Proceedings of the Royal Society of Edinburgh, **153**, 1073–1098, *2023*
DOI:10.1017/prm.2022.17

Exponential decay for the KdV equation on $\mathbb{R}$ with new localized dampings

Ming Wang
Center for Mathematical Sciences, China University of Geosciences, Wuhan 430074, P.R. China (mwang@cug.edu.cn)

Deqin Zhou
College of Mathematics and Statistics, Chongqing University, Chongqing 401331, P.R. China (deqinzhou1952@126.com)

(Received 20 July 2021; accepted 7 March 2022)

In this paper, we prove several results on the exponential decay in L^2 norm of the KdV equation on the real line with localized dampings. First, for the linear KdV equation, the exponential decay holds if and only if the averages of the damping coefficient on all intervals of a fixed length have a positive lower bound. Moreover, under the same damping condition, the exponential decay holds for the (nonlinear) KdV equation with small initial data. Finally, with the aid of certain properties of propagation of regularity in Bourgain spaces for solutions of the associated linear system and the unique continuation property, the exponential decay for the KdV equation with large data holds if the damping coefficient has a positive lower bound on E, where E is equidistributed over the real line and the complement E^c has a finite Lebesgue measure.

Keywords: KdV equation; exponential decay; localized damping

2020 *Mathematics subject classification:* Primary: 35Q53, 35B40
Secondary: 39A12

1. Introduction

In this paper, we are interested in the exponential decay property of the KdV equation on the real line $\mathbb{R}$

$$\partial_t u + \partial_x^3 u + u\partial_x u + a(x)u = 0, \quad u(x,0) = u_0(x) \in L^2(\mathbb{R}). \tag{1.1}$$

Here, $u(x, t)$ is a real-valued function on $\mathbb{R} \times \mathbb{R}^+$, the function $a(x)$ satisfies the condition

$$0 \leqslant a(x) \in L^\infty(\mathbb{R}). \tag{A1}$$

In the case $a(x) = 0$, (1.1) reduces to the classical KdV equation, which models the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems. If we multiply (1.1) by $2u$ and integrate over $\mathbb{R}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x+2\int_{\mathbb{R}}a(x)|u(x,t)|^2\mathrm{d}x=0. \tag{1.2}$$

This, and condition $a(x)\geqslant 0$, clearly implies that $\|u(\cdot,t)\|_{L^2(\mathbb{R})}\leqslant\|u_0\|_{L^2(\mathbb{R})}$ for all $t\geqslant 0$. Moreover, if $a(x)\geqslant a_0>0$ for all $x\in\mathbb{R}$, then (1.2) gives the decay bound

$$\|u(\cdot,t)\|_{L^2(\mathbb{R})}\leqslant \mathrm{e}^{-a_0 t}\|u_0\|_{L^2(\mathbb{R})},\quad \text{for all } t\geqslant 0. \tag{1.3}$$

Thus, the term au is called a damping in the literature. Now an interesting question arises naturally, whether the exponential decay as (1.3) holds if

$$a(x)\geqslant a_0>0,\quad x\in E \tag{A2}$$

is satisfied only on a subset $E\subset\mathbb{R}$? In this case, the term au is referred to a localized damping.

Similar problems for KdV type equations on bounded domains have been studied extensively, we refer to e.g. [**3**, **14**, **15**, **17**–**19**, **21**, **23**] and the survey [**24**]. But much less is known for the KdV equation on unbounded domains. For the KdV equation posed on $(x,\,t)\in\mathbb{R}_+^2$, the exponential decay of the $L^2(\mathbb{R}_+)$ norm was proved in [**16**] when (A2) holds on $E=(0,\,\delta)\bigcup(L,\,+\infty)$ for some $0<\delta<L$. The same result was obtained in [**22**] under a weaker localized damping, namely (A2) holds only on $E=(L,\,+\infty)$ for some $L>0$. For the KdV equation on $(x,\,t)\in\mathbb{R}\times\mathbb{R}_+$, namely (1.1), the exponential decay was established in [**4**] with damping on $E=(-\infty,\,-L)\bigcup(L,\,+\infty)$. If one considers the KdV equation with strong dissipation

$$\partial_t u-\partial_x^2 u+\partial_x^3 u+u\partial_x u+a(x)u=0,\quad u(x,0)=u_0(x)\in L^2(\mathbb{R}),$$

called the Korteweg–de Vries–Burgers equation, then the exponential decay holds with an indefinite damping, namely $a(x)$ may change sign, see [**5**, **7**, **9**].

In this article, the main goal is to consider the following question: To what extent the set E can be small so that the exponential decay holds for KdV equations with damping on E. First of all, we give a sufficient and necessary condition for the exponential decay of the linear KdV equation.

THEOREM 1.1. *Assume that (A1) holds. Then the following are equivalent*:

(1) *There exist constants $C,\,\lambda>0$ so that*

$$\|u(\cdot,t)\|_{L^2(\mathbb{R})}\leqslant C\mathrm{e}^{-\lambda t}\|u_0\|_{L^2(\mathbb{R})},\quad \forall t\geqslant 0$$

holds for all solutions of the initial value problem (IVP) $\partial_t u+\partial_x^3 u+a(x)u=0$, $u(x,0)=u_0(x)\in L^2(\mathbb{R})$.

(2) *There exists a constant $L>0$ so that*

$$\inf_{x\in\mathbb{R}}\int_{x-L}^{x+L}a(y)\mathrm{d}y>0. \tag{1.4}$$

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

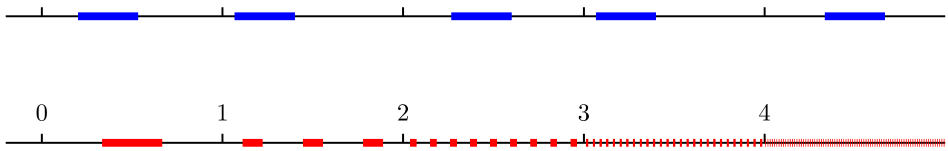

Figure 1. Comparison of thick set with NCC.

As noted in [8], under assumption (A1), condition (1.4) is equivalent to (A2) for some $a_0 > 0$ and a thick set E. Recall that (see e.g. [29]) a measurable set $E \subset \mathbb{R}$ is *thick*, if there exists $L > 0$ so that

$$\inf_{x\in\mathbb{R}} \left| E \bigcap [x - L, x + L] \right| > 0.$$

Here and below, we use $|E|$ to denote the Lebesgue measure of E. Based on theorem 1.1 and the contraction mapping principle, we give the exponential decay for the KdV equation (1.1) with small data.

THEOREM 1.2 (Decay for small data). *Assume that* (A1) *holds and* (A2) *holds on a thick set E. Then there exist constants $C > 0$, $\lambda > 0$, $\delta > 0$ such that*

$$\|u(t)\|_{L^2(\mathbb{R})} \leqslant C\mathrm{e}^{-\lambda t}\|u_0\|_{L^2(\mathbb{R})}, \quad t \geqslant 0$$

holds for all solutions of (1.1) *with data $\|u_0\|_{L^2(\mathbb{R})} \leqslant \delta$.*

To obtain the exponential decay for large data, we need to strength the damping effect. To state our result, we first introduce a set class. A set $E \subset \mathbb{R}$ is said to be satisfying the *network control condition* (NCC), named after [2], if there exist constants r, $L > 0$ so that

$$E \supset \bigcup_n (x_n - r, x_n + r), \quad \inf_n |x - x_n| \leqslant L, \text{ for all } x \in \mathbb{R}.$$

Clearly, a set satisfying NCC is a thick set, but a thick set could not satisfy NCC, see figure 1.

REMARK 1.3. In figure 1, the set consisting of red intervals is a typical thick set. It has a fixed positive Lebesgue measure on every $[n, n + 1]$, but it does not contain an interval with given length simultaneously on all $[n, n + 1]$, $n \in \mathbb{Z}$. The union of blue intervals is a typical set satisfying NCC.

The set class satisfying NCC in higher dimensions (the definition is the same except minor modifications) was first introduced to study the observability of the Kolmogorov equation [10] (see also [6] for observability of heat equations).

THEOREM 1.4 (Decay for general data). *Assume that* (A1) *holds and* (A2) *holds on E, where E satisfies NCC and the complement set E^c has a finite Lebesgue*

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

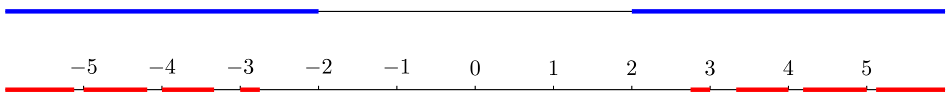

Figure 2. Damping domain considered in [4] vs damping domain considered in theorem 1.4.

measure. Then for every $R > 0$*, there exist constants* $C, \lambda > 0$ *depending only on* R *and* $a(x)$ *so that*

$$\|u(t)\|_{L^2(\mathbb{R})} \leqslant C\mathrm{e}^{-\lambda t}\|u_0\|_{L^2(\mathbb{R})}, \quad t \geqslant 0$$

holds for all solutions u *of IVP* (1.1) *with initial data satisfying* $\|u_0\|_{L^2(\mathbb{R})} \leqslant R$*.*

Note that if E is the complement of a compact set, then E satisfies NCC and $m(E^c) < \infty$. However, if

$$E = \mathbb{R}\backslash \bigcup_{0\neq k\in\mathbb{Z}} \left[k, k+\frac{1}{k^2}\right],$$

then E satisfies NCC and $m(E^c) < \infty$, but E^c cannot be contained in a compact set, see figure 2. Therefore, theorem 1.4 improves the results in [4] in the sense that the exponential decay holds with localized damping on more general sets. In fact, to obtain exponential decay results established in [4], it is assumed that the damping effect holds on the complement of a compact set (see the blue set in figure 2). On the other hand, conditions in theorem 1.4 allows no damping to occur on some small gaps at infinity (see the red set in figure 2).

Let us describe briefly the applications of conditions established on the set E and the main arguments to prove theorem 1.4.

(1) E satisfies NCC. This condition is necessary to show the propagation of regularity for the operator $\mathbb{L} = \partial_t + \partial_x^3$. Roughly speaking, let u be a solution of the equation $\mathbb{L}u = f$ with a smooth f, then u is smooth on $\mathbb{R}$ if u is smooth on E. We refer to lemma 4.3 for precise statements.

(2) $m(E^c) < \infty$. This condition is necessary to establish the compactness of sequence $u_n \in L^2(0, T; L^2(E^c))$, where u_n is the solution of IVP (1.1) with initial data u_{0n} bounded in $L^2(\mathbb{R})$. Combining the compactness and the propagation of regularity, we show that the solution of (1.1) enjoys the observability

$$\int_0^T \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x\mathrm{d}t \leqslant C \int_0^T \int_{\mathbb{R}} a(x)|u(x,t)|^2 \mathrm{d}x\mathrm{d}t.$$

Then the exponential decay follows from a standard argument.

