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RETRACTIVE TRANSFERS AND p-LOCAL FINITE GROUPS

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Abstract In this paper we explore the possibility of defining p-local finite groups in terms of transfer properties of their classifying spaces. More precisely, we consider the question, posed by Haynes Miller, of whether an equivalent theory can be obtained by studying triples (f, t, X), where X is a p-complete, nilpotent space with a finite fundamental group, $f : BS \to X$ is a map from the classifying space of a finite p-group, and t is a stable retraction of f satisfying Frobenius reciprocity at the level of stable homotopy. We refer to t as a *retractive transfer* of f and to (f, t, X) as a *retractive transfer triple over* S.

In the case where S is elementary abelian, we answer this question in the affirmative by showing that a retractive transfer triple (f, t, X) over S does indeed induce a p-local finite group over S with X as its classifying space.

Using previous results obtained by the author, we show that the converse is true for general finite p-groups. That is, for a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, the natural inclusion $\theta : BS \to X$ has a retractive transfer t, making $(\theta, t, |\mathcal{L}|^\wedge_p)$ a retractive transfer triple over S. This also requires a proof, obtained jointly with Ran Levi, that $|\mathcal{L}|^\wedge_p$ is a nilpotent space, which is of independent interest.

Keywords: p-local finite groups; fusion systems; classifying spaces; transfer

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1. Introduction

Defined in [**6**], p-local finite groups are the culmination of a programme initiated by Puig [**23**] to find a formal framework for the p-local structure of a finite group. With a finite group G, one associates a fusion system (at a prime p) consisting of all p-subgroups of G and the homomorphims between them induced by conjugation in G. Puig formalized fusion systems and identified an important subclass of fusion systems, which we now call saturated fusion systems. Fusion systems of finite groups are contained in this class, but saturated fusion systems also arise in other important contexts, most notably in modular representation theory through Brauer subpairs of blocks of group algebras, and more recently as Chevalley groups of p-compact groups [**4**].

The fusion system of a group G can be considered as an algebraic interpretation of the p-local structure of the group. One can also take a topological approach, and think of the p-local structure of G as being the p-completed classifying space $BG^\wedge_p$. By Oliver's solution [**21**, **22**] of the Martino–Priddy conjecture [**18**], these approaches are the same. That is, two groups induce the same fusion system if and only if their p-completed classifying spaces are homotopy equivalent. In fact, the fusion system can be recovered from the classifying space via a homotopy theoretic construction, which is presented in § 2.3. Therefore, one can in some sense regard $BG^{\wedge}_p$ as a classifying space of the fusion system. This suggests that, more generally, each saturated fusion system may have a unique classifying space.

A p-local finite group consists of a saturated fusion system and an associated centric linking system, a category which contains just enough information to construct a classifying space associated with the fusion system. Thus, one can think of a p-local finite group as a saturated fusion system with a chosen classifying space.

The definition of p-local finite groups is rather complicated and has the drawback that there is no straightforward concept of morphisms between p-local finite groups, so they have not yet been made to form a category in any sensible way. In this paper, we adopt the approach used by Dwyer and Wilkerson for p-compact groups [**9**], and try to develop the theory of p-local finite groups in terms of classifying spaces.

Specifically, we consider a homotopy monomorphism $f : BS \to X$ from the classifying space of a finite p-group S to a p-complete, nilpotent space X with a finite fundamental group, which is endowed with a stable retraction t satisfying Frobenius reciprocity at the level of stable homotopy. We refer to such a triple (f, t, X) as a *retractive transfer triple over* S. For a retractive transfer triple (f, t, X), we ask whether X is the classifying space of a p-local finite group. This question is addressed in §§ 4 and 5, where we answer the question in the affirmative in the case where S is elementary abelian (Theorem 3.4). Conversely, we ask whether a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ gives rise to a Frobenius transfer triple. This is indeed the case for any finite p-group S, as we show in § 6. This involves joint work with Ran Levi, in which we show that the classifying space of a p-local finite group is a torsion space and (since its fundamental group is a finite p-group) consequently a nilpotent space.

The homotopy theory of classifying spaces of elementary abelian p-groups was studied intensively in the 1990s and is now well understood through contributions by various authors, the most important being Miller's solution of the Sullivan conjecture [**19**] and Lannes's T-functor technology [**16**]. Other contributions that are related to the results and methods in this paper are the work of Goerss *et al*. [**12**], Harris and Kuhn [**13**], Henn *et al*. [**14**, **15**] and Dwyer and Wilkerson [**8**]. The work in § 5 of this paper mimics the methods used by Dwyer *et al*. in [**10**], replacing inclusions of maximal tori in compact Lie groups with the inclusion of elementary abelian p-groups in finite groups.

