ON A FAMILY OF SIMPLE ORDERED GROUPS

VLASTIMIL DLAB

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1. Introduction

In the present paper we shall consider some subgroups of (increasing) autohomeomorphisms of the closed real interval $\langle 0, 1\rangle$; mainly because of brevity, we shall defer discussing possible generalizations of our results to more general ordered fields.

Groups of (order) automorphisms of the real line, or more generally, of some ordered sets have been frequently used in constructions of groups with some specified properties (cf., e.g., [1], [4], [5]); in particular, the group of all piecewise linear automorphisms of $\langle 0, 1\rangle$ coinciding with the identical mapping in some neighbourhoods of 0 and 1 has served Chehata [1] in establishing the existence of an (algebraically) simple (linearly) ordered group.

Here, we present a whole family of (algebraically) simple (linearly) ordered groups G_H: for every subgroup H of the multiplicative group R of all positive real numbers, G_H is the group of all so-called locally right H-linear automorphisms of $\langle 0, 1\rangle$ coinciding with the identical mapping in some neighbourhoods of 0 and 1. In particular, G_H is divisible if and only if H is divisible; in fact, G_H is then, in a certain sense, strongly divisible. There is a one-to-one correspondence between the (linear) orders of G_H and those of H; thus, G_H admits only two (linear) orders if and only if H is of rank 1. Furthermore, G_R contains an isomorphic copy of any (linearly) ordered abelian group of countable "Archimedean rank" and the continuum of non-order-isomorphic (linearly) ordered free groups of rank 2.

A particular value of our approach rests on the fact that simplicity and other properties of the groups G_H can be established very easily.

2. Preliminaries

Let G be the group of all increasing autohomeomorphisms of the interval $\langle 0, 1\rangle$, i.e. the group of all continuous (strictly) increasing real functions on $\langle 0, 1\rangle$ having 0 and 1 as their fixed points. Throughout the paper, they will be denoted by small Greek letters and written on the right:

$$x(\alpha\beta) = (x\alpha)\beta \text{ for all } x \in \langle 0, 1\rangle;$$

ε will always denote the unity of G.

An open interval $(a, b) \subseteqq \langle 0, 1\rangle$ is said to be a *supporting interval* of $\alpha \in G$ if $a\alpha = a$, $b\alpha = b$ and $x\alpha \neq x$ for all $x \in (a, b)$. The (at most countable disjoint) union of all supporting intervals of α is called the *support* of α. An element of G with a single supporting interval is said to be *simple*.

Let H be a subgroup of the multiplicative group R of all positive real numbers; in particular, $H_r \subset R$ denotes, for every $r \in R$, the least divisible subgroup of R containing r. An automorphism $\alpha \in G$ is said to be *H-linear* in $\langle a, b\rangle \subseteqq \langle 0, 1\rangle$ if there is $h \in H$ such that

$$x\alpha = a\alpha + h(x-a) \text{ for all } x \in \langle a, b\rangle;$$

α is said to be *piecewise H-linear* in $\langle a, b\rangle$ if there exists a finite number of a_i's: $a = a_0 < a_1 < \cdots < a_n = b$ such that α is H-linear in each $\langle a_{i-1}, a_i\rangle$, $1 \leqq i \leqq n$.

In what follows we shall consider the subgroup $\bar{G}_H$ of G of the *locally right H-linear* automorphisms of $\langle 0, 1\rangle$, i.e. the subgroup of all $\alpha \in G$ such that, for any $a \in \langle 0, 1)$, there exists a positive real e^a_α and $h^a_\alpha \in H$ satisfying

$$x\alpha = a\alpha + h^a_\alpha(x-a) \text{ for all } x \in \langle a, a+e^a_\alpha\rangle;$$

thus, $\alpha \in \bar{G}_H$ if and only if, for any $a \in \langle 0, 1)$, there is a non-trivial interval $\langle a, a+e^a_\alpha\rangle$ in which α is H-linear.

To every $\alpha \neq \varepsilon$ of $\bar{G}_H$, let us make to correspond the (at most countable) well-ordered subset $A_\alpha = \{a_{\alpha t}\}$ of $\langle 0, 1\rangle$

$$0 \leqq a_{\alpha 1} < a_{\alpha 2} < \cdots < a_{\alpha t} < \cdots \leqq 1$$

defined as follows: $a_{\alpha t} \in A$ if and only if $a_{\alpha t}\alpha = a_{\alpha t}$ and there is no neighbourhood $N(a_{\alpha t})$ of $a_{\alpha t}$ such that $x\alpha = x$ in $N(a_{\alpha t})$. Notice that

$$a_{\alpha \tau_\alpha} = \sup_t a_{\alpha t}$$

is the greatest element of A_α and that α is simple if and only if A_α has two elements. In fact, $(a_{\alpha 1}, a_{\alpha 2})$ is always a supporting interval of α and, furthermore, provided $a_{\alpha\, t+2}$ exists, either $(a_{\alpha t}, a_{\alpha\, t+1})$ or $(a_{\alpha\, t+1}, a_{\alpha\, t+2})$ is a supporting interval of α.

Now, for every $\alpha \in \bar{G}_H$, consider the function h_α on $\langle 0, 1)$ mapping each $a \in \langle 0, 1)$ into the corresponding gradient $h^a_\alpha \in H$; as a matter of fact, we shall be particularly interested in the restriction of h_α to A_α. Thus, $h^{a_{\alpha t}}_\alpha \neq 1$ if and only if $(a_{\alpha t}, a_{\alpha\, t+1})$ is a supporting interval of α; of course, always $h^{a_{\alpha 1}}_\alpha \neq 1$. The function h_α restricted to A_α describes what we like to call the basic characteristic of α. More precisely, we shall say that α and β of $\bar{G}_H$ have the *same basic characteristic* if there is an automorphism $\varphi \in G$ of

$\langle 0, 1\rangle$ (or, what appears to be the same, $\varphi \in \bar{G}_H$ or even $\varphi \in G_{H*}$ defined below) mapping $A_\alpha = \{a_{\alpha t}\}$ onto A_β and satisfying

$$h_\alpha^{a_{\alpha t}} = h_\beta^{a_{\alpha t}\varphi}.$$

Besides the subgroup $\bar{G}_H$ of G, we shall investigate the subgroups G_{H*}, G_{*H} and G_H of $\bar{G}_H$: $\alpha \neq \varepsilon$ belongs to G_{H*}, G_{*H} or G_H if and only if $0 < a_{\alpha 1}$, $a_{\alpha \tau_\alpha} < 1$ or $0 < a_{\alpha 1} < a_{\alpha \tau_\alpha} < 1$, respectively. Evidently, all three groups are normal in $\bar{G}_H$; in fact, G_H is the only minimal normal subgroup of $\bar{G}_H$ (for the full normal structure of $\bar{G}_H$ see the diagram in Theorem 4.3.).