The notation used in this paper is standard. We only point out that we use $A \lesssim B$ to denote $A \leqslant CB$ for some constant $C > 0$, which may vary from place to place.

The paper is organized as follows. In § 2, we show theorem 1.1. The proof of theorem 1.2 is given in § 3. Finally, in § 4, the propagation of regularity and an observability inequality are established to prove theorem 1.4.

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

2. Exponential decay for linear KdV

Assume that $0 \leqslant a(x) \in L^\infty(\mathbb{R})$. Consider the linear operator $A : H^3(\mathbb{R}) \mapsto L^2(\mathbb{R})$

$$Au = \partial_x^3 u + a(x)u, \quad u \in H^3(\mathbb{R}).$$

Clearly, we have

$$(Au, u) = (\partial_x^3 u + a(x)u, u) = \int_{\mathbb{R}} a(x)|u(x)|^2 \mathrm{d}x \geqslant 0.$$

This shows that $-A$ is dissipative. According to Lumer–Phillips theorem [**20**], $-A$ generates a C_0 semigroup of contractions in $L^2(\mathbb{R})$, namely

$$\|\mathrm{e}^{-tA}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant 1, \quad \forall t \geqslant 0.$$

This will be improved to an exponential decay upper bound if a satisfies some further damping conditions.

THEOREM 2.1. *Assume that* (A1) *holds and* (A2) *holds on a thick set E. Then there exist constants C, $\lambda > 0$ depending only on a and E so that*

$$\|\mathrm{e}^{-tA}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant C\mathrm{e}^{-\lambda t}, \quad \forall t \geqslant 0.$$

Proof. The result has been stated in [**31**] without proof. We give a sketch here for the reader's convenience. Let E be a thick set. Based on the uncertainty principle of the Fourier transform, one can show that (see [**31**, lemma 2.3]) there exists a constant $c_1 > 0$ so that for all $\tau \in \mathbb{R}$, $u \in H^3(\mathbb{R})$,

$$c_1\|u\|_{L^2(\mathbb{R})} \leqslant \|(\partial_x^3 + \mathrm{i}\tau)u\|_{L^2(\mathbb{R})} + \|u\|_{L^2(E)}. \tag{2.1}$$

Since $a \in L^\infty(\mathbb{R})$, by the triangular inequality

$$\|(\partial_x^3 + \mathrm{i}\tau)u\|_{L^2(\mathbb{R})} \leqslant \|(A + \mathrm{i}\tau)u\|_{L^2(\mathbb{R})} + \|a\|_{L^\infty(\mathbb{R})}^{1/2}\|a^{1/2}u\|_{L^2(\mathbb{R})}. \tag{2.2}$$

By assumption (A2), $a(x) \geqslant a_0$ on E, we have

$$\|u\|_{L^2(E)} \leqslant a_0^{-(1/2)}\|a^{1/2}u\|_{L^2(\mathbb{R})}. \tag{2.3}$$

It follows from (2.1)–(2.3) that for some $c_2 > 0$

$$c_1\|u\|_{L^2(\mathbb{R})} \leqslant \|(A + \mathrm{i}\tau)u\|_{L^2(\mathbb{R})} + c_2\|a^{1/2}u\|_{L^2(\mathbb{R})}. \tag{2.4}$$

Moreover, taking the $L^2(\mathbb{R})$ inner product and using (A1), we find

$$\int_{\mathbb{R}} a(x)|u|^2\mathrm{d}x = \mathbf{Re}((A + \mathrm{i}\tau)u, u) \leqslant \|u\|_{L^2(\mathbb{R})}\|(A + \mathrm{i}\tau)u\|_{L^2(\mathbb{R})}. \tag{2.5}$$

Plugging (2.5) into (2.4) and using Cauchy–Schwartz, we infer that for all $\varepsilon > 0$,

$$c_1\|u\|_{L^2(\mathbb{R})} \leqslant \|(A + \mathrm{i}\tau)u\|_{L^2(\mathbb{R})} + c_2\varepsilon\|u\|_{L^2(\mathbb{R})} + c_2\varepsilon^{-1}\|(A + \mathrm{i}\tau)u\|_{L^2(\mathbb{R})}. \tag{2.6}$$

Taking $\varepsilon = \varepsilon_0 > 0$ small enough such that $c_2\varepsilon_0 \leqslant c_1/2$, then we deduce from (2.6) that

$$\|u\|_{L^2(\mathbb{R})} \leqslant c_3\|(A + \mathrm{i}\tau)u\|_{L^2(\mathbb{R})}, \quad \forall \tau \in \mathbb{R},$$

where $c_3 > 0$ is a constant independent of τ. This implies that the resolvent set $\rho(A) \supset \mathrm{i}\mathbb{R}$ and $\sup_{\tau\in\mathbb{R}} \|(A + \mathrm{i}\tau)^{-1}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < \infty$. Thus according to Gearhart–Pruss–Huang criteria [**11**], the desired decay holds. □

Proof of theorem 1.1. Assume that (A1) holds. We divide the proof into two steps.

(2)⟹(1). First, following [**8**], we show that the condition (2) implies that (A2) holds on a thick set E. In fact, if (2) holds, then there exists $\gamma > 0$, so that

$$\inf_{x\in\mathbb{R}} \int_{x-L}^{x+L} a(y)\mathrm{d}y \geqslant \gamma > 0. \tag{2.7}$$

Fix $x \in \mathbb{R}$. For every $\varepsilon > 0$, we consider the set

$$\Sigma_\varepsilon = \{y \in [x - L, x + L] : 0 \leqslant a(y) < \varepsilon\}.$$

It follows from (2.7) that

$$\begin{aligned}\gamma \leqslant \int_{x-L}^{x+L} a(y)\mathrm{d}y &\leqslant \int_{[x-L,x+L]\bigcap\Sigma_\varepsilon} a(y)\mathrm{d}y + \int_{[x-L,x+L]\bigcap\Sigma_\varepsilon^c} a(y)\mathrm{d}y \\ &\leqslant 2L\varepsilon + \|a\|_{L^\infty(\mathbb{R})}|\Sigma_\varepsilon^c|. \end{aligned}\tag{2.8}$$

Choose $\varepsilon = \varepsilon_0 := \gamma/4L$. It follows from (2.8) and the definition of Σ_ε^c that

$$|y \in [x - L, x + L] : a(y) \geqslant \varepsilon_0| \geqslant \frac{\gamma}{2\|a\|_{L^\infty(\mathbb{R})}} > 0.$$

Since x can be chosen arbitrarily, we conclude that the set $\{y \in \mathbb{R} : a(y) \geqslant \varepsilon_0\}$ is a thick set. Now according to theorem 2.1, (1) holds.

(1)⟹(2). Assume that (1) holds for some constants C, $\lambda > 0$, then

$$\|\mathrm{e}^{-tA}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant C\mathrm{e}^{-\lambda t}, \quad \forall t \geqslant 0,$$

where $A = \partial_x^3 + a(x)$ with domain $D(A) = H^3(\mathbb{R})$. According to Gearhart–Pruss–Huang criteria [**11**], we have

$$\sup_{\tau\in\mathbb{R}} \|(A + \mathrm{i}\tau)^{-1}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < \infty.$$

In particular, letting $\tau = 0$, this implies $\|A^{-1}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < \infty$. In other words, there exists a constant $c > 0$ so that

$$c\|f\|_{L^2(\mathbb{R})} \leqslant \|(\partial_x^3 + a(x))f\|_{L^2(\mathbb{R})}, \quad \forall f \in L^2(\mathbb{R}). \tag{2.9}$$

Now we are going to test (2.9) with a sequence of functions

$$f_\varepsilon(x) = \varepsilon^{1/4}\mathrm{e}^{-\varepsilon x^2}, \quad \varepsilon > 0, x \in \mathbb{R}.$$

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

Fix $\varepsilon > 0$. Clearly,

$$\|f_\varepsilon\|_{L^2(\mathbb{R})} \geqslant c_0 \tag{2.10}$$

for some constant $c_0 > 0$ independent of ε. Moreover, we compute

$$\partial_x^3(\mathrm{e}^{\varphi}) = \mathrm{e}^{\varphi}\Big((\partial_x\varphi)^3 + 3\partial_x\varphi\partial_x^2\varphi + \partial_x^3\varphi\Big),$$

which implies that

$$\partial_x^3 f_\varepsilon = \varepsilon^{1/4}\mathrm{e}^{-\varepsilon x^2}(-8\varepsilon^3 x^3 + 12\varepsilon^2 x).$$

This gives that for some constant $c_1 > 0$ independent of ε

$$\|\partial_x^3 f_\varepsilon\|_{L^2(\mathbb{R})} \leqslant c_1\varepsilon^{3/2}. \tag{2.11}$$

It follows from (2.9)–(2.11) that

$$cc_0 \leqslant c_1\varepsilon^{3/2} + \|af_\varepsilon\|_{L^2(\mathbb{R})}. \tag{2.12}$$

Choosing $\varepsilon = \varepsilon_0$ such that $c_1\varepsilon_0^{3/2} = cc_0/2$ in (2.12), we find

$$\frac{cc_0}{2} \leqslant \|af_{\varepsilon_0}\|_{L^2(\mathbb{R})} \leqslant \|af_{\varepsilon_0}\|_{L^2(|x|\leqslant L)} + \|af_{\varepsilon_0}\|_{L^2(|x|>L)}, \tag{2.13}$$

for every $L > 0$. Note that for some $c_2 > 0$ we have

$$\|af_{\varepsilon_0}\|_{L^2(|x|>L)} \leqslant \varepsilon_0^{1/4}\|a\|_{L^\infty(\mathbb{R})}\left(\int_{|x|\geqslant L}\mathrm{e}^{-2\varepsilon_0 x^2}\,\mathrm{d}x\right)^{1/2} \leqslant c_2\|a\|_{L^\infty(\mathbb{R})}\mathrm{e}^{-(\varepsilon_0/2)L^2}. \tag{2.14}$$

Choosing $L = L_0$ so that $c_2\|a\|_{L^\infty(\mathbb{R})}\mathrm{e}^{-(\varepsilon_0/2)L^2} \leqslant cc_0/4$, we deduce from (2.13)–(2.14) that

$$\frac{cc_0}{4} \leqslant \|af_{\varepsilon_0}\|_{L^2(|x|\leqslant L_0)}. \tag{2.15}$$

Squaring both sides of (2.15) and using Cauchy–Schwarz inequality, we infer that

$$\left(\frac{cc_0}{4}\right)^2 \leqslant \|a\|_{L^\infty(\mathbb{R})}\int_{|x|\leqslant L_0} a(x)f_{\varepsilon_0}^2(x)\mathrm{d}x \leqslant \sqrt{\varepsilon_0}\|a\|_{L^\infty(\mathbb{R})}\int_{|x|\leqslant L_0} a(x)\mathrm{d}x. \tag{2.16}$$

It follows from (2.16) that

$$\int_{|x|\leqslant L_0} a(x)\mathrm{d}x \geqslant \gamma \tag{2.17}$$

with $\gamma = \left(\frac{cc_0}{4}\right)^2 \frac{1}{\sqrt{\varepsilon_0}\|a\|_{L^\infty(\mathbb{R})}} > 0$.

