diameter of $\widetilde{\Xi}(k,l,m)$ is at most $(k+5)/2+l+2$.*

Proof We proceed as in the proof of Proposition 4.1. Every vertex in the graph of $\widetilde{\Xi}(k,l,m)$ is distant by at most $l/2+1$ of a blue vertex. By construction, there are exactly $k+8$ blue vertices in this graph and the subgraph they induce is made up of a cycle of length 8 corresponding to the boundary of the octagonal facet of $\widetilde{\Xi}(k,l,m)$, and of a simple path of length $k+1$ whose extremities are two vertices of that cycle. As a consequence, two blue vertices are distant by at most $(k+5)/2$ in the graph of $\widetilde{\Xi}(k,l,m)$ and we obtain an upper bound of $(k+5)/2+l+2$ on the diameter of that graph, as desired. ∎

According to Lemma 2.1, when taking the Minkowski sum of $\widetilde{\Xi}(k,l,m)$ with the polygon Π, the only faces whose geometry is affected are the vertical facets of $\widetilde{\Xi}(k,l,m)$ and the faces of the polytope P_F for each of the vertical facets F of $\Theta(k,l)$ that are disjoint from M. Consider such a facet F of $\Theta(k,l)$. By construction, every facet of P_F is parallel to an edge of Π. In particular, according to Lemma 2.1, the Minkowski sum with Π affects the facets of P_F as shown on the right of Figure 6. Note that each of the blue vertices of P_F will be copied $m+1$ times. Each of these copies will be thought of as a blue vertex and represented as a blue square. The two vertical facets of $\widetilde{\Xi}(k,l,m)$ obtained by cutting in half a vertical facet of $\Xi(k,l,m)$ also each gain exactly m new blue vertices. The vertical octagonal facet of $\widetilde{\Xi}(k,l,m)$ remains an octagon after the Minkowski sum with Π, although its two vertical edges are longer by the length of e. All the vertices of that deformed octagon will still be considered blue vertices. It follows that $\widetilde{\Xi}(k,l,m)+\Pi$ has $k(m+1)+8$ blue vertices. By construction, the subgraph induced by these blue vertices in the graph of $\widetilde{\Xi}(k,l,m)+\Pi$ is made up of a simple path of length $k(m+1)+1$ tied at each end to the graph of the octagonal facet. In particular, that subgraph has diameter at least $k(m+1)/2$. A portion of this subgraph is depicted in Figure 7. We will show that, when l is large enough, the long geodesics in the graph of $\widetilde{\Xi}(k,l,m)+\Pi$ will mostly visit blue vertices.

Proposition 4.5 *If $l \geq 2m+8$, then the Minkowski sum of $\widetilde{\Xi}(k,l,m)$ with Π has diameter at least $k(m+1)/2$.*

Proof We will proceed as for Proposition 4.2. As observed above, the subgraph induced by the blue vertices in the graph of $\widetilde{\Xi}(k,l,m)+\Pi$ has diameter at least $k(m+1)/2$. As a consequence, we only need to find a map λ that takes each vertex of $\widetilde{\Xi}(k,l,m)+\Pi$ to a blue vertex in such a way that blue vertices are sent to themselves and any two adjacent vertices are sent either to adjacent vertices or to the same vertex.

First, consider the facets of $\widetilde{\Xi}(k,l,m)+\Pi$ sketched on the right of Figure 6. The way λ affects the vertices of these facets is shown on the left and in the center of Figure 7. The sketch has been deformed for clarity, which does not matter since λ is only combinatorial. Observe that the red and green vertices are arranged in horizontal layers bounded by a blue vertex on the left and on the right. There are $l-1$ red or green vertices in each of these layers. There is an additional layer made up of the two blue vertices of a horizontal edge of $\widetilde{\Xi}(k,l,m)+\Pi$, shown below the red vertices on the left of the figure and above the green vertices in the center of the figure. The map λ takes the first green or red vertex in a layer (from the left or from the right of the layer) to the blue vertex closest to it. The second green or red vertex in a layer will be sent to the blue vertex closest to it in the next layer and so on. Upon reaching the layer made up of a single horizontal edge of $\widetilde{\Xi}(k,l,m)+\Pi$, vertices will all be sent to the vertex of this edge closest to them in the graph of $\widetilde{\Xi}(k,l,m)+\Pi$. If $l \geq 2m+4$, then λ takes adjacent vertices to either adjacent or identical blue vertices. Note that since l is even,