3. Preparatory results

PROPOSITION 3.1. (cf. [5], [7]). *Two non-unity elements α and β of $\bar{G}_H$, G_{H*}, G_{*H} or G_H are conjugate in the respective subgroup if and only if they have the same basic characteristic.*

In particular, if α and β of $\bar{G}_H$ have the same basic characteristic, then

$$\beta = \Psi^{-1}\alpha\Psi \tag{3.1}$$

with $\Psi \in G_H$ if $a_{\alpha \tau_\alpha} < 1$ and $\Psi \in G_{H*}$ otherwise.

PROOF. First, suppose that α and β are conjugate in G, i.e. that

$$\beta = \varphi^{-1}\alpha\varphi \text{ for a certain } \varphi \in G.$$

Let $A_\alpha = \{a_{\alpha t}\}$. Then, evidently,

$$A_\beta = \{a_{\beta t}\}, \text{ where } a_{\beta t} = a_{\alpha t}\varphi.$$

Moreover,

$$h_\beta^{a_{\alpha t}\varphi} = h_{\varphi^{-1}}^{a_{\alpha t}\varphi}\, h_\alpha^{a_{\alpha t}}\, h_\varphi^{a_{\alpha t}} = (h_\varphi^{a_{\alpha t}})^{-1}\, h_\alpha^{a_{\alpha t}}\, h_\varphi^{a_{\alpha t}} = h_\alpha^{a_{\alpha t}},$$

and thus, α and β have the same basic characteristic.

Now, assume that α and β have the same basic characteristic, i.e. that there is $\varphi \in G$ such that

$$A_\alpha = \{a_{\alpha t}\},\ A_\beta = \{a_{\alpha t}\beta\} \text{ and } h_\alpha^{a_{\alpha t}} = h_\beta^{a_{\alpha t}\varphi}.$$

In order to construct an element Ψ of $\bar{G}_H$ such that (3.1) holds, it is evidently sufficient to construct, for each supporting interval $(a_{\alpha t}, a_{\alpha t+1})$ of α, a function Ψ_t of G which is locally right H-linear in $\langle a_{\alpha t}, a_{\alpha t+1}\rangle$ and satisfies, moreover,

$$a_{\alpha t}\Psi_t = a_{\alpha t}\varphi, \qquad a_{\alpha t+1}\Psi_t = a_{\alpha t+1}\varphi$$

and

$$\Psi_t^{-1}\alpha\Psi_t = \beta \text{ in } \langle a_{\alpha t}\varphi, a_{\alpha t+1}\varphi\rangle.$$

For, then the continuous function Ψ defined by

$$x\Psi = x\Psi_t \text{ in each} (a_{\alpha t}, a_{\alpha t+1})$$

and piecewise H-linear in each complementary interval of the support of α in $\langle 0, 1\rangle$ belongs to $\bar{G}_H$ and satisfies (3.1); moreover, since our Ψ_1 will satisfy $x\Psi_1 = x$ in a neighbourhood of $a_{\alpha 1}$, Ψ can be taken from G_H if $a_{\alpha\tau_\alpha} < 1$ and from G_{H*} otherwise.

Thus, to complete the proof, we are going to construct a function Ψ_t with the appropriate properties. First, if $a_{\alpha t} > 0$, let

$$x\Psi_t = (a_{\alpha t}\varphi)a_{\alpha t}^{-1}x \text{ in } \langle 0, a_{\alpha t}\rangle$$

and, if $a_{\alpha t+1} < 1$,

$$x\Psi_t = 1+(1-a_{\alpha t+1}\varphi)(1-a_{\alpha t+1})^{-1}x \text{ in } \langle a_{\alpha t+1}, 1\rangle.$$

Without loss of generality, assume that

$$h_\alpha^{a_{\alpha t}} > 1. \tag{3.2}$$

Let α and β be linear in $\langle a_{\alpha t}, a_{\alpha t}+e_1\rangle$ and $\langle a_{\alpha t}\varphi, a_{\alpha t}\varphi+e_2\rangle$, respectively. Take $e > 0$ satisfying the relation

$$h_\alpha^{a_{\alpha t}} \cdot e \leqq \min(e_1, e_2)$$

and define

$$x\Psi_t = x+a_{\alpha t}\varphi - a_{\alpha t} \text{ for } x \in \langle a_{\alpha t}, a_{\alpha t}+e\rangle.$$

Thus, for

$$x \in \langle a_{\alpha t}\varphi, a_{\alpha t}\varphi+e\rangle,$$
$$x\Psi_t^{-1} = x+a_{\alpha t}-a_{\alpha t}\varphi$$

and

$$x(\Psi_t^{-1}\alpha\Psi_t) = [a_{\alpha t}+h_\alpha^{a_{\alpha t}}(x-a_{\alpha t}\varphi)]\Psi_t = x\beta.$$

Now, consider the increasing sequence $\{a_n\}$ defined by

$$a_n = (a_{\alpha t}+e)\alpha^{n-1} \quad \text{for} \quad n \geqq 1.$$

In view of (3.2), we have

$$\lim_n a_n = a_{\alpha t+1}.$$

Hence, for every $x \in \langle a_{\alpha t}+e, a_{\alpha t-1})$, there is (a unique) $n_0 \geqq 1$ such that $a_{n_0} < x \leqq a_{n_0+1}$, and we complete the definition of Ψ_t by putting

$$x\Psi_t = x(\alpha^{-n_0}\Psi_t\beta^{n_0}).$$

It is a matter of routine to check that Ψ_t possesses all the required properties and thus to complete the proof of Proposition 3.1.

LEMMA 3.2. *Let $\varepsilon \neq \omega \in \bar{G}_H$. If $a_{\omega 1} > 0$, then – for any given $0 \leqq a < a_{\omega 1}$, $a_{\omega\tau_\omega} \leqq b \leqq 1$ and $1 \neq h \in H$ – there are simple elements ρ and σ of $\bar{G}_H$ such that*

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$$a_{\rho 1} = a_{\sigma 1} = a, \quad h_\rho^a = h_\sigma^a = h, \quad a_{\rho \tau_\rho} = a_{\sigma \tau_\sigma} = b$$

and

$$\omega = \sigma^{-1}\rho.$$

On the other hand, if $\varepsilon \neq \omega = \sigma^{-1}\rho$ *with simple elements* ρ *and* σ *of* $\bar{G}_H$ *such that*

$$h_\rho^{a_{\rho 1}} = h_\sigma^{a_{\sigma 1}} = h, \quad h \neq h_\omega^{a_{\omega 1}}, \quad h \neq (h_\omega^{a_{\omega 1}})^{-1},$$

then

$$0 \leqq a_{\rho 1} = a_{\sigma 1} < a_{\omega 1} \text{ and } \max(a_{\rho \tau_\rho}, a_{\sigma \tau_\sigma}) \geqq a_{\omega \tau_\omega}.$$