2. A quick review of p-local finite groups

In this section we give a brief overview of the theory of p-local finite groups. Most of this material is found in [**6**]. In this section and throughout the paper, p is a fixed prime.

2.1. Some definitions and terminology

We begin by recalling some terminology regarding p-local finite groups.

Definition 2.1. A *fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphism sets $\hom_{\mathcal{F}}(P, Q)$ satisfy the following conditions:

(i) $\hom_S(P, Q) \subseteq \hom_{\mathcal{F}}(P, Q) \subseteq \operatorname{inj}(P, Q)$ for all $P, Q \leqslant S$;

(ii) every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

Here $\hom_S(P, Q)$ is the set of group homomorphims induced by conjugation by elements in S.

Before stating the next definition, we need to introduce some additional terminology and notation. We say that two subgroups $P, P' \leqslant S$ are *$\mathcal{F}$-conjugate* if they are isomorphic in $\mathcal{F}$. A subgroup $P \leqslant S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geqslant |C_S(P')|$ for every $P' \leqslant S$ which is $\mathcal{F}$-conjugate to P. Similarly, P is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geqslant |N_S(P')|$ for every $P' \leqslant S$ which is $\mathcal{F}$-conjugate to P.

Definition 2.2. A fusion system $\mathcal{F}$ over a p-group S is *saturated* if the following two conditions hold:

(i) if $P \leqslant S$ is fully normalized in $\mathcal{F}$, then P is also fully centralized, and $\operatorname{aut}_S(P)$ is a Sylow subgroup of $\operatorname{aut}_{\mathcal{F}}(P)$;

(ii) if $P \leqslant S$ and $\varphi \in \hom_{\mathcal{F}}(P, S)$ are such that $\varphi(P)$ is fully centralized, then φ extends to $\bar{\varphi} \in \hom_{\mathcal{F}}(N_\varphi, S)$, where

$$N_\varphi = \{g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in \operatorname{aut}_S(\varphi(P))\}.$$

There is a class of subgroups of S of special interest to us, defined as follows.

Definition 2.3. Let $\mathcal{F}$ be a fusion system over a p-group S. A subgroup $P \leqslant S$ is $\mathcal{F}$-centric if $C_S(P') \leqslant P'$ for every P' that is $\mathcal{F}$-conjugate to P. Let $\mathcal{F}^{\mathrm{c}}$ denote the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of S.

Remark 2.4. The condition $C_S(P') \leqslant P'$ in the previous definition is equivalent to the condition $C_S(P') = Z(P')$.

Definition 2.5. Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system associated with* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi : \mathcal{L} \to \mathcal{F}^{\mathrm{c}},$$

and distinguished monomorphisms $P \xrightarrow{\delta_P} \operatorname{aut}_{\mathcal{L}}(P)$, for each $\mathcal{F}$-centric subgroup $P \leqslant S$, which satisfy the following conditions.

(i) The functor π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, the centre $Z(P)$ acts freely on $\operatorname{mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leqslant \operatorname{aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\operatorname{mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \hom_{\mathcal{F}}(P, Q).$$

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(ii) For each $\mathcal{F}$-centric subgroup $P \leqslant S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{aut}_{\mathcal{F}}(P)$.

(iii) For each $f \in \mathrm{mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\delta_P(g)"] & Q \arrow[d, "\delta_Q(\pi(f)(g))"] \\ P \arrow[r, "f"] & Q \end{tikzcd}$$

We can now finally define our objects of study.

Definition 2.6. A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over a finite p-group S and $\mathcal{L}$ is a centric linking system associated with $\mathcal{F}$. The *classifying space* of the p-local finite group is the p-completed geometric realization $|\mathcal{L}|_p^\wedge$.

Note that a p-local finite group comes equipped with a natural inclusion

$$\theta : BS \to |\mathcal{L}|_p^\wedge.$$

One of the main questions in the theory of p-local finite groups concerns the existence and uniqueness of a centric linking system associated with a given saturated fusion system. In [**6**, § 3], Broto *et al*. have developed an obstruction theory to address this question.