Now for every $x_0 \in \mathbb{R}$, if we testing (2.9) with

$$f_\varepsilon(x) = \varepsilon^{1/4}\mathrm{e}^{-\varepsilon(x-x_0)^2},$$

then, similar to (2.17), we have

$$\int_{|x-x_0|\leqslant L_0} a(x)\mathrm{d}x \geqslant \gamma. \tag{2.18}$$

This shows that (2) holds. Thus the proof is complete. □

3. Exponential decay with small data

First we recall some estimates in Bourgain spaces. Let $s,\, b \in \mathbb{R}$, the Bourgain spaces $X^{s,b}$ are defined by the norm

$$\|u\|_{X^{s,b}} := \left(\int_{\mathbb{R}^2} (1+|\xi|)^{2s}(1+|\tau-\xi^3|)^{2b}|\widehat{u}(\xi,\tau)|^2 \mathrm{d}\xi \mathrm{d}\tau\right)^{1/2} < \infty,$$

where $\widehat{u}(\xi,\, \tau)$ is the space–time Fourier transform of u, given by

$$\widehat{u}(\xi,\tau) = \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}(x\xi+t\tau)} u(x,t)\mathrm{d}x\mathrm{d}t.$$

For an open interval I on $\mathbb{R}$, the restriction in time Bourgain spaces $X_I^{s,b}$ are endowed with the norm

$$\|u\|_{X_I^{s,b}} := \inf_{v\in X^{s,b}} \Big\{ \|v\|_{X^{s,b}}, v(\cdot) = u(\cdot) \text{ on } I \Big\}.$$

Let $\{W(t)\}_{t\in\mathbb{R}}$ be the Airy group, given by

$$(W(t)u_0)(x) = \mathrm{e}^{-t\partial_x^3} u_0 = c \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{e}^{\mathrm{i}t\xi^3} \widehat{u_0}(\xi)\mathrm{d}\xi, \tag{3.1}$$

where c is an absolute constant.

Lemma 3.1. *Assume that* $I = (0,\, \delta)$ *with* $0 < \delta \leqslant 1$ *and* $s \geqslant 0$.

(1) *If* $b > 1/2$, *then*

$$\|W(t)u_0\|_{X_I^{s,b}} \lesssim_b \|u_0\|_{H^s(\mathbb{R})}, \tag{3.2}$$

$$\left\| \int_0^t W(t-\tau) f(\cdot,\tau)\mathrm{d}\tau \right\|_{X_I^{s,b}} \lesssim_b \|f\|_{X_I^{s,b-1}}. \tag{3.3}$$

(2) *If* $-(1/2) < b \leqslant b' < 1/2$, *then*

$$\|u\|_{X^{s,b}_{(-\delta,\delta)}} \lesssim_{s,b,b'} \delta^{b'-b} \|u\|_{X_I^{s,b'}}. \tag{3.4}$$

(3) *If* $1/2 < b \leqslant b' \leqslant 3/4$, *then*

$$\|\partial_x(uv)\|_{X_I^{s,b'-1}} \lesssim_{s,b} \|u\|_{X_I^{s,b}} \|v\|_{X_I^{s,b}}. \tag{3.5}$$

Proof. See [30]. □

The item (1) of lemma 3.1 can be understood as some estimates of the Airy group $W(t)$ in Bourgain spaces. Now we give some similar estimates of e^{-tA}, $A = \partial_x^3 + a(x)$, based on lemma 3.1.

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

LEMMA 3.2. *Assume that $a \in L^\infty(\mathbb{R})$, $I = (0, \delta)$ and $b \in (1/2, 1]$. Then for some small $\delta = \delta(b, \|a\|_{L^\infty}) > 0$, we have*

$$\|\mathrm{e}^{-tA}u_0\|_{X_I^{0,b}} \lesssim_b \|u_0\|_{L^2(\mathbb{R})}, \tag{3.6}$$

$$\left\| \int_0^t \mathrm{e}^{-(t-\tau)A}\partial_x(uv)(\tau)\mathrm{d}\tau \right\|_{X_I^{0,b}} \lesssim_b \|u\|_{X_I^{0,b}}\|v\|_{X_I^{0,b}}. \tag{3.7}$$

Proof. By Duhamel formula we have

$$\mathrm{e}^{-At}u_0 = W(t)u_0 - \int_0^t W(t-s)(a\mathrm{e}^{-As}u_0)\mathrm{d}s. \tag{3.8}$$

Taking $X_I^{0,b}$ norm on both sides of (3.8), using (1) of lemma 3.1, we find

$$\begin{aligned} \|\mathrm{e}^{-tA}u_0\|_{X_I^{0,b}} &\leqslant \|W(t)u_0\|_{X_I^{0,b}} + \left\| \int_0^t W(t-\tau)(a\mathrm{e}^{-\tau A}u_0)\mathrm{d}\tau \right\|_{X_I^{0,b}} \\ &\leqslant C\|u_0\|_{L^2(\mathbb{R})} + C\|a\mathrm{e}^{-tA}u_0\|_{X_I^{0,b-1}}, \end{aligned} \tag{3.9}$$

where $C = C(b) > 0$. Since $b \leqslant 1$, noting $X^{0,0} = L^2_{t,x}$, we have

$$\|a\mathrm{e}^{-tA}u_0\|_{X_I^{0,b-1}} \leqslant \|a\mathrm{e}^{-tA}u_0\|_{X_I^{0,0}} \leqslant \|a\|_{L^\infty}\|\mathrm{e}^{-tA}u_0\|_{X_I^{0,0}} \leqslant C'\delta^b\|\mathrm{e}^{-tA}u_0\|_{X_I^{0,b}}, \tag{3.10}$$

where in the last step we used (2) of lemma 3.1, and $C' > 0$ is a constant depending on b and $\|a\|_{L^\infty}$, but independent of δ. It follows from (3.9)–(3.10) that

$$\|\mathrm{e}^{-tA}u_0\|_{X_I^{0,b}} \leqslant C\|u_0\|_{L^2(\mathbb{R})} + CC'\delta^b\|\mathrm{e}^{-tA}u_0\|_{X_I^{0,b}}. \tag{3.11}$$

If we take δ small such that $CC'\delta^b \leqslant 1/2$, then the last term of (3.11) can be absorbed by the left hand side, we have

$$\|\mathrm{e}^{-tA}u_0\|_{X_I^{0,b}} \leqslant 2C\|u_0\|_{L^2(\mathbb{R})}.$$

This proves (3.6) for such δ.

To prove (3.7), we apply the identity

$$\begin{aligned} \int_0^t \mathrm{e}^{-(t-\tau)A}f(\tau)\mathrm{d}\tau = \int_0^t W(t-\tau)f(\tau)\mathrm{d}\tau \\ - \int_0^t W(t-\tau)a(x)\left(\int_0^\tau \mathrm{e}^{-(\tau-\tau')A}f(\tau')\mathrm{d}\tau'\right)\mathrm{d}\tau \end{aligned}$$

with $f = \partial_x(uv)$, similar to the above argument, we obtain

$$\begin{aligned} &\left\| \int_0^t \mathrm{e}^{-(t-\tau)A}\partial_x(uv)(\tau)\mathrm{d}\tau \right\|_{X_I^{0,b}} \\ &\quad \leqslant C\|u\|_{X_I^{0,b}}\|v\|_{X_I^{0,b}} + C\|a\|_{L^\infty}\delta^b \left\| \int_0^t \mathrm{e}^{-(t-\tau)A}\partial_x(uv)(\tau)\mathrm{d}\tau \right\|_{X_I^{0,b}}. \end{aligned}$$

This proves (3.7) for δ small enough. □

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

Lemma 3.2 holds for the interval I with small length. Now we extend it to general intervals.

LEMMA 3.3. *Assume that* (A1) *holds and* (A2) *holds on a thick set* E. *Let* $T>0$, $I=(0,\,T)$ *and* $b\in(1/2,\,1]$. *Then there exists* $C=C(b,\,\|a\|_{L^\infty})>0$ *we have*

$$\|\mathrm{e}^{-tA}u_0\|_{X_I^{0,b}}\leqslant C\|u_0\|_{L^2(\mathbb{R})}, \tag{3.12}$$

$$\left\|\int_0^t \mathrm{e}^{-(t-\tau)A}\partial_x(uv)(\tau)\mathrm{d}\tau\right\|_{X_I^{0,b}}\leqslant C(T+1)\|u\|_{X_I^{0,b}}\|v\|_{X_I^{0,b}}. \tag{3.13}$$

Proof. In the case $T\leqslant\delta$, (3.12) and (3.12) follows from (3.6) and (3.7), respectively. So we assume $T>\delta$ now. The proof then mainly relies on the following inequality: for all $s\geqslant 0$, $b\in(1/2,\,1]$, $\delta>0$ and $t_0\in\mathbb{R}$,

$$\|u\|_{X^{s,b}_{(t_0,t_0+2\delta)}}\lesssim\|u\|_{X^{s,b}_{(t_0,t_0+\delta)}}+\|u\|_{X^{s,b}_{(t_0+\delta,t_0+2\delta)}}. \tag{3.14}$$

See [**32**, lemma 6.2] for a proof.

To prove (3.12), choose an integer $k\geqslant 1$ so that $k\delta\geqslant T$. Then $(0,\,T)\subset(0,\,k\delta)$, using (3.14) repeatedly, we find

$$\|\mathrm{e}^{-tA}u_0\|_{X^{0,b}_{(0,T)}}\lesssim\sum_{j=1}^k\|\mathrm{e}^{-tA}u_0\|_{X^{0,b}_{((j-1)\delta,j\delta)}}. \tag{3.15}$$

But by (3.6) again, we have

$$\begin{aligned}\|\mathrm{e}^{-tA}u_0\|_{X^{0,b}_{((j-1)\delta,j\delta)}}&=\|\mathrm{e}^{-tA}\mathrm{e}^{-(j-1)\delta A}u_0\|_{X^{0,b}_{(0,\delta)}}\\&\lesssim_b\|\mathrm{e}^{-(j-1)\delta A}u_0\|_{L^2(\mathbb{R})}\lesssim\mathrm{e}^{-(j-1)\delta\lambda}\|u_0\|_{L^2(\mathbb{R})},\end{aligned}$$

where in the last step we used theorem 2.1. This, together with (3.15), gives

$$\|\mathrm{e}^{-tA}u_0\|_{X^{0,b}_{(0,T)}}\lesssim\sum_{j=1}^k\mathrm{e}^{-(j-1)\delta\lambda}\|u_0\|_{L^2(\mathbb{R})}\leqslant C\|u_0\|_{L^2(\mathbb{R})}$$

with some $C>0$ depending on δ and b. This proves (3.12).