PROOF. Define $\omega^* \in \bar{G}_H$ by

$$x\omega^* = x\omega \text{ for } x\omega \geqq x \text{ and } x\omega^* = x\omega^{-1} \text{ otherwise;}$$

thus, $x\omega^* \geqq x$ for all $x \in \langle 0, 1\rangle$. Put $h^* = \max(h, h^{-1})$. Now, denote by σ^* an element of $\bar{G}_H$ such that

$$\begin{aligned} x\sigma^* &= x \text{ in } \langle 0, a\rangle, \\ x\sigma^* &= a+h^*(x-a) \text{ in } \langle a, a+e\rangle \text{ for } 0 < e < a_{\omega 1}-a, \\ x\omega^* &< x\sigma^* \leqq b \text{ in } \langle a+e, b\rangle \end{aligned}$$

and

$$x\sigma^* = x \text{ in } \langle b, 1\rangle.$$

It is evident that such elements of $\bar{G}_H$ exist: Consider the greatest subintervals $\langle c_1, c_2\rangle$ of $\langle a_{\omega 1}, b\rangle$ such that ω^* is linear in $\langle c_1, c_2\rangle$; for each such interval put

$$c_i\sigma^* = \tfrac{1}{2}(b+c_i\omega^*), \qquad i = 1, 2,$$

and σ^* piecewise H-linear in $\langle c_1, c_2\rangle$; also, in $\langle a_1+e, a_{\omega 1}\rangle$, let σ^* be piecewise H-linear with $(a+e)\sigma^* = a+h^*e$ and $a_{\omega 1}\sigma^* = \frac{1}{2}(b+a_{\omega 1})$.

Now, if $h^* = h^j$, $j = \pm 1$, put $\sigma = (\sigma^*)^j$ and $\rho = \sigma\omega$. It is routine to check that β satisfies all the requirements. The other part of Lemma 3.2 is obvious.

PROPOSITION 3.3. *Let* $\alpha \neq \varepsilon$ *be a non-simple element of* $\bar{G}_H$. *Then, there exists* $\Psi \in \bar{G}_H$ *such that*

$$a_{\alpha 1} < a_{\Psi 1}, \quad a_{\Psi \tau_\Psi} \leqq a_{\alpha \tau_\alpha} \quad (i.e.\ \Psi \in G_{H*})$$

and

$$\beta = \alpha\Psi^{-1}\alpha\Psi$$

satisfies

$$A_\beta = \{a_{\alpha 1}, a_{\beta 2}, a_{\alpha \tau_\alpha}\} \text{ with } h_\beta^{a_{\alpha 1}} = (h_\alpha^{a_{\alpha 1}})^2.$$

Moreover, given an arbitrary element $1 \neq h \in H$, Ψ *can always be chosen so that*

$$h_\beta^{a_{2\beta}} = h$$

provided that two sequences $\{x_n\}$ *and* $\{y_n\}$ *exist such that*

$$\lim_n x_n = \lim_n y_n = a_{\alpha\,\tau_\alpha}$$

and

$$x_n\alpha > x_n, \quad y_n\alpha < y_n \quad \textit{for } n \geqq 1.$$

PROOF. Since the proof of the statement in each case to be considered has the same basic idea, we present here only the proof of the last assertion in the case $h_\alpha^{a_{\alpha 1}} < 1$.

First, assume that $h > 1$. Let

$$(3.3) \qquad a_{\alpha 1} < a_{\alpha 2} \leqq b_1 < c_1 \leqq \cdots \leqq b_n < c_n \leqq \cdots, \lim_n b_n = a_{\alpha\,\tau_\alpha}$$

be a sequence of supporting intervals of α such that

$$x_\alpha > x \text{ in each } (b_n, c_n), \qquad n \geqq 1.$$

Define in each (b_n, c_n) a subinterval $\langle u_n, v_n\rangle$ such that

$$v_n < u_n\alpha$$

and, moreover, such that α is linear in $\langle b_1, v_1\rangle$ and $u_n = v_n$ if, for $n \geqq 2$, $c_{n-1} = b_n$. Also, let $v_0 \in (a_{\alpha 1}, a_{\alpha 2})$ be a number such that α is linear in $\langle a_{\alpha 1}, v_0\rangle$; put $u_0 = \frac{1}{2}(a_{\alpha 1}+v_0)$.

Let $\bar\alpha \in \bar G_H$ be defined by

$$x\bar\alpha = x\alpha \text{ for } x \in \langle a_{\alpha 1}, a_{\alpha 2}\rangle,\ x \in \langle b_n, c_n\rangle, \qquad n \geqq 1$$

and

$$x\bar\alpha = x \text{ otherwise.}$$

Furthermore, define $\beta \in \bar G_H$ in the following manner:

(a) β is linear in $\langle a_{\alpha 1}, u_0\rangle$, $\langle u_0, w_0\rangle$, $\langle w_0, u_1\alpha^{-1}\rangle$ $\langle u_1\alpha^{-1}, z_0\rangle$ and $\langle z_0, u_1\rangle$ with

$$a_{\alpha 1}\beta = a_{\alpha 1}, \quad (u_1\alpha^{-1})\beta = u_1\alpha^{-2}, \quad u_1\beta = u_1,$$

$$h_\beta^{a_{\alpha 1}} = h_\alpha^{a_{\alpha 1}}, \qquad h_\beta^{u_1\alpha^{-1}} = h\cdot(h_\alpha^{b_1})^{-1},$$

$$h_\beta^{u_0} \leqq \frac{v_0\alpha - a_{\alpha 1}}{2(b_1-u_0)}, \quad h_\beta^{u_0} \in H,$$

$$h_\beta^{w_0} \geqq \frac{u_1\alpha^{-2}-v_0\alpha}{u_1\alpha^{-1}-b_1}, \quad h_\beta^{w_0} \in H,$$

and

$$h_\beta^{z_0} \geqq \frac{h}{h-1}, \quad h_\beta^{z_0} \in H.$$

(b) for $n \geqq 1$, β is linear in $\langle v_n, w_n\rangle$, $\langle w_n, z_n\rangle$ and $\langle z_n, u_{n+1}\rangle$ with

$$w_n = \tfrac{1}{2}(v_n+c_n), \quad v_n\bar{\beta} = v_n, \quad u_{n+1}\bar{\beta} = u_{n+1},$$
$$h_{\bar{\beta}}^{v_n} = h_{\alpha}^{b_n},$$
$$h_{\bar{\beta}}^{w_n} \geqq \frac{b_{n+1}-w_n}{c_n-w_n}, \quad h_{\bar{\beta}}^{w_n} \in H,$$

and

$$h_{\bar{\beta}}^{z_n} \leqq \min\left(\frac{u_{n+1}-b_{n+1}}{u_{n+1}-c_n}, \frac{u_{n+1}-[v_n+h_{\alpha}^{b_n}(w_n-v_n)]}{u_{n+1}-w_n}\right), \quad h_{\bar{\beta}}^{z_n} \in H.$$

(c) $x\bar{\beta} = x$ otherwise.