2.2. The fusion system of a group

In this section we will discuss the fusion system arising from a Sylow subgroup inclusion $S \leqslant G$. This section serves as motivation for the discussion in the previous section as well as being of independent interest.

Definition 2.7. Let G be a finite group. The *fusion system of* G is the category $\mathcal{F}(G)$, whose objects are the p-subgroups of G and whose morphism sets are given by

$$\hom_{\mathcal{F}(G)}(P, Q) = \hom_G(P, Q)$$

for all p-subgroups $P, Q \leqslant G$.

For a p-subgroup $S \leqslant G$, the *fusion system of* G *over* S is the full subcategory $\mathcal{F}_S(G) \subseteq \mathcal{F}(G)$, whose objects are the subgroups of S.

If S is a Sylow subgroup of G, then the inclusion of $\mathcal{F}_S(G)$ in $\mathcal{F}(G)$ is an equivalence of categories, since every p-subgroup of G is conjugate to a subgroup of S.

Proposition 2.8 (Broto *et al*. [6, Proposition 1.3]). *Let G be a finite group and let S be a p-subgroup. Then the fusion system $\mathcal{F}_S(G)$ of G over S is saturated if and only if S is a Sylow subgroup.*

The centric linking system of a finite group was initially introduced in [**5**] as a powerful tool with which to study homotopy equivalences between p-completed classifying spaces of finite groups. The p-centric subgroups of a finite group G are the p-subgroups $P \leqslant G$ whose centre $Z(P)$ is a p-Sylow subgroup of the centraliser $C_G(P)$. This notion of centricity is equivalent to that introduced in Definition 2.3 in the sense that if $S \leqslant G$ is a Sylow subgroup, then a subgroup $P \leqslant S$ is p-centric if and only if it is $\mathcal{F}_S(G)$-centric.

For the following definition, we recall that if a group $P \leqslant G$ is p-centric, then one can write

$$C_G(P) = Z(P) \times C'_G(P),$$

where $C'_G(P) \leqslant G$ has order prime to p. The notation $C'_G(P)$ will be used in the definition. In addition, for subgroups $P, Q \leqslant G$, $N_G(P, Q)$ denotes the transporter:

$$N_G(P, Q) = \{g \in G \mid gPg^{-1} \leqslant Q\}.$$

Definition 2.9. Let G be a finite group. The *centric linking system of* G is the category $\mathcal{L}(G)$, whose objects are the p-centric subgroups of G and whose morphism sets are given by

$$\mathrm{mor}_{\mathcal{L}(G)}(P, Q) = N_G(P, Q)/C'_G(P)$$

for all p-subgroups $P, Q \leqslant G$.

For a p-subgroup $S \leqslant G$, the *centric linking system of* G *over* S is the full subcategory $\mathcal{L}_S(G) \subseteq \mathcal{L}(G)$ whose objects are the subgroups of S that are p-centric in G.

In the case of a Sylow inclusion $S \leqslant G$, the centric linking system $\mathcal{L}_S(G)$ is a centric linking system associated with the saturated fusion system $\mathcal{F}_S(G)$, and we have the following proposition, which serves as a motivating example for the definition of a p-local finite group.

Proposition 2.10. *Let S be a Sylow subgroup of a finite group G. Then the triple $(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$ is a p-local finite group over S. Furthermore, the natural map*

$$\theta : BS \to |\mathcal{L}_S(G)|^\wedge_p$$

is homotopy equivalent to the p-completed inclusion

$$BS \to BG^\wedge_p$$

as a space under BS.

2.3. Homotopy theoretic constructions of fusion systems

In this section we recall how a map $f : BS \to X$, from the classifying space of a finite p-group S to a space X, induces a fusion system $\mathcal{F}_{S,f}(X)$ over S. In general this fusion system is not saturated.

The following definition is motivated by the fact that two group homomorphisms $\varphi, \psi : G \to H$ between finite groups are H-conjugate if and only if the induced maps of classifying spaces are freely homotopic.

Definition 2.11. For any space X, any p-group S, and any map $f : BS \to X$, define $\mathcal{F}_{S,f}(X)$ to be the category whose objects are the subgroups of S, and whose morphisms are given by

$$\hom_{\mathcal{F}_{S,f}}(P,Q) = \{\varphi \in \operatorname{inj}(P,Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi\}$$

for each $P, Q \leqslant S$.