To show (3.13), we choose k so that $(k-1)\delta<T\leqslant k\delta$. Then by (3.14) and (3.7),

$$\begin{aligned}\left\|\int_0^t\mathrm{e}^{-(t-\tau)A}\partial_x(uv)(\tau)\mathrm{d}\tau\right\|_{X^{0,b}_{(0,T)}}&\lesssim\sum_{j=1}^k\left\|\int_0^t\mathrm{e}^{-(t-\tau)A}\partial_x(uv)(\tau)\mathrm{d}\tau\right\|_{X^{0,b}_{((j-1)\delta,j\delta)}}\\&\lesssim_b\sum_{j=1}^k\|u\|_{X^{0,b}_{((j-1)\delta,j\delta)}}\|v\|_{X^{0,b}_{((j-1)\delta,j\delta)}}\\&\lesssim k\|u\|_{X^{0,b}_{(0,T)}}\|v\|_{X^{0,b}_{(0,T)}}\\&\leqslant\left(\frac{T}{\delta}+1\right)\|u\|_{X^{0,b}_{(0,T)}}\|v\|_{X^{0,b}_{(0,T)}}.\end{aligned}$$

This proves (3.13). □

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

Now we can prove theorem 1.2.

Proof of theorem 1.2. By Duhamel principle, we can rewrite the KdV equation (1.1) into an integral form

$$u(t) = \mathrm{e}^{-tA}u_0 - \int_0^t \mathrm{e}^{-(t-s)A}(uu_x)(s)\mathrm{d}s. \tag{3.16}$$

Taking $L^2(\mathbb{R})$ norm on both sides of (3.16), using theorem 2.1 and the embedding $X^{0,b}_{(0,T)} \hookrightarrow L^\infty(0,T; L^2(\mathbb{R}))$ when $b > 1/2$, we find for all $T > 0$

$$\begin{aligned}\|u(T)\|_{L^2(\mathbb{R})} &\leqslant C_0\mathrm{e}^{-\lambda T}\|u_0\|_{L^2(\mathbb{R})} + C\left\|\int_0^t \mathrm{e}^{-(t-s)A}(uu_x)(s)\mathrm{d}s\right\|_{X^{0,b}_{(0,T)}} \\ &\leqslant C_0\mathrm{e}^{-\lambda T}\|u_0\|_{L^2(\mathbb{R})} + C_1(T+1)\|u\|^2_{X^{0,b}_{(0,T)}}\end{aligned} \tag{3.17}$$

for some $C_0 > 1/2$, where in the last step we used (3.13).

Now fix a large $T > 0$ so that

$$C_0\mathrm{e}^{-\lambda T} = \frac{1}{2}\mathrm{e}^{-(\lambda/2)T}. \tag{3.18}$$

Then (3.17) becomes

$$\|u(T)\|_{L^2(\mathbb{R})} \leqslant \frac{1}{2}\mathrm{e}^{-(\lambda/2)T}\|u_0\|_{L^2(\mathbb{R})} + C_1(T+1)\|u\|^2_{X^{0,b}_{(0,T)}}. \tag{3.19}$$

Moreover, taking $X^{0,b}_{(0,T)}$ norm on both sides of (3.16), using lemma 3.3, we obtain

$$\|u\|_{X^{0,b}_{(0,T)}} \leqslant C_2\|u_0\|_{L^2(\mathbb{R})} + C_3(T+1)\|u\|^2_{X^{0,b}_{(0,T)}}. \tag{3.20}$$

Now consider the map Γ

$$\Gamma u = \mathrm{e}^{-tA}u_0 - \int_0^t \mathrm{e}^{-(t-s)A}(uu_x)(s)\mathrm{d}s \tag{3.21}$$

on the ball

$$\mathcal{B} = \left\{u : \|u\|_{X^{0,b}_{(0,T)}} \leqslant 2C_2\|u_0\|_{L^2(\mathbb{R})}\right\}.$$

The estimate (3.20) shows that if $u \in \mathcal{B}$ then

$$\|\Gamma u\|_{X^{0,b}_{(0,T)}} \leqslant C_2\|u_0\|_{L^2(\mathbb{R})} + 4C_2^2C_3(T+1)\|u_0\|^2_{L^2(\mathbb{R})}. \tag{3.22}$$

Moreover, if $u,\, v \in \mathcal{B}$, then

$$\|\Gamma u - \Gamma v\|_{X^{0,b}_{(0,T)}} \leqslant 4C_2C_3(T+1)\|u_0\|_{L^2(\mathbb{R})}\|u-v\|_{X^{0,b}_{(0,T)}}. \tag{3.23}$$

Thanks to (3.22)–(3.23), if $\|u_0\|_{L^2(\mathbb{R})}$ is small enough, say,

$$\|u_0\|_{L^2(\mathbb{R})} \leqslant \frac{1}{8C_2C_3(T+1)} := \delta_1, \tag{3.24}$$

then $\Gamma\mathcal{B} \subset \mathcal{B}$ and $\|\Gamma u - \Gamma v\|_{X^{0,b}_{(0,T)}} \leqslant (1/2)\|u-v\|_{X^{0,b}_{(0,T)}}$, thus Γ is a contraction mapping on $\mathcal{B}$. So equation (3.16) has a unique solution $u \subset \mathcal{B}$. This, together

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

with the bound (3.19), gives that

$$\|u(T)\|_{L^2(\mathbb{R})} \leqslant \frac{1}{2}\mathrm{e}^{-(\lambda/2)T}\|u_0\|_{L^2(\mathbb{R})} + 4C_2^2C_1(T+1)\|u_0\|^2_{L^2(\mathbb{R})}. \tag{3.25}$$

Assume further that

$$\|u_0\|_{L^2(\mathbb{R})} \leqslant \frac{1}{8C_2^2C_1(T+1)}\mathrm{e}^{-(\lambda/2)T} := \delta_2, \tag{3.26}$$

so that $4C_2^2C_1(T+1)\|u_0\|^2_{L^2(\mathbb{R})} \leqslant (1/2)\mathrm{e}^{-(\lambda/2)T}\|u_0\|_{L^2(\mathbb{R})}$, then (3.25) becomes

$$\|u(T)\|_{L^2(\mathbb{R})} \leqslant \mathrm{e}^{-(\lambda/2)T}\|u_0\|_{L^2(\mathbb{R})}.$$

By induction, we find that for all $n \geqslant 1$

$$\|u(nT)\|_{L^2(\mathbb{R})} \leqslant \mathrm{e}^{-(\lambda/2)nT}\|u_0\|_{L^2(\mathbb{R})}$$

if $\|u_0\|_{L^2(\mathbb{R})} \leqslant \delta = \min\{\delta_1,\, \delta_2\}$. Then by the semigroup property, we infer that

$$\|u(t)\|_{L^2(\mathbb{R})} \leqslant C'\mathrm{e}^{-\lambda' t}\|u_0\|_{L^2(\mathbb{R})}, \quad \forall t \geqslant 0$$

for some constants C', $\lambda' > 0$. This completes the proof of theorem 1.2. □

4. Exponential decay for general data

4.1. A unique continuation

This subsection is devoted to the following unique continuation property (UCP) of the KdV equation, which is a key step to establish the exponential decay for general data.

PROPOSITION 4.1. *Let $T > 0$ and E be a set satisfying NCC. Assume that $u \in X^{0,(1/2)+}_{(0,T)}$ is a solution of the KdV equation*

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad (x, t) \in \mathbb{R} \times (0, T)$$

and $u(x,\, t) = 0$ on $E \times (0,\, T)$. Then

$$u(x, t) = 0, \quad \text{for } x \in \mathbb{R},\ t \in (0, T).$$

The proof of this result relies on the following unique continuation property, stated explicitly in [**33**], that follows from the results in [**25**].

LEMMA 4.2. *Let $T > 0$. Assume that $u \in L^\infty(0,\, T; H^3(\mathbb{R}))$ is a solution of the KdV equation*

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad (x, t) \in \mathbb{R} \times (0, T)$$

and $u = 0$ on an open subset of $\mathbb{R} \times (0,\, T)$, then

$$u(x, t) = 0, \quad \text{for } x \in \mathbb{R},\ t \in (0, T).$$

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

Note that proposition 4.1 does not follow from lemma 4.2 directly, since the solution, $u \in X^{0,(1/2)+}_{(0,T)}$, has not higher enough regularity. To overcome this difficulty, we need to first establish the propagation of regularity for the operator $\partial_t + \partial_x^3$ on $\mathbb{R}$.

LEMMA 4.3. *Let $T > 0$, $r \geqslant 0$ and $f \in X^{r,-(1/2)}_{(0,T)}$. Let $u \in X^{r,1/2}_{(0,T)}$ be a solution of*

$$\partial_t u + \partial_x^3 u = f.$$

If $E \subset \mathbb{R}$ satisfies NCC and $u \in L^2_{loc}(0, T; H^{r+1/2}(E))$, then $u \in L^2_{loc}(0, T; H^{r+1/2}(\mathbb{R}))$.

Roughly speaking, lemma 4.3 means that the solution of the linear KdV equation has higher regularity on a set satisfying NCC, then the solution automatically has higher regularity on the whole space $\mathbb{R}$. This result is inspired by the work of [**14**], in which the propagation of regularity for the linear KdV equation on torus $\mathbb{T}$ is proved, when E is an open subset of $\mathbb{T}$. In [**14**], the proof relies on a partition of unity

$$1 = \sum_j \chi(x - x_j)$$

where χ is a smooth cutoff function supported on the open set E, and the sum is taken over for finite terms. However, the set E, satisfying NCC, may have complicated structure, so it is not clear whether the corresponding partition of unity exists or not. Even though, we shall use the following lemma instead, which is sufficient for our purpose.