Hence, $\bar{\alpha}$ and $\bar{\beta}$ have the same characteristic and therefore there is, in view of Proposition 3.1, $\Psi \in G_{H*}$ such that

$$\bar{\beta} = \Psi^{-1}\bar{\alpha}\Psi.$$

Again, it is routine to check that $\beta = \alpha\Psi^{-1}\alpha\Psi$ has two supporting intervals $(a_{\alpha 1}, u_1\alpha^{-2})$ and $(u_1\alpha^{-2}, a_{\alpha\tau_\alpha})$ with

$$h_{\beta}^{a_{\alpha 1}} = (h_{\alpha}^{a_{\alpha 1}})^2 < 1 \text{ and } h_{\beta}^{u_1\alpha^{-2}} = h > 1.$$

Similarly, we deal with the case when $h < 1$. Then, we assume that (3.3) satisfies

$$x\alpha > x \text{ in } (b_1, c_1) \text{ and } x\alpha < x \text{ in each } (b_n, c_n), \qquad n \geqq 2.$$

Again, we define $\bar{\beta}$ in a similar way as before with the exception of the interval $\langle a_{\alpha 1}, u_2\rangle$; there, $\bar{\beta}$ is linear in $\langle a_{\alpha 1}, u_0\rangle$, $\langle u_0, w_0\rangle$, $\langle w_0, u_1\alpha^{-1}\rangle$, $\langle u_1\alpha^{-1}, u_1)$, $\langle u_1, z_0\rangle$, $\langle z_0, u_2\rangle$ with

$$a_{\alpha 1}\bar{\beta} = a_{\alpha 1}, \quad (u_1\alpha^{-1})\bar{\beta} = u_1\alpha^{-2}, \quad u_2\bar{\beta} = u_2,$$
$$h_{\bar{\beta}}^{a_{\alpha 1}} = h_{\alpha}^{a_{\alpha 1}}, \quad h_{\bar{\beta}}^{u_1\alpha^{-1}} = h \cdot (h_{\alpha}^{b_1})^{-1},$$

$h_{\bar{\beta}}^{u_0}$, $h_{\bar{\beta}}^{u_1} \in H$ sufficiently small and $h_{\bar{\beta}}^{w_0}$, $h_{\bar{\beta}}^{z_0} \in H$ sufficiently large. The proof can be then easily completed.

PROPOSITION 3.4. (i) *Let $\varepsilon \neq \omega \in G_H$. Then, for any $\varepsilon \neq \alpha \in \bar{G}_H$, there are κ and λ of G_H such that*

$$\omega = (\kappa^{-1}\alpha\kappa)^{-1}\lambda^{-1}\alpha\lambda.$$

(ii) *Let $\omega \in G_{H*}\setminus G_H$. Then, for any $\alpha \in \bar{G}_H\setminus G_{*H}$, there are κ, λ, μ and ν of G_{H*} such that*

$$\omega = (\kappa^{-1}\alpha\kappa\mu^{-1}\alpha\mu)^{-1}(\lambda^{-1}\alpha\lambda\nu^{-1}\alpha\nu).$$

(iii) *Let $\omega \in \bar{G}_H\setminus G_{H*}$. Then ω belongs to the normal closure of $\alpha \in \bar{G}_H$ in $\bar{G}_H$ if and only if the cyclic subgroup of R generated by h_{α}^{0} contains h_{ω}^{0} and, if $a_{\omega\tau_\omega} = 1$, $a_{\alpha\tau_\alpha} = 1$, as well.*

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PROOF. (i) This is an immediate consequence of Proposition 3.1 and Lemma 3.2. Indeed let $\bar{\alpha} \in \bar{G}_H$ be given by

$$x\bar{\alpha} = x\alpha \text{ for } x \in (a_{\alpha 1}, a_{\alpha 2}) \text{ and } x\bar{\alpha} = x \text{ otherwise.}$$

Then, there are $\bar{\kappa}$ and $\bar{\lambda}$ of G_H such that $\omega = \sigma^{-1}\rho$ with

$$\sigma = \bar{\kappa}^{-1}\bar{\alpha}\bar{\kappa}, \quad \rho = \bar{\lambda}^{-1}\bar{\alpha}\bar{\lambda} \quad \text{and} \quad a_{\sigma\tau_\sigma} = a_{\rho\tau_\rho} = a_{\omega\tau_\omega} \; (= a_{\alpha 2}\bar{\kappa} = a_{\alpha 2}\bar{\lambda}).$$

Define κ and λ of G_H to satisfy

$$x\kappa = x\bar{\kappa} \text{ and } x\lambda = x\bar{\lambda} \text{ for } x \in \langle 0, a_{\alpha 2}\rangle$$

and to be identical and piecewise H-linear otherwise. Then

$$\omega = (\kappa^{-1}\alpha\kappa)^{-1}\lambda^{-1}\alpha\lambda.$$

(ii) First, according to Proposition 3.3, there is $\Psi \in G_{H*}$ such that

$$\bar{\alpha} = \alpha\Psi^{-1}\alpha\Psi$$

has a supporting interval of the form $(a, 1)$. Then, Proposition 3.1 and Lemma 3.2 can be applied as in the previous case (i):

$$\omega = (\bar{\kappa}^{-1}\bar{\alpha}\bar{\kappa})^{-1}\,\bar{\lambda}^{-1}\bar{\alpha}\bar{\lambda};$$

finally, put $\kappa = \bar{\kappa}$, $\mu = \Psi\bar{\kappa}$, $\lambda = \bar{\lambda}$, $\nu = \Psi\bar{\lambda}$.

(iii) This part follows readily from Proposition 3.1.

EXAMPLE 3.5 As a matter of fact, very often we can make a stronger conclusion in the case (ii) similar to that of (i). However, the following example illustrates that, in general, such a conclusion does not hold:

Let $(a, 1)$, $0 < a < 1$, be a supporting interval of ω with $h^a_\omega > 1$; let $x\alpha \geqq x$ for all $x \in \langle b, 1\rangle$, $0 < b < 1$, and 1 be an accumulation point of the set of all x's such that $x\alpha = x$. Assume that, under these conditions, there are κ and λ of $\bar{G}_R$ such that

$$\omega = (\kappa^{-1}\alpha\kappa)^{-1}\lambda^{-1}\alpha\lambda.$$

First, there is evidently $0 < c < 1$ such that

$$c\alpha = c \text{ and } c\lambda > \max(a, b\kappa).$$

Therefore, since

$$x\kappa^{-1}\alpha\kappa \geqq x \text{ for all } x \in \langle b\kappa, 1\rangle,$$

$$y = (c\lambda)\kappa^{-1}\alpha\kappa \geqq c\lambda > a.$$

Thus, by a simple calculation, we get

$$y\omega = [(c\lambda)(\kappa^{-1}\alpha\kappa)](\kappa^{-1}\alpha\kappa)^{-1}(\lambda^{-1}\alpha\lambda) = c\alpha\lambda = c\lambda \leqq y,$$

a contradiction of $h^a_\omega > 1$.

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Lemma 3.6. *Let* $\alpha \in \bar{G}_H$ *be simple with the supporting interval* (a, b) *and* $h^a_\alpha = h_0$; let $h \in H$. *Then, there exists* $\beta \in \bar{G}_H$, *unique in* $\bar{G}_R$, *satisfying*

(a) $\alpha\beta = \beta\alpha$;

(b) $h^a_\beta = h$;

(c) *each supporting interval of* β *has a non-empty intersection with* (a, b).