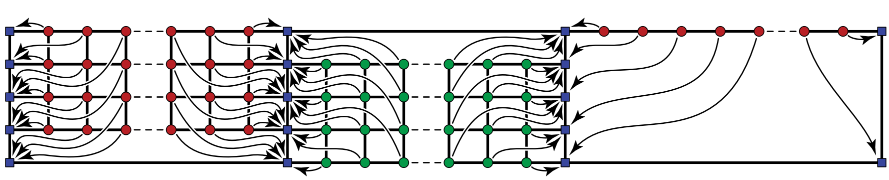

Figure 7: (Color online.) The map λ for the polytope $\widetilde{\Xi}(k,l,m)$.

there is a vertex in the center of each layer. This vertex can be sent indifferently to any of the two vertices of the horizontal edge of $\widetilde{\Xi}(k,l,m)+\Pi$ in the last layer.

Now consider a facet of $\widetilde{\Xi}(k,l,m)+\Pi$ whose intersection with M is an edge. The map λ affects the vertices of this facet as shown on the right of Figure 7, where the vertices that belong to M are depicted on the right. Note that several vertices may be sent to any of the blue vertices at the bottom of the facet in case l grows large. By construction, this facet has $l/2-1$ red or green vertices and $m+3$ blue vertices. Therefore, λ takes adjacent vertices to either adjacent or identical blue vertices as soon as l is greater than or equal to $2m+8$.

It remains to explain where λ sends the vertices of the horizontal grey facets. Any vertex of these facets that belongs to M is blue and, therefore, is sent to itself by λ. For any other vertex of these facets, the transformation is similar to what is shown in Figure 4: if such a vertex is adjacent to a red or a green vertex v, then its image by λ will be $\lambda(v)$. Otherwise, its image by λ is any of the blue vertices it is adjacent to.

This defines a map λ that sends blue vertices to themselves and any two adjacent vertices of $\widetilde{\Xi}(k,l,m)+\Pi$ to either adjacent or identical blue vertices, as desired. ∎

We obtain the following theorem by combining Propositions 4.4 and 4.5.

Theorem 4.6 *If $l \geq 2m+4$, then*

$$\lim_{k\to\infty}\frac{\delta(\widetilde{\Xi}(k,l,m)+\Pi)}{\delta(\widetilde{\Xi}(k,l,m))}=m+1.$$

In other words, the Minkowski sum with Π multiplies the diameter of $\widetilde{\Xi}(k,l,m)$ by the number of vertices of Π, even though both of these quantities can grow arbitrarily large. This might come as a surprise. Indeed, while a geodesic in the graph of Π never visits more than half of the vertices, the geodesics in the graph of $\widetilde{\Xi}(k,l,m)+\Pi$ will visit an arbitrarily large number of copies of each vertex of Π. This proves that the bound stated by Theorem 3.5 is sharp even when the diameter of one summand is arbitrarily large, and the other summand is a line segment or an arbitrarily large polygon. Note that by taking consecutive prisms over $\Xi(k,l)$ and $\widetilde{\Xi}(k,l,m)$, one obtains that for any fixed dimension d greater than 2, this bound remains sharp when one summand is d-dimensional and its diameter is arbitrarily large, while the other summand is a line segment or an arbitrarily large polygon.

Further note that when both summands have dimension at most 2, the diameter of their Minkowski sum is better behaved, since it is always at most, and can be equal to, the sum of the diameters of the two summands.

This begs the question whether the bound provided by Theorem 3.5 remains sharp when both summands are high dimensional.

Question 4.7 Do there exist two polytopes P and Q, both of dimension at least 3 such that $\delta(P)$ and $f_0(Q)$ are arbitrarily large, while the ratio between $\delta(P+Q)$ and $\delta(P)f_0(Q)$ gets arbitrarily close to 1?

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