LEMMA 4.4. *Assume that E satisfy NCC. Then there exist constants $L_0 > 0$, $m_0 \in \mathbb{N}$ and a smooth function $\chi \in C^\infty(\mathbb{R})$ such that the support supp $\chi \subset E$, $|\partial_x^k \chi| \leqslant C_k$ for all $k \in \mathbb{N}$, and*

$$\sum_{\ell=-m_0}^{m_0} \chi^2(x + \ell L_0) \geqslant 1, \qquad \forall x \in \mathbb{R}. \tag{4.1}$$

Proof. By definition, for some constants r, $L > 0$ we have

$$E \supset \bigcup_n (x_n - r, x_n + r), \qquad \inf_{n\in\mathbb{Z}} |x - x_n| \leqslant L, \text{ for all } x \in \mathbb{R}.$$

Without loss of generality, we assume that $L > 2r$. Clearly, there exists $n_0 \in \mathbb{Z}$ so that $|x_{n_0}| \leqslant L$. Then

$$(x_{n_0} - r, x_{n_0} + r) \subset \left(-\frac{3L}{2}, \frac{3L}{2}\right).$$

We set

$$I_0 := (x_{n_0} - r, x_{n_0} + r).$$

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

Similarly, for every $j \in \mathbb{Z}$, there exists an interval

$$I_j := (x_{n_j} - r, x_{n_j} + r) \subset \left(6jL - \frac{3L}{2}, 6jL + \frac{3L}{2}\right). \tag{4.2}$$

Clearly, we have

$$E \supset \bigcup_{j\in\mathbb{Z}} I_j, \tag{4.3}$$

$$\operatorname{dist}(I_j, I_{j+1}) \geqslant 3L, \text{ for all } j \in \mathbb{Z}. \tag{4.4}$$

Now let $\chi_0 \in C_c^\infty(I_0)$ (the set of smooth functions with compact support) so that

$$\chi_0(x) = \begin{cases} 1, & \text{if } x \in \left[x_{n_0} - \frac{r}{2}, x_{n_0} + \frac{r}{2}\right] \\ 0, & \text{if } |x - x_{n_0}| \geqslant r. \end{cases} \tag{4.5}$$

Moreover, we define for every $j \in \mathbb{Z}$

$$\chi_j(x) = \chi_0(x - x_{n_j}), \quad x \in \mathbb{R} \tag{4.6}$$

and

$$\chi(x) = \sum_{j\in\mathbb{Z}} \chi_j(x), \quad x \in \mathbb{R}. \tag{4.7}$$

Now we show that χ enjoys the desired property. First, $\operatorname{supp}\chi \subset E$ follows from (4.3) and (4.5)–(4.7). Moreover, for every $x \in \mathbb{R}$, by (4.2), all terms in the sum of (4.7) vanish except at most one term, so

$$|\partial_x^k \chi(x)| \leqslant |\partial_x^k \chi_0(x)| \leqslant C_k, \quad \forall k \in \mathbb{N},$$

and at the same time we have

$$\chi^2(x) = \sum_{j\in\mathbb{Z}} \chi_j^2(x), \quad x \in \mathbb{R}. \tag{4.8}$$

Since $\chi_0 = 1$ on a subinterval of $[-6L, 6L]$ with length r, then for some $\mathbb{N} \ni m_0 \geqslant 6L/r$ we have

$$\sum_{\ell=-m_0}^{m_0} \chi_0^2(x + \ell r) \geqslant 1, \qquad \text{for all } x \in [-6L, 6L]. \tag{4.9}$$

Similarly,

$$\sum_{\ell=-m_0}^{m_0} \chi_j^2(x + \ell r) \geqslant 1, \qquad \text{for all } x \in [6(j-1)L, 6(j+1)L]. \tag{4.10}$$

Let $L_0 = r$. It follows from (4.8)–(4.10) that

$$\sum_{\ell=-m_0}^{m_0} \chi^2(x + \ell L_0) \geqslant 1, \qquad \forall x \in \mathbb{R}.$$

This completes the proof. □

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

Proof of lemma 4.3. Fix $T > 0$. We assume, without loss of generality, that $0 \leqslant r < 1$. Otherwise, we consider the equation

$$\partial_t(\partial_x^k u) + \partial_x^3(\partial_x^k u) = \partial_x^k f$$

instead, where k is a positive integer. We divide the proof into three steps.

Step 1. Let $\phi \in C_c^\infty(0, T)$ and $\varphi \in C^\infty(\mathbb{R})$ so that $|\partial_x^k \varphi| \leqslant C_k$, $\forall k \in \mathbb{N}$. We claim that

$$\left|(\phi(t)J^{2r-1}(\partial_x\varphi)\partial_x^2 u, u)\right| \leqslant C. \tag{4.11}$$

Here and in the rest of the proof, J^s is defined by the Fourier transform as $\widehat{J^s f} = (1+|\xi|)^s \widehat{f}(\xi)$, $(\cdot, \cdot)$ denotes the inner product in $L^2(0, T; L^2(\mathbb{R}))$.

In fact, let $\mathbb{L} = \partial_t + \partial_x^3$ and $\mathbb{A} = \phi(t)J^{2r-1}\varphi$. Then using Parseval's identity, we have

$$(\mathbb{L}u, \mathbb{A}^* u) + (\mathbb{A}u, \mathbb{L}u) = ([\mathbb{A}, \partial_x^3]u, u) - (\phi'(t)J^{2r-1}\varphi u, u), \tag{4.12}$$

where $\mathbb{A}^* = \varphi(x)J^{2r-1}\phi(t)$ is the dual operator of $\mathbb{A}$, the commutator $[A, B] = AB - BA$ as usual. Since $\mathbb{L}u = f \in X_{(0,T)}^{r,-(1/2)}$, we infer that

$$\begin{aligned}|(\mathbb{L}u, \mathbb{A}^* u) + (\mathbb{A}u, \mathbb{L}u)| &\leqslant \|f\|_{X_{(0,T)}^{r,-(1/2)}}(\|\mathbb{A}u\|_{X_{(0,T)}^{-r,(1/2)}} + \|\mathbb{A}^* u\|_{X_{(0,T)}^{0,1/2}})\\ (\text{by lemma A.5}) &\leqslant C\|f\|_{X_{(0,T)}^{r,-(1/2)}}\|u\|_{X_{(0,T)}^{r,1/2}} \leqslant C.\end{aligned}$$

By (A.1) in the appendix, we also have $|(\phi'(t)J^{2r-1}\varphi u, u)| \leqslant C$. So by (4.12),

$$|([\mathbb{A}, \partial_x^3]u, u)| \leqslant C. \tag{4.13}$$

A direct computation gives that

$$[\mathbb{A}, \partial_x^3] = -3\phi(t)J^{2r-1}(\partial_x\varphi)\partial_x^2 - 3\phi(t)J^{2r-1}(\partial_x^2\varphi)\partial_x - \phi(t)J^{2r-1}\partial_x^3\varphi.$$

Similar to (A.1), we have

$$\left|\left(3\phi(t)J^{2r-1}(\partial_x^2\varphi)\partial_x u + \phi(t)J^{2r-1}\partial_x^3\varphi)u, u\right)\right| \leqslant C\|u\|^2_{L^2_{loc}(0,T;H^r(\mathbb{R}))} \leqslant C.$$

Thus, the claim (4.11) follows from (4.13).

Step 2. For any $\phi \in C_c^\infty(0, T)$, $\chi \in C^\infty(\mathbb{R})$ with support $\operatorname{supp}\chi \subset E$ and $|\partial_x^k \chi| \leqslant C_k$, we claim that

$$\left|(\phi(t)J^{2r-1}\chi^2\partial_x^2 u, u)\right| \leqslant C. \tag{4.14}$$

In fact, we rewrite

$$(\phi(t)J^{2r-1}\chi^2\partial_x^2 u, u) = I_1 + I_2$$

where

$$\begin{aligned}I_1 &= (\phi(t)J^{r-(3/2)}\chi\partial_x^2 u, J^{r+1/2}\chi u),\\ I_2 &= (\phi(t)J^{r-(3/2)}\chi\partial_x^2 u, [\chi, J^{r+1/2}]u) + (\phi(t)[J^{r-(3/2)}, \chi]\chi\partial_x^2 u, J^{r+1/2}u).\end{aligned}$$

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

From assumption $u \in L^2_{loc}(0, T; H^{r+1/2}(E))$, we infer $\chi\partial_x^2 u \in L^2_{loc}(0, T; H^{r-(3/2)}(\mathbb{R}))$, thus

$$|I_1| \leqslant C\|\phi(t)J^{r-(3/2)}\chi\partial_x^2 u\|_{L^2(0,T;L^2(\mathbb{R}))}\|u\|_{L^2_{loc}(0,T;H^{r+1/2}(E))} \leqslant C.$$

Moreover, using the fact $u \in L^2(0, T; H^r(\mathbb{R}))$ and (A.2)–(A.3), one can show that $|I_2| \leqslant C$. This proves the claim (4.14).

Step 3. Complete the proof. Let $\chi \in C^\infty$ be the cutoff function constructed in lemma 4.4. For every $x_0 \in \mathbb{R}$, define a function φ by Fourier transform

$$\widehat{\varphi}(\xi) = \frac{1 - \mathrm{e}^{\mathrm{i}x_0\xi}}{\mathrm{i}\xi}\widehat{\chi^2}(\xi).$$

By the Fourier inversion, this implies that

$$\partial_x\varphi(x) = \chi^2(x) - \chi^2(x + x_0), \quad x \in \mathbb{R}. \tag{4.15}$$

We apply (4.11) with $\partial_x\varphi$ given by (4.15), and use (4.14) to find that

$$\left|(\phi(t)J^{2r-1}\chi^2(\cdot{+}x_0)\partial_x^2 u, u)\right| \leqslant C. \tag{4.16}$$

Since $0 \leqslant r < 1$ we infer

$$|(\phi(t)J^{2r-1}\chi^2(\cdot{+}x_0)u, u)| \leqslant C.$$

Noting $J^2 = 1 - \partial_x^2$ we get from (4.16) that

$$\left|(\phi(t)J^{2r-1}\chi^2(\cdot{+}x_0)J^2 u, u)\right| \leqslant C. \tag{4.17}$$

We rewrite

$$\begin{aligned}&(\phi(t)J^{2r-1}\chi^2(\cdot{+}x_0)J^2 u, u)\\ &\quad= \left(\phi(t)J^{r+1/2}u, \chi^2(\cdot{+}x_0)J^{r+1/2}u\right) + \left(\phi(t)J^{r+1/2}u, [J^{r-(3/2)}, \chi^2(\cdot{+}x_0)]J^2 u\right)\end{aligned}$$

and use the bound $|(\phi(t)J^{r+1/2}u,\ [J^{r-(3/2)},\ \chi^2(\cdot + x_0)]J^2 u)| \leqslant C\|u\|^2_{L^2_{loc}(0,\,T;H^r(\mathbb{R}))} \leqslant C$ (follows from (A.4)), we deduce from (4.17) that

$$\int_0^T \int_{\mathbb{R}} \phi(t)|\chi(x + x_0)J^{r+1/2}u|^2 \mathrm{d}x\mathrm{d}t \leqslant C. \tag{4.18}$$

Applying (4.18) with $x_0 = \{lL_0\}_{l=-m_0}^{m_0}$, noting (4.1), we conclude that

$$\int_0^T \phi(t)\int_{\mathbb{R}} |J^{r+1/2}u|^2 \mathrm{d}x\mathrm{d}t \leqslant \sum_{l=-m_0}^{m_0} \int_0^T \int_{\mathbb{R}} \phi(t)|\chi(x + lL_0)J^{r+1/2}u|^2 \mathrm{d}x\mathrm{d}t \leqslant C.$$

This shows that $u \in L^2_{loc}(0, T; H^{r+1/2}(\mathbb{R}))$, and completes the proof. □

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

REMARK 4.5. Very recently, Panthee and Vielma Leal have established in [**19**] the propagation of regularity in Bourgain's spaces for the Benjamin equation on a periodic domain.