If $h = 1$, *then* $\beta = \varepsilon$; *otherwise,* β *is simple with the supporting interval* (a, b). *Moreover, if*

$$h = h_0^r \text{ for a rational } r = m/n,\ n > 0,$$

then β *is the (unique) solution of the equation*

$$\xi^n = \alpha^m.$$

Proof. We shall sketch the proof in the case $h_0 > 1$; the basic idea is that of the proof of Proposition 3.1. Let $c > a$ be such that α is linear in $\langle a, c\rangle$ and take

$$0 < a_1 < \min\,[c, a+h^{-1}h_0(c-a)];$$

put

$$a_n = a_1\alpha^{n-1} \text{ for } n \geqq 1.$$

Evidently, $\lim_n a_n = b$ and, for every $x \in (a_1, b)$, there is a unique $n_0 \geqq 1$ such that $a_{n_0} < x \leqq a_{n_0+1}$. Now, one can easily see that β defined by

$$x\beta = a+h(x-a) \text{ for } x \in \langle a, a_1\rangle,$$

$$x\beta = x\alpha^{-n}\beta\alpha^n \text{ for } x \in (a_n, a_{n+1}\rangle, \qquad n \geqq 1,$$

and

$$x\beta = x \text{ otherwise}$$

satisfies (a), (b), (c) and is by these three properties uniquely determined. The rest of Lemma 3.6 follows easily.

Proposition 3.7. *Let* $\alpha \in \bar{G}_H$ *and* r *be a real number. Let* $(h^{a_{\alpha t}}_\alpha)^r \in H$ *for every supporting interval* $(a_{\alpha t}, a_{\alpha t+1})$ *of* α. *Then there exists* $\beta \in \bar{G}_H$, *unique in* $\bar{G}_R$, *satisfying*

(a) $\alpha\beta = \beta\alpha$;

(b) *each supporting interval of* β *intersects non-trivially some supporting interval of* α;

(c) *for each* $(a_{\alpha t}, a_{\alpha t+1})$, $h^{a_{\alpha t}}_\beta = (h^{a_{\alpha t}}_\alpha)^r$.

As a consequence, α *and* β *have the same supporting intervals. Also, if* $r = m/n$, $n > 0$, *is rational, then* $\beta = \alpha^{m/n}$ *is the (unique) solution of the equation*

$$\xi^n = \alpha^m.$$

In general, for any $\alpha \in \bar{G}_R$ *and any real* r, *a unique* β *satisfying* (a), (b), (c) *always exists in* $\bar{G}_R$; *by definition, put*

$$\beta = \alpha^r.$$

Then,

$$\alpha^r\alpha^s = \alpha^{r+s} = \alpha^s\alpha^r$$

and

$$(\alpha^r)^s = \alpha^{rs} = (\alpha^s)^r$$

for all real numbers r *and* s.

Proof. The first part follows from the preceding Lemma 3.6 applied to each supporting interval of α. But then we get readily the remaining relations for the "powers". For, $\beta = \alpha^r\alpha^s$ and $\beta = \alpha^s\alpha^r$ satisfy (a), (b) and (c) with

$$h_{\beta}^{a_\alpha t} = h_{\alpha^r}^{a_\alpha t} h_{\alpha^s}^{a_\alpha t} = h_{\alpha^s}^{a_\alpha t} h_{\alpha^r}^{a_\alpha t} = (h_{\alpha}^{a_\alpha t})^{r+s};$$

hence, in view of uniqueness,

$$\alpha^r\alpha^s = \alpha^s\alpha^r = \alpha^{r+s}.$$

Similarly, apply the first part of this Proposition 3.7 to α^r: Both $\beta_1 = (\alpha^r)^s$, $\beta_2 = \alpha^{rs}$ satisfy (i) and (ii); furthermore,

$$h_{\beta_1}^{a_\alpha t} = (h_{\alpha^r}^{a_\alpha t})^s = [(h_{\alpha}^{a_\alpha t})^r]^s = (h_{\alpha}^{a_\alpha t})^{rs} = h_{\beta_2}^{a_\alpha t}.$$

Hence, $\beta_1 = \beta_2$, as required.

Lemma 3.8. (cf.[2]) *Let*

$$W(\xi, \eta) = \xi^{k_1}\eta^{l_1}\xi^{k_2}\eta^{l_2} \cdots \xi^{k_n}\eta^{l_n},$$

where all the integers k_i, l_i $(1 \leqq i \leqq n)$ *with a possible exception of* l_n *are non-zero, be a given "word". Let* $0 \leqq a < b \leqq 1$. *Then, for every* $H \subseteq R$, *there exist* α *and* β *in* $\bar{G}_H$ *such that*

$$a_{\alpha 1} = a_{\beta 1} = a, \quad a_{\alpha \tau_\alpha} = a_{\beta \tau_\beta} = b$$

and

$$W(\alpha, \beta) \neq \varepsilon.$$

Proof. First, put

$$k = |k_1 l_1 k_2 l_2 \cdots k_n l_n^*| \geqq 1,$$

where $l_n^* = 1$ if $l_n = 0$ and $l_n^* = l_n$ otherwise. Choose $h_0 \in H$ such that $h_0 > 3$. Furthermore, put

$$h = h_0^k > 3^k \geqq 3,$$

and denote by c a number satisfying $h \leqq c$; finally, choose

$$0 < e < \frac{b-a}{2nc},$$

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put $a_0 = a$ and

$$a_q = a+q(h-1)e \text{ for } 2 \leqq q \leqq 2n-1;$$

clearly, $a_0 < a_2 < \cdots < a_{2n-1} < b$.

Now, in the interval $\langle a_q, a_q+2(h-1)e\rangle$ define the function Ψ_q by

$$x\Psi_q = a_q+h(x-a_q) \text{ for } x \in \langle a_q, a_q+e\rangle,$$

$$x\Psi_q = a_q+he+h^{-1}(x-a_q-e) \text{ for } x \in \langle a_q+e, a_q+e+he\rangle$$

and

$$x\Psi_q = x \text{ otherwise.}$$

By means of these functions then define $\bar{\alpha} \in \bar{G}_H$ and $\bar{\beta} \in \bar{G}_H$ in the following way:

(i) $x\bar{\alpha} = x$ in $\langle 0, a\rangle$ and $\langle b, 1\rangle$,

$x\bar{\alpha} = x\Psi_{2(i-1)}$ in $\langle a_{2(i-1)}, a_{2i}\rangle$ for $1 \leqq i \leqq n-1$,

$x\bar{\alpha} = a_{2(n-1)}+h(x-a_{2(n-1)})$ in $\langle a_{2(i-1)}, a_{2(i-1)}+e\rangle$

and

$x\bar{\alpha} \neq x$, $\bar{\alpha}$ piecewise H-linear otherwise.

