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A NOTE ON RADICAL EXTENSIONS OF RINGS

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All rings are associative. A ring T is said to be radical over a subring R if for every $t \in T$, there exists a natural number $n(t)$ such that $t^{n(t)} \in R$.

In [1] Faith showed that if T is radical over R and T is primitive, then R is primitive. We might then ask if the same is true if prime is substituted for primitive. This is not in general true if T does not have a unity element or if char $T \neq 0$. However, we do have

THEOREM 1. *Suppose T is radical over R, T and R have a unity element*, char $T=0$, *and T is prime. Then R is prime.*

The above theorem follows easily from the following

THEOREM 2. *Suppose that the ring T is radical over a subring R, R and T have a common unity element, and T is torsion-free as a Z-module. Then $T_{Z^*}=R_{Z^*}$, where R_{Z^*} is the localization of R at the nonzero integers.*

In proving theorem 2, we use the following

THEOREM 3. (*Kaplansky* [2]) *Suppose that a field K is radical over a proper subfield F. Then K has prime characteristic, and is either purely inseparable over F, or algebraic over its prime subfield.*

Proof of theorem 2. We prove the theorem by assuming that T and R are Q-algebras, and showing that $R=T$.

We first show that every nilpotent element of T lies in R. Suppose $x \in T \backslash R$ is nilpotent. From the sequence $x, x^2, x^3, \ldots$, choose k maximal such that $x^k \notin R$. Since T is radical over R, $\exists\, n$ such that $(1+x^k)^n \in R$. Then $1+nx^k+\cdots+x^{kn} \in R$, from which we deduce that $x^k \in R$, a contradiction.

Now suppose that T is a commutative Artinian Q-algebra and that R is also Artinian. Since the Jacobson radical of an Artinian ring is nilpotent, we have $J(T)=J(R)=J$. T/J is a finite direct product of fields, and is radical over R/J. By Kaplansky's theorem $T/J=R/J$, hence $T=R$.

Now let T be arbitrary. Suppose $x \in T$. Then $Q[x]=A$ is radical over $Q[x] \cap R=B$. If A is finite dimensional, then $A=B$, by the above result. If x is transcendental over Q, localize A at B^*, the nonzero elements of B. Since A is radical over B, A_{B^*} is a field, radical over the field B_{B^*}. Hence, once again, $A_{B^*}=B_{B^*}$. Take $r \in B$, $r\neq 0$, such that $rx \in B$, and let $s=r^n$, where $x^n \in B$. We now have $sA \subset B$. Let $I=(s)$, and note A/I is radical over B/I. However, A/I is Artinian, and so the problem is reduced to the previous case, thus $A/I=B/I$. Since $I\subset B$, $A=B$, and therefore $R=T$. This completes the proof.

Although we have $T_{Z^*}=R_{Z^*}$, we do not necessarily have $R=T$. Let $T=Z[x]$ and let R be the subring generated by $\{1, 2x, x^2, x^3, \ldots\}$. Then $t^2 \in R$ for every $t \in T$, but $R\neq T$.

We conclude this paper with an example of a prime ring T, without unity, radical over a subring R which is not prime, where char. $T=0$. Let F be the free Z-algebra on countably many noncommuting variables, $x_1, x_2, \ldots$. We assume that Z is not embedded in F. Since F is countable, we can order the elements, $f_1, f_2, \ldots$. Let S be the set of monomials occurring as terms in the set $\{f_k^k : k=1, 2, \ldots\}$, and let S' be the multiplicative closure of S. Let E be the subring of F generated by S, and let I be the two-sided ideal generated by $\{x_1 s x_1 : s \in S'\}$. Set $T=F/I$ and $R=E/E \cap I$. That T is radical over R, and that R is not semiprime follows easily from our construction. Let m_1 and m_2 be nonzero monomials in T, and let h be an integer such that x_h does not occur in any generator g of I with (degree g)$\leqq$(degree m_1+degree m_2+1). Then $m_1 x_h m_2 \notin I$, hence $m_1 x_h m_2 \neq 0$. It quickly follows that T is prime.

References

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