COROLLARY 4.6. *Let $T > 0$ and $u \in X^{0,(1/2)+}_{(0,T)}$ be a solution of the KdV equation*

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad (x, t) \in \mathbb{R} \times (0, T). \tag{4.19}$$

If E satisfies NCC and $u(x, t) = 0$ for $(x, t) \in E \times (0, T)$, then $u \in L^2_{loc}(0, T; H^\infty(\mathbb{R}))$.

REMARK 4.7. Here and below $X^{0,(1/2)+} = X^{0,(1/2)+\varepsilon}$ with an arbitrarily small $\varepsilon > 0$.

Proof. Rewrite the KdV equation as

$$\partial_t u + \partial_x^3 u = f$$

with $f = -u\partial_x u$. Since $u \in X^{0,(1/2)+}_{(0,T)}$, by the bilinear estimate (3.5),

$$\|f\|_{X^{0,-(1/2)+}_{(0,T)}} \leqslant C\|u\|^2_{X^{0,(1/2)+}_{(0,T)}} \leqslant C.$$

Since $u(x, t) = 0$ for $(x, t) \in E \times (0, T)$, we have of course $u \in L^2_{loc}(0, T; H^\infty(E))$. By lemma 4.3, we obtain

$$u \in L^2_{loc}(0, T; H^{1/2}(\mathbb{R})).$$

Let $t_0 \in (0, T)$ so that $u(t_0) \in H^{1/2}(\mathbb{R})$. Then we find the solution u of (4.19) satisfies

$$u \in X^{1/2,(1/2)+}_{(0,T)}.$$

Similarly, using lemma 4.3 repeatedly, we conclude that $u \in L^2_{loc}(0, T; H^\infty(\mathbb{R}))$. □

Proof of proposition 4.1. According to corollary 4.6, we know that $u \in L^2_{loc}(0, T; H^\infty(\mathbb{R}))$. Since $u(x, t) = 0$ for $(x, t) \in E \times (0, T)$ and E contains an open set in $\mathbb{R}$, so $u = 0$ on an open set in $\mathbb{R} \times (0, T)$, then by the UCP in lemma 4.2, we conclude that $u \equiv 0$. □

4.2. Proof of theorem 1.4

Let $W(t) = \mathrm{e}^{-t\partial_x^3}$ be the Airy group. Then we have the sharp Kato smoothing effect [**12**, theorem 4.1]

$$\|\partial_x W(t) u_0\|_{L^\infty_x L^2_t(\mathbb{R}^2)} \lesssim \|u_0\|_{L^2(\mathbb{R})}. \tag{4.20}$$

The estimate (4.20) can be reformulated in Bourgain space as (see [**13**, p. 5])

$$\|\partial_x u\|_{L^\infty_x L^2_t(\mathbb{R}^2)} \lesssim \|u\|_{X^{0,(1/2)+}}. \tag{4.21}$$

The bound (4.21) will be used to derive the compactness of some sequences later.

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

Proposition 4.8. *Assume that* $0 \leqslant a(x) \in L^\infty(\mathbb{R})$*. Then the IVP* (1.1) *has a unique global solution* $u \in C([0, \infty); L^2(\mathbb{R}))$*. Moreover, for every* $T > 0$

$$\|u\|_{X^{0,(1/2)+}_{(0,T)}} \leqslant C(T, \|a\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^2(\mathbb{R})}) < \infty.$$

Proof. Thanks to lemma 3.1, using the contraction principle one can show that there exists a unique solution $u \in X^{0,(1/2)+}_{(0,\delta)}$ of (1.1) with the bound

$$\|u\|_{X^{0,(1/2)+}_{(0,\delta)}} \leqslant 2\|u_0\|_{L^2(\mathbb{R})},$$

where the life span

$$\delta = \delta(\|a\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^2(\mathbb{R})}) > 0$$

is small enough. Multiplying (1.1) by u and integrating over $\mathbb{R}$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x + \int_{\mathbb{R}} a(x)|u(x,t)|^2 \mathrm{d}x = 0,$$

which, together with the fact $a(x) \geqslant 0$, implies that

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \leqslant \|u_0\|_{L^2(\mathbb{R})}, \quad \forall t \geqslant 0. \tag{4.22}$$

Thus we can take $u(\delta)$ as a new data, to find a solution on $(\delta, 2\delta)$ so that $\|u\|_{X^{0,(1/2)+}_{(\delta,2\delta)}} \leqslant 2\|u_0\|_{L^2(\mathbb{R})}$. Repeat this process, we find that for every $T > 0$

$$\|u\|_{X^{0,(1/2)+}_{(0,T)}} \leqslant C(T, \|a\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^2(\mathbb{R})}).$$

This completes the proof. □

Corollary 4.9. *Assume that* $0 \leqslant a(x) \in L^\infty(\mathbb{R})$*. Let* u *be the solution of* (1.1) *obtained in proposition* 4.8*. Then for every* $T > 0$ *and for every measurable set* $\Omega \subset \mathbb{R}$

$$\int_0^T \int_\Omega |\partial_x u(x,t)|^2 \mathrm{d}x\mathrm{d}t \leqslant |\Omega| C(T, \|a\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^2(\mathbb{R})}), \tag{4.23}$$

where $|\Omega|$ *denotes the Lebesgue measure of* Ω*.*

Proof. Fix $T > 0$. Combining (4.21) and proposition 4.8 we obtain

$$\|\partial_x u\|_{L^\infty_x L^2_t(\mathbb{R}^2)} \lesssim C(T, \|a\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^2(\mathbb{R})}).$$

From this, we use Hölder inequality to find

$$\|\partial_x u\|_{L^2_x(\Omega) L^2_t(0,T)} \leqslant |\Omega|^{1/2} \|\partial_x u\|_{L^\infty_x(\Omega) L^2_t(\mathbb{R})} \leqslant C.$$

Then we conclude (4.23) by Fubini theorem. □

Now we prove an observability inequality, which means that we can recover the solution of the KdV equation if we observe the solution on $E \times (0, T)$ when E satisfies NCC and E^c has a finite Lebesgue measure.

LEMMA 4.10. *Let R, $T > 0$ and let E satisfy NCC, $|E^c| < \infty$. Assume that* (A1) *and* (A2) *hold. Then there exist a constant $C = C(R, T, a) > 0$ such that for all $\|u_0\|_{L^2(\mathbb{R})} \leqslant R$, the solution u of the IVP* (1.1) *satisfies*

$$\int_0^T \int_{\mathbb{R}} u^2(x,t)\mathrm{d}x\mathrm{d}t \leqslant C \int_0^T \int_{\mathbb{R}} a(x)u^2(x,t)\mathrm{d}x\mathrm{d}t. \tag{4.24}$$

Proof. Following [**4**], we argue by contradiction. Assume that there exists a sequence solutions u_k of the KdV equation (1.1), with initial data $\|u_{0k}\|_{L^2(\mathbb{R})} \leqslant R$, such that

$$\lim_{k\to\infty} \frac{\int_0^T \int_{\mathbb{R}} a(x)u_k^2(x,t)\mathrm{d}x\mathrm{d}t}{\int_0^T \int_{\mathbb{R}} u_k^2(x,t)\mathrm{d}x\mathrm{d}t} = 0. \tag{4.25}$$

Define

$$\alpha_k = \|u_k\|_{L^2(\mathbb{R}\times(0,T))}, \quad v_k(x,t) = \frac{u_k(x,t)}{\alpha_k}. \tag{4.26}$$

Then

$$\|v_k\|_{L^2(\mathbb{R}\times(0,T))} = 1, \quad k \in \mathbb{N} \tag{4.27}$$

and v_k is a solution of

$$(v_k)_t + (v_k)_{xxx} + \alpha_k v_k (v_k)_x + a(x)v_k = 0, \quad (x,t) \in \mathbb{R}\times\mathbb{R}^+ \tag{4.28}$$

with initial data

$$v_k(x,0) = u_{0k}(x)/\alpha_k.$$

It follows from (4.25)–(4.26) that

$$\lim_{k\to\infty} \int_0^T \int_{\mathbb{R}} a(x)v_k^2(x,t)\mathrm{d}x\mathrm{d}t = 0. \tag{4.29}$$

This, since $a(x) \geqslant a_0 > 0$ for all $x \in E$, gives that

$$\lim_{k\to\infty} \int_0^T \int_E v_k^2(x,t)\mathrm{d}x\mathrm{d}t = 0. \tag{4.30}$$

Moreover, multiplying (4.28) with v_k and integrating over $\mathbb{R}\times(0, t)$ and changing the order of integration, we obtain

$$\int_{\mathbb{R}} v_k^2(x,t)\mathrm{d}x + 2\int_0^t \int_{\mathbb{R}} a(x)v_k^2(x,\tau)\mathrm{d}x\mathrm{d}\tau = \int_{\mathbb{R}} v_{0k}^2\mathrm{d}x. \tag{4.31}$$

Integrating (4.31) with respect to t over $[0, T]$, we obtain

$$\int_{\mathbb{R}} v_{0k}^2\mathrm{d}x \leqslant \frac{1}{T}\int_0^T \int_{\mathbb{R}} v_k^2(x,t)\mathrm{d}x\mathrm{d}t + 2\int_0^T \int_0^t \int_{\mathbb{R}} a(x)v_k^2\mathrm{d}x\mathrm{d}t.$$

This, together with (4.27) and (4.29), shows that

$$\int_{\mathbb{R}} v_{0k}^2(x)\mathrm{d}x \leqslant C(T, \|a\|_{L^\infty}, R). \tag{4.32}$$

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

Furthermore, it follows from the bound (4.22) that

$$\int_0^T \int_{\mathbb{R}} u_k^2(x,t)\mathrm{d}x\mathrm{d}t \leqslant T \int_{\mathbb{R}} u_{0k}^2(x)\mathrm{d}x.$$

This, together with (4.26), gives that

$$\alpha_k \leqslant \left(T \int u_{0k}^2(x)\mathrm{d}x\right)^{1/2} \leqslant T^{1/2}R \tag{4.33}$$

since $\|u_{0k}\|_{L^2(\mathbb{R})} \leqslant R$ for all k. Thanks to (4.32) and (4.33), by the well posedness of (4.28), we have

$$\|v_k\|_{X_{(0,T)}^{0,(1/2)+}} \leqslant C(T, \|a\|_{L^\infty}, R). \tag{4.34}$$