(ii) $x\bar{\beta} = x$ in $\langle 0, a\rangle$, $\langle a+2he, a+3(h-1)e\rangle$ and $\langle b, 1\rangle$,

$x\bar{\beta} = x\Psi_{2i-1}$ in $\langle a_{2i-1}, a_{2i+1}\rangle$ for $2 \leqq i \leqq n-1$,

$x\bar{\beta} = a+h(x-a)$ in $\langle a, a+e\rangle$,

$x\bar{\beta} = x+(h-1)e$ in $\langle a+e, a+he\rangle$,

$x\bar{\beta} = a_{2n-1}+h(x-a_{2n-1})$ in $\langle a_{2n-1}, a_{2n-1}+e)$

and

$x\bar{\beta} \neq x$, $\bar{\beta}$ piecewise H-linear otherwise.

Finally, apply Lemma 3.6, and define $\alpha \in \bar{G}_H$ by

$$x\alpha = x\bar{\alpha}^{-k_i} \text{ in } \langle a_{2(i-1)}, a_{2i}\rangle \text{ for } 1 \leqq i \leqq n-1$$

and

$$x\alpha = x\bar{\alpha}^{-k_n} \text{ otherwise.}$$

Similarly, take $\beta \in \bar{G}_H$ satisfying

$$x\beta = x\bar{\beta}^{-l_1} \text{ in } \langle a, a_3\rangle,$$

$$x\beta = x\bar{\beta}^{-l_i} \text{ in } \langle a_{2i-1}, a_{2i+1}\rangle \text{ for } 2 \leqq i \leqq n-1$$

and

$$x\beta = x\bar{\beta}^{-l_n^*} \text{ otherwise.}$$

A routine calculation shows that

$$(a+e)W(\alpha, \beta) = a+e+2ne(h-1) \neq a+e,$$

as required.

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4. Theorems

In this final section, we shall derive – using the results of the preceding section – the main properties of the groups G_H, $H \subseteqq R$. As a matter of fact, formulations of the results in § 3 allow many statements on G_H to be extended to $\bar{G}_H$ etc.

THEOREM 4.1. *For any subgroup $H \subseteqq R$, G_H can be linearly ordered. In fact, there is a one-to-one correspondence between the linear orders of H and those of G_H. Hence, G_H has only two linear orders if and only if H is of rank* 1, *i.e. if and only if H is a subgroup of H_r for a suitable $r \in R$.*

PROOF. Let $P(H)$ be the positive cone of a linear order in H. Define $P(G_H)$ by

$$\omega \in P(G_H) \text{ if and only if } \omega = \varepsilon \text{ or } h_\omega^{a\omega^{-1}} \in P(H).$$

In view of Proposition 3.1, $P(G_H)$ is the positive cone of a linear order of G_H.

On the other hand, let $P(G_H)$ be the positive cone of a linear order of G_H. Evidently, $P(G_H)$ contains simple elements: let $\sigma \in P(G_H)$ be simple, (a, b) its supporting interval and $h_\sigma^a = h_0$. We are going to show that any $\omega \in G_H$ with $h_\omega^{a\omega^{-1}} = h_0$ belongs to $P(G_H)$. First, take $\Psi_0 \in G_H$ such that

$$\beta = \sigma^2 \Psi_0^{-1} \sigma \Psi_0 \in P(G_H)$$

satisfies

$$A_\beta = \{a, b, c\}, \quad h_\beta^a = h_0^2, \quad h_\beta^b = h_0.$$

Now, given $1 \neq h \in H$, there are, according to Proposition 3.3, always three elements α, Ψ_1 and Ψ_2 of G_H such that

$$\beta_1 = \alpha \Psi_1^{-1} \alpha \Psi_1 \text{ and } \beta_2 = \alpha \Psi_2^{-1} \alpha \Psi_2$$

satisfy

$$A_{\beta_1} = \{a, b, c\}, \quad h_{\beta_1}^a = h_0^2, \quad h_{\beta_1}^a = h_0$$

and

$$A_{\beta_2} = \{a, b, c\}, \quad h_{\beta_2}^a = h_0^2, \quad h_{\beta_2}^a = h.$$

In view of Proposition 3.1, $\beta_1 \in P(G_H)$ and thus, necessarily, $\alpha \in P(G_H)$. Hence, $\beta_2 \in P(G_H)$. Since, again by Propositions 3.1 and 3.3, there is $\Psi \in G_H$ such that $\omega \Psi^{-1} \omega \Psi$ and β_2 are conjugate, we get $\omega \in P(G_H)$, as required.

From here we deduce immediately that the correspondence

"$h \in H$ belongs to $P(H)$ if and only if all $\omega \in G_H$
with $h_\omega^{a\omega^{-1}} = h$ belong to $P(G_H)$"

between the linear orders of the groups H and G_H is one-to-one.

The rest of Theorem 4.1 follows easily.

Another consequence of Proposition 3.3 is the following

THEOREM 4.2. *Let $F_H \subsetneqq G_H$ be the subgroup of all elements of G_H which are piecewise H-linear* (in $\langle 0, 1\rangle$). *Thus, for every $\varepsilon \neq \alpha \in F_H$, α is linear in an interval $\langle a, a_{\alpha\tau_\alpha}\rangle$; put $h_\alpha = h^a_\alpha$. Let $P(H)$ be the positive cone of a certain linear order of H. Define the linear order in F_H by*

$$\varepsilon \neq \omega \in P(F_H) \text{ if and only if } h_\omega \in P(H).$$

This (partial) order of G_H cannot be extended to a linear order of G_H, i.e. G_H is not an O^*-group in the sense of [3].

PROOF. Again, given $h \in P(H)$, there are, by Propositions 3.1 and 3.3, α, Ψ_1 and Ψ_2 of G_H such that

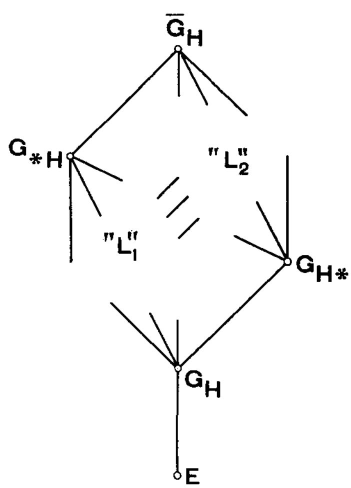
$$\beta_1 = \alpha\Psi_1^{-1}\alpha\Psi_1 \text{ and } \beta_2 = \alpha\Psi_2^{-2}\alpha\Psi_2$$

belong to F_H and

$$h_{\beta_1} = h = (h_{\beta_2})^{-1}.$$

Thus, $\beta_1 \in P(F_H)$ whilst $\beta_2 \notin P(F_H)$; consequently, there is no linear order of G_H extending the given order of F_H.

THEOREM 4.3. *For any subgroup $H \subsetneqq R$, G_H is (algebraically) simple. In fact, the following diagram*



where $L_1 \cong L_2$ are isomorphic to the lattice of all subgroups of H describes the full normal structure of $\bar{G}_H$ ($G_{*H}/G_H \cong \bar{G}_H/G_{H*} \cong H$).