It is easy to see that $\mathcal{F}_{S,f}$ is indeed a fusion system, although it need not be saturated. In the case where $\mathcal{F}_{S,f}$ is saturated, however, one can obtain a candidate $\mathcal{L}^{\mathrm{c}}_{S,\theta}(|\mathcal{L}|^{\wedge}_p)$ for an associated centric linking system by retaining information about the homotopies giving the equivalence $f|_{BP} \simeq f|_{BQ} \circ B\varphi$ in the definition above (see [**6**] for details).

Theorem 2.12 (Broto *et al.* [6, Theorems 7.4 and 7.5]). *For a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, the fusion system $\mathcal{F}_{S,\theta}(|\mathcal{L}|^{\wedge}_p)$ is saturated and $\mathcal{L}_{S,\theta}(|\mathcal{L}|^{\wedge}_p)$ is a centric linking system associated with $\mathcal{F}_{S,\theta}(|\mathcal{L}|^{\wedge}_p)$. Furthermore, the p-local finite groups $(S, \mathcal{F}, \mathcal{L})$ and $(S, \mathcal{F}_{S,\theta}(|\mathcal{L}|^{\wedge}_p), \mathcal{L}^{\mathrm{c}}_{S,\theta}(|\mathcal{L}|^{\wedge}_p))$ are isomorphic.*

2.4. *p*-local finite groups over abelian groups

We conclude this review by classifying the p-local finite groups over an abelian p-group S. The resulting classification shows that the strict equivalence classes of p-local finite groups over S are in a bijective correspondence with the subgroups $W \leqslant \operatorname{aut}(S)$ of order prime to p, under the assignment

$$W \mapsto (S, \mathcal{F}_S(W \ltimes S), \mathcal{L}_S(W \ltimes S)),$$

where $W \ltimes S$ is the semi-direct product. Here we say that two p-local finite groups are strictly equivalent if they have the same fusion system and their linking systems are isomorphic. In particular, there are no exotic p-local finite groups over abelian p-groups.

We begin with the following lemma, which describes precisely how the conditions in Definition 2.2 are simplified under the assumption that S is abelian. As this result is obvious to the experienced reader and proving it straight from the definition is an excellent exercise for those new to p-local finite groups, the proof is left to the reader.

Lemma 2.13. *Let $\mathcal{F}$ be a fusion system over a finite abelian p-group S. Then $\mathcal{F}$ is saturated if and only if the following two conditions are satisfied:*

(i) $\operatorname{aut}_{\mathcal{F}}(S)$ *has order prime to p;*

(ii) *every $\varphi \in \hom_{\mathcal{F}}(P,Q)$ is the restriction of some $\tilde{\varphi} \in \operatorname{aut}_{\mathcal{F}}(S)$.*

The following proposition follows easily.

Proposition 2.14. *If S is an abelian finite p-group, then the assignment $W \mapsto \mathcal{F}_S(W \ltimes S)$ gives a bijective correspondence between subgroups $W \leqslant \operatorname{aut}(S)$ of order prime to p and saturated fusion systems over S.*

Being the fusion system of a group, the fusion system $\mathcal{F}_S(W \ltimes S)$ has an obvious associated centric linking system $\mathcal{L}^{\mathrm{c}}_S(W \ltimes S)$. This is in fact the only associated centric linking system and we have the following classification result.

Proposition 2.15. *If S is an abelian finite p-group, then the assignment*

$$W \mapsto (S, \mathcal{F}(W \ltimes S), \mathcal{L}_S^c(W \ltimes S))$$

gives a bijective correspondence between subgroups $W \leqslant Aut(S)$ of order prime to p and strict equivalence classes of p-local finite groups over S. In particular, there are no exotic p-local finite groups over S.

Proof. When S is abelian, there are no proper centric subgroups. Therefore, the obstruction to uniqueness of centric linking systems [**6**, § 3] simplifies to the cohomology group $H^2(W; S)$. Now use a transfer argument to show that $H^*(W; S)$ vanishes for $* > 0$. □

3. Retractive transfer triples

In this section we introduce retractive transfer triples. First we make precise the setting we are working in. Cohomology will always be taken to be with $\mathbb{F}_p$-coefficients unless otherwise specified. The following definition is a homotopy generalization of group monomorphisms.

Definition 3.1. A map $f : Y \to X$ between two topological spaces is a *homotopy monomorphism at p* if its induced map in cohomology makes $H^*(Y)$ a finitely generated $H^*(X)$-module. In the special case where $Y = BP$ is the classifying space of a finite p-group, we say that f is a *p-subgroup inclusion*.