By our assumption, the complement set E^c has finite Lebesgue measure, $|E^c| < \infty$. Thanks to (4.32) and (4.33), we can apply corollary 4.9 to obtain that

$$\int_0^T \int_{E^c} |\partial_x v_k(x,t)|^2\mathrm{d}x\mathrm{d}t \leqslant C|E^c| < \infty, \quad \forall k \in \mathbb{N}. \tag{4.35}$$

Combining (4.27) and (4.35) we find that v_k is uniformly bounded in $L^2(0, T; H^1(E^c))$. Also, using equation (4.28), we get that $(v_k)_t$ is uniformly bounded in $L^2(0, T; H^{-2}(E^c))$. Since $|E^c| < \infty$, it is easy to see that

$$\lim_{x\to\infty} \left|E^c \bigcup \left(x - \frac{1}{2}, x + \frac{1}{2}\right)\right| = 0,$$

then according to [**1**, theorem 2.8], the embedding $H^1(E^c) \hookrightarrow L^2(E^c)$ is compact. Thus, by Aubin–Lions theorem, there exists a subsequence, still denoted by v_k, so that $v_k \to v$ in $L^2((0, T) \times E^c)$. On the other hand, it follows from (4.30) that $v_k \to 0$ in $L^2((0, T) \times E)$. These show that

$$v_k \to v \text{ strongly in } L^2(0,T;L^2(\mathbb{R})) \tag{4.36}$$

with $\|v\|_{X_{(0,T)}^{0,(1/2)+}} \leqslant C$ (by (4.34)) and

$$v(x,t) = 0, \text{ for } (x,t) \in E \times (0,T). \tag{4.37}$$

We assume that $\alpha_k \to \alpha \geqslant 0$. Let $\phi(x, t)$ be a function such that $\phi \in C([0, T]; H^3(\mathbb{R}))$, $\phi_t \in C([0, T]; L^2(\mathbb{R}))$ with $\phi|_{t=0} = \phi|_{t=T} = 0$. Testing (4.28) with

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

ϕ we get

$$\int_0^T \int_{\mathbb{R}} \left(v_k \partial_t \phi + v_k \partial_x^3 \phi - a(x) v_k \phi + \alpha_k \frac{v_k^2}{2} \partial_x \phi \right) \mathrm{d}x\mathrm{d}t = 0. \tag{4.38}$$

Now, taking the limit as $k \to \infty$ in (4.38), and applying (4.36)–(4.37) we arrive at

$$\int_0^T \int_{\mathbb{R}} \left(v \partial_t \phi + v \partial_x^3 \phi + \alpha \frac{v^2}{2} \partial_x \phi \right) \mathrm{d}x\mathrm{d}t = 0, \tag{4.39}$$

where we used the fact that $\int_0^T \int_{\mathbb{R}} a(x) v_k(x,\, t) \phi(x,\, t) \mathrm{d}x\mathrm{d}t \to 0$ as $k \to \infty$, which follows from (4.29) and the inequality

$$\left| \int_0^T \int_{\mathbb{R}} a(x) v_k(x,t) \phi(x,t) \mathrm{d}x\mathrm{d}t \right| \leqslant \left(\int_0^T \int_{\mathbb{R}} a(x) v_k^2(x,t) \mathrm{d}x\mathrm{d}t \right)^{1/2} \|a^{1/2}\phi\|_{L^2(0,T;L^2(\mathbb{R}))}.$$

The identity (4.39) means that $v(x,\, t) \in X_{(0,T)}^{0,(1/2)+}$ is a weak solution of

$$\partial_t v + \partial_x^3 v + \alpha v \partial_x v = 0, \quad (x,t) \in \mathbb{R} \times (0,T). \tag{4.40}$$

Moreover, by (4.37) we have $v|_{E\times(0,T)} = 0$.

If $\alpha = 0$, then equation (4.40) becomes $\partial_t v + \partial_x^3 v = 0$. Since $v = 0$ on an open set in $(x,\, t) \in \mathbb{R} \times (0,\, T)$, according to [**33**, corollary 3.1], we have $v \equiv 0$.

If $\alpha > 0$, set $V(x,\, t) = v(cx,\, c^3 t)$ with $c = \alpha^{-(1/2)}$, then $V \in X_{(0,c^{-3}T)}^{0,(1/2)+}$ is a solution of

$$\partial_t V + \partial_x^3 V + V \partial_x V = 0, \quad (x,t) \in \mathbb{R} \times (0, c^{-3}T).$$

Moreover, we have $V|_{c^{-1}E\times(0,\, c^{-3}T)} = 0$, where $c^{-1}E = \{c^{-1}x : x \in E\}$. It is easy to see that $c^{-1}E$ also satisfies NCC. Then by the unique continuation property in proposition 4.1 for the KdV equation, we get $V \equiv 0$ and thus $v \equiv 0$.

In both cases, we arrive at the conclusion $v \equiv 0$. However, this contradicts to (4.27). Therefore, (4.24) holds. □

Proof of theorem 1.4. Let $T > 0$ and $\|u_0\|_{L^2(\mathbb{R})} \leqslant R$. Multiplying (1.1) by u and integrating over $\mathbb{R} \times (0,\, T)$, we get

$$\int_{\mathbb{R}} |u(x,T)|^2 \mathrm{d}x + 2 \int_0^T \int_{\mathbb{R}} a(x) |u(x,t)|^2 \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} |u_0(x)|^2 \mathrm{d}x. \tag{4.41}$$

Since E satisfies NCC, it follows from lemma 4.10 that

$$\int_0^T \int_{\mathbb{R}} a(x) |u(x,t)|^2 \mathrm{d}x\mathrm{d}t \geqslant c \int_0^T \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x\mathrm{d}t, \tag{4.42}$$

where $c > 0$ is a constant depending only on T, R and the damping coefficient $a(x)$. Moreover, since $a(x) \geqslant 0$, we have $\|u(\cdot,\, t)\|_{L^2(\mathbb{R})} \leqslant \|u(\cdot,\, t')\|_{L^2(\mathbb{R})}$ for all $t \geqslant t'$, which implies that

$$\int_0^T \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x\mathrm{d}t \geqslant T \int_{\mathbb{R}} |u(x,T)|^2 \mathrm{d}x. \tag{4.43}$$

Combining (4.41)–(4.43), we infer that

$$\int_{\mathbb{R}} |u(x,T)|^2 \mathrm{d}x \leqslant \alpha \int_{\mathbb{R}} |u_0(x)|^2 \mathrm{d}x$$

with $\alpha = 1/(1+2cT) \in (0,\, 1)$. By iteration we have for all $n \geqslant 1$

$$\int_{\mathbb{R}} |u(x,nT)|^2 \mathrm{d}x \leqslant \alpha^n \int_{\mathbb{R}} |u_0(x)|^2 \mathrm{d}x.$$

This gives the exponential decay clearly. □

Acknowledgements

We owe our sincere thanks to the referees for their careful reading and insightful comments which have deeply improved the presentation of this work. Wang is supported by the National Natural Science Foundation of China under grant No. 12171442, and the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan) under grant No. CUGSX01. Zhou is supported by the National Natural Science Foundation of China under grant No. 11901067, the Natural Science Foundation of Chongqing under grant No. cstc2020jcyj-msxmX0623, and the Fundamental Research Funds for the Central Universities under grant No. 2021CDJQY-009.

Appendix A. Appendix

In this section, we prove some technical estimates used in lemma 4.3.

Let $b(x,\, \xi)$ be a smooth function on $\mathbb{R}^2$. Define the pseudo-differential operator

$$b(x,D)f(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi} b(x,\xi)\widehat{f}(\xi)\mathrm{d}\xi, \quad f \in \mathscr{S}(\mathbb{R}),$$

where $D = \mathrm{i}^{-1}\partial_x$, $\widehat{f}$ denotes the Fourier transform of f, $\mathscr{S}(\mathbb{R})$ the Schwartz class. The function $b(x,\, \xi)$ is called the symbol of the operator $b(x,\, D)$. In particular, letting $b(x,\, \xi) = (1+|\xi|^2)^{s/2}$, $s \in \mathbb{R}$, we recover the definition of the fractional Laplacian $J^s = (1-\partial_x^2)^{s/2}$. The Sobolev space $H^s(\mathbb{R})$ is an Hilbert space endowed with the norm

$$\|f\|_{H^s(\mathbb{R})} = \|J^s f\|_{L^2(\mathbb{R})}.$$

Let $m \in \mathbb{R}$. We say a smooth function $b(x,\, \xi)$ belongs to S^m if for all $\alpha,\, \beta \in \mathbb{N}$

$$|\partial_x^\beta \partial_\xi^\alpha b(x,\xi)| \leqslant C_{\alpha\beta}(1+|\xi|)^m, \quad x,\xi \in \mathbb{R}.$$

An important result on the class S^m is given in the following lemma, see [27, proposition 5.5, p. 20].

Lemma A.1. *Let m, $s \in \mathbb{R}$ and $b \in S^m$. Then $b(x, D)$ is bounded from $H^s(\mathbb{R})$ to $H^{s-m}(\mathbb{R})$.*

We say $\varphi \in C_b^\infty$ if for all $\alpha \in \mathbb{N}$

$$|\partial_x^\alpha \varphi(x)| \leqslant C_\alpha, \text{ for all } x \in \mathbb{R}.$$

Lemma A.2. *Let $m \in \mathbb{R}$ and $\varphi \in C_b^\infty$. Then*

$$\|\varphi f\|_{H^m(\mathbb{R})} \leqslant C\|f\|_{H^m(\mathbb{R})}.$$

Proof. It is equivalent to show that

$$\|\varphi J^{-m} f\|_{H^m(\mathbb{R})} \leqslant C\|f\|_{L^2(\mathbb{R})}.$$

Note that the symbol of φJ^{-m} is $b(x, \xi) = \varphi(x)(1 + |\xi|^2)^{-m/2}$, and it is easy to see that $b(x, \xi) \in S^{-m}$, then the desired bound follows from lemma A.1. □

Lemma A.3. *Let m, $s \in \mathbb{R}$ and $\varphi \in C_b^\infty$. Then*

$$\|[\varphi, J^m]f\|_{H^s(\mathbb{R})} \leqslant C\|f\|_{H^{m+s-1}(\mathbb{R})}.$$

Proof. By the definition of commutator, we have

$$[\varphi, J^m]f = \varphi(x)J^m - J^m(\varphi f) := \big(b_1(x, D) - b_2(x, D)\big)f,$$

where $b_1(x, \xi) = \varphi(x)(1 + |\xi|^2)^{-m/2}$ and $b_2(x, \xi)$ is the symbol of $J^m(\varphi\cdot)$. According to the calculus of pseudo-differential operators, see e.g. [**26**, theorem 2, p. 237],

$$b_2(x, \xi) = b_1(x, \xi) + c(x, \xi)$$

with $c(x, \xi) \in S^{-m-1}$. Thus the symbol of $[\varphi, J^m]$, equals to $c(x, \xi)$, belongs to S^{-m-1}. Then the lemma follows from lemma A.1. □

Now we prove the results used in the proof of lemma 4.3.