PROOF. The first part of Theorem 4.3 follows immediately from Proposition 3.4 (i). As a matter of fact, the diagram is a straightforward consequence of Proposition 3.4, as well.

THEOREM 4.4. *The group G_H is divisible if and only if $H \subsetneqq R$ is divisible. In fact, G_H is then strongly divisible in the following sense: Every equation*

$$(4.1) \quad \xi^{n_1}\alpha_1\xi^{n_2}\alpha_2 \cdots \xi^{n_q}\alpha_q = \varepsilon \text{ with } n_i > 0, \alpha_i \in G_H \text{ for } 1 \leqq i \leqq q$$

possesses a unique solution ξ_1 *in* G_H.

Also, G_R *is "continuously" divisible in the sense that every equation*

$$\xi^r = \alpha \text{ with a real exponent } r \text{ and } \alpha \in G_R$$

has a (unique) solution in G_R.

PROOF. Again, the first and the last parts of Theorem 4.4 follow immediately from Proposition 3.7.

However, the statement on solvability of (4.1) (in the case when H is divisible) needs an independent proof: We shall construct a solution ξ_1 of (4.1) and – at the same time – show that it is unique. It is evident that, without loss of generality, we can assume $n_i = 1$ for all $1 \leqq i \leqq q$. Moreover, throughout the proof we shall always denote the products

$$\xi_x\alpha_1\xi_x\alpha_2 \cdots \xi_x\alpha_{i-1} \ (1 \leqq i \leqq q), \text{ where } \xi_x \in G_H,$$

by $W_i(\xi_x)$; in particular, $W_1(\xi_x) = \varepsilon$.

Discarding the trivial case, assume that $\alpha_i \neq \varepsilon$ for at least one i and put

$$a = \min_{\alpha_i \neq \varepsilon} (a_{\alpha_i 1}) > 0.$$

Notice that $\xi_a = \varepsilon$ satisfies (4.1) in the interval $\langle 0, a\rangle$; moreover, any $\xi \in G_H$ satisfying (4.1) in $\langle 0, a\rangle$ is necessarily identical with ε in this interval.

Let N_1 be the set of all $1 \leqq i \leqq q$ such that $a_{\alpha_i 1} = a$. Define $h_a \in H$ as the (unique) real number satisfying

$$h_a = \prod_{i \in N_1} (h^a_{\alpha_i})^{-1/q}.$$

Consider an element $\Psi_1 \in G_H$ such that

$$x\Psi_1 = x \text{ in } \langle 0, a\rangle$$

and

$$x\Psi_1 = a + h_a(x-a) \text{ in } \langle a, a+e_1\rangle \text{ for a suitable } e_1 > 0.$$

Clearly, there is $b > a$ such that $\xi_b = \Psi$ satisfies (4.1) in $\langle 0, b\rangle$ and

$$(b \leqq)\, b^* = \max_{1 \leqq i \leqq q} [bW_i(\xi_b)] \leqq a + e_1.$$

Moreover, if any other $\xi \in G_H$ satisfies (4.1) in $\langle 0, b\rangle$, then ξ necessarily coincides with ξ_b in $\langle 0, b^*\rangle$.

Now, denote by I the (non-empty) subset of $(0, 1\rangle$ of all x such that there is an element $\xi_x \in G_H$ satisfying (4.1) in $\langle 0, x\rangle$. Evidently, if $x_1 < x_2$ and $x_2 \in I$, then $x_1 \in I$. Furthermore, for each $x \in I$ and the corresponding ξ_x define, for every $z \in \langle 0, x\rangle$

$$(z \leqq) z^*(\xi_x) = \max_{1 \leqq i \leqq q} [z W_i(\xi_x)].$$

Notice that always $z^*(\xi_x) < 1$ unless $z = x = 1$ and that $z_1 < z_2$ implies $z_1^*(\xi) < z_2^*(\xi)$.

First, we are going to show that, for $0 < x_1 \leqq x_2$, the corresponding functions ξ_{x_1} and ξ_{x_2} coincide in the interval $\langle 0, x_1^*(\xi_{x_1})\rangle$. In order to prove the assertion, denote by J the set of all $y \in \langle 0, x_1^*(\xi_{x_1})\rangle$ such that

$$x\xi_{x_1} = x\xi_{x_2} \text{ for } x \in \langle 0, y\rangle.$$

Certainly, since every $y \leqq \min(x_1^*(\xi_{x_1}), b)$ belongs to J, J is non-empty. Also, in view of continuity of the functions ξ_{x_1} and ξ_{x_2}, we get necessarily

$$\sup_{y \in J} y = y_0 \in J.$$

We claim that $y_0 = x_1^*(\xi_{x_1})$. Assuming the contrary, i.e. $y_0 < x_1^*(\xi_{x_1})$, we deduce

$$h_{\xi_{x_1}}^{y_0} \neq h_{\xi_{x_2}}^{y_0}. \tag{4.2}$$

Define

$$x_0 = \min_{1 \leqq i \leqq q} [y_0 W_i^{-1}(\xi_{x_1})] \leqq y_0.$$

Clearly, $y_0 = x_0^*(\xi_{x_1})$ and thus, since $y_0 < x_1^*(\xi_{x_1})$, we have $x_0 < x_1$. Now, denoting by N_2 the set of all $1 \leqq i \leqq q$ for which

$$x_0 < y_0 W_i^{-1}(\xi_{x_1}), \text{ i.e. } x_0 W_i(\xi_{x_1}) < y_0,$$

and by $m \geqq 1$ the number of the remaining i's, we can easily calculate that $h_{\xi_{x_1}}^{y_0}$ and $h_{\xi_{x_2}}^{y_0}$ must satisfy the following equation for h

$$\prod_{i=1}^{q} h_{\alpha_i}^{x_0[W_1(\xi_{x_1})\xi_{x_1}]} \prod_{i \in N_2} h_{\xi_{x_1}}^{x_0 W_i(\xi_{x_1})} h^m = 1, \tag{4.3}$$

and must therefore be equal, in contradiction to (4.2). Hence

$$x\xi_{x_1} = x\xi_{x_2} \text{ for } x \in \langle 0, x_1^*(\xi_{x_1})\rangle.$$

Therefore, also $x_1^*(\xi_{x_2}) = x_1^*(\xi_{x_1})$. As a consequence, if $x \in I$, then there is a unique $x \leqq x^* \leqq 1$ such that a function ξ_x satisfying (4.1) in $\langle 0, x\rangle$ is uniquely determined in $\langle 0, x^*\rangle$ and $x^* = x^*(\xi_x)$. In particular, this yields the uniqueness of the solution of (4.1) in G_H provided that a solution exists. Moreover, if $x_1, x_2 \in I$, then $x_1 < x_2$ implies $x_1^* < x_2^*$ and, for any $x < x_1^*$, there is a unique w such that

$$w \in I \text{ and } w^* = x.$$

Now, put

$$s = \sup_{x \in I} x \text{ and } s' = \sup_{x \in I} x^*. \tag{4.4}$$

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Clearly, $0 < s \leqq s' \leqq 1$. We want to show $s = s' = 1$.