The analogy with group monomorphisms is that a group homomorphism $\varphi : P \to G$ from a finite p-group P to a finite group G is a monomorphism if and only $B\varphi$ is a homotopy monomorphism at p. There are other definitions of homotopy monomorphisms in the literature, but these are equivalent in the setting in which we are working. Since the finite generation hypothesis is the only one we need, we avoid complication by considering only this definition. As the prime p is fixed throughout, we will refer to these concepts simply as 'homotopy monomorphism' and 'subgroup inclusion'.

We will demand some additional structure on our subgroup inclusions, namely that they allow a transfer with properties similar to that of the transfer of a Sylow subgroup inclusion.

Definition 3.2. Let $f : Y \to X$ be a map of spaces. A *retractive transfer* of f is a stable map $t : \Sigma^\infty_+ X \to \Sigma^\infty_+ Y$ such that $\Sigma^\infty_+ f \circ t \simeq \mathrm{id}_{\Sigma^\infty_+ X}$, and the following diagram commutes up to homotopy:

$$\begin{tikzcd} \Sigma^\infty_+ X \arrow[r, "\Delta"] \arrow[d, "t"] & \Sigma^\infty_+ X \wedge \Sigma^\infty_+ X \arrow[d, "1\wedge t"] \\ \Sigma^\infty_+ Y \arrow[r, "(f\wedge 1)\circ\Delta"] & \Sigma^\infty_+ X \wedge \Sigma^\infty_+ Y \end{tikzcd} \tag{3.1}$$

The objects that will be the focus of our attention are defined as follows.

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Definition 3.3. A *retractive transfer triple* over a finite p-group S is a triple (f,t,X), where X is a connected, p-complete, nilpotent space with finite fundamental group, f is a subgroup inclusion $BS \to X$ and t is a retractive transfer of f.

Since the space X in the above definition is p-complete with finite fundamental group, it follows that $\pi_1(X)$ is a finite p-group [**3**, **9**].

For a retractive transfer triple (f,t,X) over a finite p-group S, we ask the following questions.

- Is the fusion system $\mathcal{F}_{S,f}(X)$ saturated?
- If so, does there exist an associated centric linking system $\mathcal{L}$? Is it unique?
- If an associated centric linking system exists, then what is the relation between the classifying space $|\mathcal{L}|^\wedge_p$ and X? Are they equivalent as objects under BS?

In the course of the following sections we will answer these questions affirmatively in the case when S is an elementary abelian p-group, proving the following theorem.

Theorem 3.4. *Let (f,t,X) be a retractive transfer triple over be a finite elementary abelian p-group V, and set*

$$W := \operatorname{aut}_{\mathcal{F}_{V,f}(X)}(V) = \{\varphi \in \operatorname{aut}(V) \mid f \circ \varphi \simeq f\}.$$

The following then hold.

(i) *W has order prime to p.*

(ii) *$\mathcal{F}_{V,f}(X)$ is equal to the saturated fusion system $\mathcal{F}_V(W \ltimes V)$.*

(iii) *$\mathcal{F}_{V,f}(X)$ has an associated centric linking system, which is unique up to isomorphism, with classifying space $B(W \ltimes V)^\wedge_p$.*

(iv) *There is a homotopy equivalence $h : B(W \ltimes V)^\wedge_p \xrightarrow{\simeq} X$ making the following diagram commute up to pointed homotopy:*

$$\begin{tikzcd} & B(W \ltimes V)^\wedge_p \arrow[dd, "h", "\simeq"'] \\ BV \arrow[ur, "Bi"] \arrow[dr, "f"'] & \\ & X \end{tikzcd}$$

Thus, the triple (f,t,X) induces a p-local finite group $(V, \mathcal{F}_{V,f}(X), \mathcal{L}^{\mathrm{c}}_{V,f}(X^\wedge_p))$ over V with classifying space X.

Proof. The proof is by forward referencing. Lannes's theorem [**16**] shows that W is the group of automorphisms of V that act trivially on $H^*(X)$ when regarded as a subring of $H^*(BV)$ under f^*. By Proposition 4.11, W has order prime to p, proving (i). By Corollary 4.12, X has the cohomology type of $B(W \ltimes V)$ as objects under BV. By Proposition 5.1 there is a map $B(W \ltimes V) \to X$ realizing that cohomology isomorphism and making the uncompleted version of the diagram in (iv) commute up to pointed homotopy. This map becomes a homotopy equivalence upon p-completion, proving (iv). Part (ii) follows directly from (i) and (iv), and (iii) then follows from Proposition 2.15. $\square$

4. Cohomology type of retractive transfer triples

In this section we discuss the cohomological structure of a retractive transfer triple (f, t, X) over a finite p-group S. We first discuss general properties in § 4.1. We then specialize to the case where S is elementary abelian in § 4.2 and show that in this case $H^*(X)$ is a ring of invariants of $H^*(BS)$ under the action of a group of order prime to p.