Lemma A.4. *Let $r \geqslant 0$ and φ, $\chi \in C_b^\infty$. Then the following bounds hold:*

$$|(J^{2r-1}\varphi u, u)| \leqslant C\|u\|^2_{H^r(\mathbb{R})}, \tag{A.1}$$

$$|(J^{r-(3/2)}\chi\partial_x^2 u, [\chi, J^{r+1/2}]u)| \leqslant C\|u\|^2_{H^r(\mathbb{R})}, \tag{A.2}$$

$$|([J^{r-(3/2)}, \chi]\chi\partial_x^2 u, J^{r+1/2}u)| \leqslant C\|u\|^2_{H^r(\mathbb{R})}, \tag{A.3}$$

$$|(J^{r+1/2}u, [J^{r-(3/2)}, \chi^2]J^2 u)| \leqslant C\|u\|^2_{H^r(\mathbb{R})}. \tag{A.4}$$

Proof. By Cauchy–Schwarz inequality and lemma A.1, we have

$$|(J^{2r-1}\varphi u, u)| = |(J^{r-1}\varphi u, J^r u)| \leqslant \|J^{r-1}\varphi u\|_{L^2(\mathbb{R})}\|J^r u\| \leqslant C\|u\|^2_{H^r(\mathbb{R})}.$$

This proves (A.1). Since

$$\begin{aligned}|(J^{r-(3/2)}\chi\partial_x^2 u, [\chi, J^{r+1/2}]u)| &= |(J^{r-2}\chi\partial_x^2 u, J^{1/2}[\chi, J^{r+1/2}]u)| \\ &\leqslant \|\chi\partial_x^2 u\|_{H^{r-2}(\mathbb{R})}\|[\chi, J^{r+1/2}]u\|_{H^{1/2}(\mathbb{R})},\end{aligned}$$

we also have $\|\chi\partial_x^2 u\|_{H^{r-2}(\mathbb{R})} \leqslant C\|u\|_{H^r(\mathbb{R})}$ by lemma A.1, and $\|[\chi, J^{r+1/2}]u\|_{H^{1/2}(\mathbb{R})} \leqslant C\|u\|_{H^r(\mathbb{R})}$ by lemma A.2. Thus (A.2) holds.

Similarly, one can show that (A.3) and (A.4) hold. □

Finally, we provide a multiplication property of Bourgain space $X^{s,b}$.

LEMMA A.5. *Let $-1 \leqslant b \leqslant 1$, $s \in \mathbb{R}$ and $\varphi \in C_b^\infty$. Then for all $u \in X^{s,b}$*

$$\|\varphi(x)u\|_{X^{s-2|b|,b}} \lesssim \|u\|_{X^{s,b}}. \tag{A.5}$$

Similarly, for every $T > 0$, we have $\|\varphi(x)u\|_{X^{s-2|b|,b}_{(0,T)}} \lesssim \|u\|_{X^{s,b}_{(0,T)}}$.

Proof. The proof is the same as that in [**14**, lemma 3.4], we only give a sketch here. By duality and interpolation arguments, it suffices to consider the cases $b = 0$ and $b = 1$.

In the case $b = 0$, (A.5) follows from lemma A.2 clearly.

In the case $b = 1$, we first observe that

$$\begin{aligned}\|\varphi(x)u\|_{X^{s-2,1}} &\lesssim \|\varphi u\|_{X^{s-2,0}} + \|(\partial_t + \partial_x^3)(\varphi u)\|_{X^{s-2,0}} \\ &\leqslant \Upsilon + \|\varphi(\partial_t + \partial_x^3)u\|_{X^{s-2,0}},\end{aligned} \tag{A.6}$$

where $\Upsilon = \|\varphi u\|_{X^{s-2,0}} + \|3\partial_x\varphi\partial_x^2 u + 3\partial_x^2\varphi\partial_x u + \partial_x^3\varphi u\|_{X^{s-2,0}}$. Using lemma A.2 again, we deduce from (A.6) that

$$\|\varphi u\|_{X^{s-2,1}} \lesssim \|u\|_{X^{s,0}} + \|(\partial_t + \partial_x^3)u\|_{X^{s-2,0}} \lesssim \|u\|_{X^{s,1}}.$$

Thus (A.5) also holds. □

REMARK A.6. If $\phi \in C_c^\infty(\mathbb{R})$, then $\phi(t)$ maps $X^{s,b}$ into $X^{s,b}$, see [**28**, lemma 2.11, p. 101]. In other words, the Bourgain space is stable with the multiplication by a compact supported smooth function of time t. However, as lemma A.5 indicates, some regularity index is lost with the multiplication by a smooth function of spatial variable x. The loss is unavoidable, see the example in [**14**, lemma 3.4]. Anyway, the lemma is sufficient for our purpose in this paper.

References

1 M. Berger and S. Martin. Embedding theorems and quasi-linear elliptic boundary value problems for unbounded domains. *Trans. Am. Math. Soc.* **172** (1972), 261–278.

2 N. Burq and R. Joly. Exponential decay for the damped wave equation in unbounded domains. *Commun. Contemp. Math.* **18** (2016), 1650012.

https://doi.org/10.1017/prm.2022.17 Published online by Cambridge University Press

3 R. A. Capistrano-Filho. Weak damping for the Korteweg–de Vries equation. *Electron. J. Qual. Theory Differ. Equ.* **43** (2021), 25.

4 M. M. Cavalcanti, V. N. D. Cavalcanti, A. Faminskii and F. Natali. Decay of solutions to damped Korteweg–de Vries type equation. *Appl. Math. Optim.* **65** (2012), 221–251.

5 M. M. Cavalcanti, V. N. D. Cavalcanti, V. Komornik and J. H. Rodrigues. Global well-posedness and exponential decay rates for a KdV–Burgers equation with indefinite damping. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31** (2014), 1079–1100.

6 Y. L. Duan, L. J. Wang and C. Zhang. Observability inequalities for the heat equation with bounded potentials on the whole space. *SIAM J. Control Optim.* **58** (2020), 1939–1960.

7 F. A. Gallego and A. F. Pazoto. On the well-posedness and asymptotic behaviour of the generalized Korteweg–de Vries–Burgers equation. *Proc. Roy. Soc. Edinburgh Sect. A* **149** (2019), 219–260.

8 W. Green. On the energy decay rate of the fractional wave equation on $\mathbb{R}$ with relatively dense damping. *Proc. Am. Math. Soc.* **148** (2020), 4745–4753.

9 V. Komornik and C. Pignotti. Well-posedness and exponential decay estimates for a Korteweg–de Vries–Burgers equation with time-delay. *Nonlinear Anal.* **191** (2020), 111646.

10 J. Le Rousseau and I. Moyano. Null controllability of the Kolmogorov equation in the whole phase space. *J. Differ. Equ.* **260** (2016), 3193–3233.

11 F. L. Huang. Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. *Ann. Differ. Equ.* **1** (1985), 43–56.

12 C. E. Kenig, G. Ponce and L. Vega. Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.* **40** (1991), 33–69.

13 C. E. Kenig, G. Ponce and L. Vega. The Cauchy problem for the Korteweg–de Vries equation in Sobolev spaces of negative indices. *Duke Math. J.* **71** (1993), 1–21.

14 C. Laurent, L. Rosier and B.-Y. Zhang. Control and stabilization of the Korteweg–de Vries equation on a periodic domain. *Commun. Partial Differ. Equ.* **35** (2010), 707–744.

15 F. Linares and A. F. Pazoto. On the exponential decay of the critical generalized Korteweg–de Vries equation with localized damping. *Proc. Am. Math. Soc.* **135** (2007), 1515–1522.

16 F. Linares and A. F. Pazoto. Asymptotic behavior of the Korteweg–de Vries equation posed in a quarter plane. *J. Differ. Equ.* **246** (2009), 1342–1353.

17 G. P. Menzala, C. F. Vasconcellos and E. Zuazua. Stabilization of the Korteweg–de Vries equation with localized damping. *Q. Appl. Math.* **60** (2002), 111–129.

18 M. Panthee and F. Vielma Leal. On the controllability and stabilization of the linearized Benjamin equation on a periodic domain. *Nonlinear Anal. Real World Appl.* **51** (2020), 102978.

19 M. Panthee and F. Vielma Leal. On the controllability and stabilization of the Benjamin equation on a periodic domain. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **38** (2021), 1605–1652.

20 A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, 44, pp. 13–15 (Springer-Verlag, 1983).

21 A. F. Pazoto. Unique continuation and decay for the Korteweg–de Vries equation with localized damping. *ESAIM Control Optim. Calc. Var.* **11** (2005), 473–486.

22 A. F. Pazoto and L. Rosier. Uniform stabilization in weighted Sobolev spaces for the KdV equation posed on the half-line. *Discrete Contin. Dyn. Syst. B* **14** (2010), 1511–1535.

23 L. Rosier. Exact boundary controllability for the Korteweg–de Vries equation on a bounded domain. *ESAIM Control Optim. Calc. Var.* **2** (1997), 33–55.

24 L. Rosier and B. Y. Zhang. Control and stabilization of the Korteweg–de Vries equation: recent progresses. *J. Syst. Sci. Complex.* **22** (2009), 647–682.

25 J.-C. Saut and B. Scheurer. Unique continuation for some evolution equations. *J. Differ. Equ.* **66** (1987), 118–139.

26 E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory* (Princeton University Press, Princeton, New Jersey, 1993).

27 M. E. Taylor, Partial differential equations II, Qualitative studies of linear equations. 2nd ed., Applied Mathematical Sciences, Vol. 116 (Springer-Verlag, New York, 2011).

28 T. Tao, Nonlinear dispersive equations: local and global analysis, volume 106 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006.

29 G. Wang, M. Wang, C. Zhang and Y. Zhang. Observable set, observability, interpolation inequality and spectral inequality for the heat equation in $\mathbb{R}^n$. *J. Math. Pure Appl.* **126** (2019), 144–194.

30 M. Wang and J. Huang. The global attractor for the weakly damped KdV equation on $\mathbb{R}$ has a finite fractal dimension. *Math. Methods Appl. Sci.* **43** (2020), 4567–4584.

31 M. Wang and D. Zhou. Exponential decay for the linear KdV with a rough localized damping. *Appl. Math. Lett.* **120** (2021), 107264.

32 Y. Wu. The Cauchy problem of the Schrödinger–Korteweg–de Vries system. *Differ. Int. Equ.* **23** (2010), 569–600.

33 B. Y. Zhang. Unique continuation for the Korteweg–de Vries equation. *SIAM J. Math. Anal.* **32** (1992), 55–71.