Using an indirect argument again, suppose that $s < 1$. Define the function $\Psi_2 \in G_H$ as follows: First of all, for $x \in \langle 0, s')$,

$$x\Psi_2 = x\xi_w, \text{ where } w^* = x$$

and

$$s'\Psi_2 = \lim_{x \to s'_-} x\Psi_2.$$

Since Ψ_2 (for the time being defined in $\langle 0, s' \rangle$ only) satisfies the equation (4.1) in $\langle 0, s)$ and is uniquely determined in $\langle 0, s')$, we deduce, in view of continuity, that Ψ_2 satisfies (4.1) in $\langle 0, s \rangle$ and is uniquely determined in $\langle 0, s^* \rangle$, where $s^* = s'$. Also, since $s < 1$, necessarily $s^*\Psi_2 < 1$ and we can extend Ψ_2 on $\langle 0, 1 \rangle$ in such a way – by calculating the gradient $h^{s^*}_{\Psi_2}$ from a relation similar to (4.3) – that Ψ_2 satisfies (4.1) in an interval $\langle 0, s+e_2 \rangle$ with a suitable $e_2 > 0$. Hence, $s+e_2 \in I$ — a contradiction of (4.4); consequently, $s = s^* = 1$, as required.

Thus, there exists a solution $\xi_1 \in G_{H*}$ of (4.1) and – as shown above – it is unique. Moreover, one can see immediately that

$$\min_{\alpha_i \neq \varepsilon} (a_{\alpha_1 1}) \leqq a_{\xi_1 1} \text{ and } a_{\xi_1 \tau_{\xi_1}} \leqq \max_{\alpha_i \neq \varepsilon} (a_{\alpha_i \tau_{\alpha_i}}).$$

Hence, ξ_1 belongs to G_H.

The proof of Theorem 4.4 is completed.

Corollary 4.5. *For every divisible $H \subseteqq R$, G_H is an (algebraically) simple divisible (linearly) ordered group; in particular, G_{H_r} is an (algebraically) simple divisible group admitting only two linear orders.*

Theorem 4.6. *Let $\varepsilon \neq \alpha$ be an element of G_H; let $\langle 0, a_{\alpha 1} \rangle$, $\langle a_{\alpha \tau_\alpha}, 1 \rangle$ and $\langle a_m, b_m \rangle$, $m \in M$, be all non-trivial disjoint closed intervals complementary to the support of α; let $1 \leqq n_0 \leqq \aleph_0$ be the number of the supporting intervals of α. Then, the centre Z_α of the centraliser $C(\alpha)$ of α in G_H is isomorphic to a direct product of n_0 copies of H and*

$$C(\alpha)/Z_\alpha \cong G_1 \times \prod_{m \in M} \bar{G}_m \times G_2,$$

where $G_1 \cong G_{H*}$, $G_2 \cong G_{*H}$ and $\bar{G}_m \cong \bar{G}_H$ for all $m \in M$.

Proof. By Lemma 3.6 and Proposition 3.7, every supporting interval of $\beta \in C(\alpha)$ is either a supporting interval of α – and β is then in that supporting interval a (real) power of α, or does not intersect the support of α. On the other hand, any element whose support does not intersect the support of α belongs evidently to $C(\alpha)$. Thus, define

$$\omega \in G_1 \text{ if and only if } a_{\omega \tau_\omega} \leqq a_{\alpha 1},$$
$$\omega \in G_m \text{ if and only if } a_m \leqq a_{\omega 1} < a_{\omega \tau_\omega} \leqq b_m,\ m \in M,$$
$$\omega \in G_2 \text{ if and only if } a_{\alpha \tau_\alpha} \leqq a_{\omega 1},$$

and Theorem 4.6 follows.

We conclude the paper with the following two simple remarks.

REMARK 4.7. (cf. [6]). *With respect to its "natural" order (i.e. the order corresponding to the natural order of R), the group G_R contains the continuum of non-order-isomorphic linearly ordered free subgroups of rank 2.*

This is an immediate consequence of Lemma 3.8: Let

$$0 < a_0 < b_0 < a_1 < a_2 < \cdots < a_n < \cdots < 1$$

be an arbitrary sequence and $W_n(\xi, \eta)$ – the set of all "words" in ξ, η. For every $n \geqq 1$, let α_n and β_n be two elements of G_R whose supports are $I_n = (a_n, a_{n+1})$ and for which $W_n(\alpha_n, \beta_n) \neq \varepsilon$. Moreover, let α_0 be a (simple) element of G_R with the supporting interval $I_0 = (a_0, b_0)$ and

$$\beta_0 = \alpha_0^r \text{ for a real number } r \in R. \tag{4.5}$$

Now, denote by $F(\alpha, \beta)$ the (free) subgroup of G_R generated by α and β defined by

$$x\alpha = x\alpha_n, \quad x\beta = x\beta_n \quad \text{for} \quad x \in I_n, \quad n \geqq 0$$

and

$$x\alpha = x\beta = x \text{ otherwise.}$$

Evidently, the number $r \in R$ in (4.5) can be chosen in continuum different ways so that the resulting linearly ordered subgroups $F(\alpha, \beta)$ are non-order-isomorphic.

It is apparent from the proof that there is an infinite number of non-order isomorphic linearly ordered free subgroups of rank 2 in G_H for every $H \subseteqq R$. Also, let us point out here the trivial fact that G_R (with its "natural" order) contains an isomorphic copy of any abelian linearly ordered group of countable "Archimedean rank" (cf. [8]).

REMARK 4.8. *Let $H \subseteqq R$ and $0 < a < 1$. Then the subgroup $G_H^a \subseteqq G_H$ consisting of all $\alpha \in G_H$ such that $a\alpha = a$ is a maximal subgroup of G_H.*

We offer here a proof which retains its validity for every "transitive" subgroup of automorphisms of $(0, 1)$.

Let β be an arbitrary element of $G_H \setminus G_H^a$; without loss of generality, assume that $a\beta^{-1} > a$. In order to prove maximality of G_H^a in G_H it is obviously sufficient to show that any $\omega \in G_H$ such that $a\omega^{-1} > a$ belongs to the group generated by G_H^a and β. Take

$$0 < \max(a\beta^{-1}, a\omega^{-1}) < b < 1$$

and denote by α a piecewise H-linear automorphism of $\langle 0, 1 \rangle$ satisfying

$$(a\omega^{-1})\alpha = a\beta^{-1}$$

and

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$$x\alpha = x \text{ in } \langle 0, a\rangle \text{ and } \langle b, 1\rangle.$$

Now, since

$$a[(\alpha\beta)^{-1}\omega] = a\beta^{-1}\alpha^{-1}\omega = a,$$

$\gamma = (\alpha\beta)^{-1}\omega$ belongs to G_H^a and thus

$$\omega = \alpha\beta\gamma \text{ with } \alpha \text{ and } \gamma \text{ from } G_H^a,$$

as required.

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Department of Mathematics
Institute of Advanced Studies
Australian National University
Canberra, A.C.T.

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