4.1. The general case

Applying the cohomology functor $H^*(\,\cdot\,;\mathbb{F}_p)$ to (3.1), we get maps

$$H^*(X) \xrightarrow{f^*} H^*(BS) \xrightarrow{t^*} H^*(X)$$

with the following properties:

(C1) $t^* \circ f^* = \mathrm{id}$;

(C2) t^* is $H^*(X)$-linear (Frobenius reciprocity);

(C3) t^* is a morphism of unstable modules over the Steenrod algebra;

(C4) f^* is a morphism of unstable algebras over the Steenrod algebra.

Hence, $H^*(X)$ is a direct summand of $H^*(BS)$ as a $H^*(X)$-module and as a module over the Steenrod algebra. (C1) allows us to regard $H^*(X)$ as a subring of $H^*(BS)$ and we will often do so without further comment.

These properties are quite restrictive and the question of which unstable subalgebras $R^* \subset H^*(BS)$ over the Steenrod algebra admit a splitting $H^*(BS) \to R^*$ as R^*-modules and unstable modules over the Steenrod algebra is interesting in itself. However, we focus our attention on p-local finite groups.

The following finiteness properties of retractive transfer triples will be needed later.

Lemma 4.1. *Let S be a finite p-group and let (f, t, X) be a Frobenius transfer triple over S. Then $H^*(X)$ is Noetherian and in particular X is of finite $\mathbb{F}_p$-type.*

Proof. By [**9**, Lemma 2.6], this follows from (C1), (C2) and the classical result that $H^*(BS)$ is Noetherian [**11**, **26**]. $\square$

Lemma 4.2. *Let S be a finite p-group and let (f,t,X) be a Frobenius transfer triple over S. Then X is of $\mathbb{Z}_{(p)}$-finite type.*

Proof. By the universal coefficient theorem, it suffices to show that X is of finite $\mathbb{F}_p$-type and of finite $\mathbb{Q}$-type. The former is Lemma 4.1, above. The latter is deduced in a similar way: by a transfer argument, BS has trivial $\mathbb{Q}$-cohomology. As in the $\mathbb{F}_p$-coefficient case, $H^*(X;\mathbb{Q})$ is a direct summand of $H^*(BS;\mathbb{Q})$. Hence, X also has trivial $\mathbb{Q}$-cohomology and we are done. □

4.2. The elementary abelian case

In this subsection, we restrict ourselves to the case where S is an elementary abelian finite p-group V. In this case, we use a theorem of Goerss *et al*. [**12**], based on the celebrated work of Adams and Wilkerson [**2**], to prove that if (f,t,X) is a retractive transfer triple over V, then $H^*(X)$ is a ring of invariants $H^*(BV)^W$ for a subgroup $W \leqslant \operatorname{aut}(V)$ of order prime to p. Furthermore, we show that the group W may be taken to be the group of automorphisms of V that act trivially on $H^*(X)$. We consider only the case of an odd prime. The results still hold true at the prime 2 and the proofs proceed in more or less the same way, but are simpler at times. As pointed out to the author by Nick Kuhn, these results can also be obtained, possibly more directly, as a consequence of [**15**].

In [**2**], Adams and Wilkerson study the following category.

Definition 4.3. Let $\mathcal{AW}$ be the category of evenly graded unstable algebras R over the Steenrod algebra that are integral domains.

Adams and Wilkerson also make precise the notions of 'algebraic extension' and 'algebraic closure' in this setting and prove the following.

Proposition 4.4 (Adams and Wilkerson [2, Proposition 1.5]). *Every object R in $\mathcal{AW}$ has an algebraic closure H in $\mathcal{AW}$. If R has finite transcendence degree, then so does H.*

Theorem 4.5 (Adams and Wilkerson [2, Theorem 1.6]). *The objects H in $\mathcal{AW}$ that are algebraically closed and of finite transcendence degree are precisely the polynomial algebras $\mathbb{F}_p[x_1,\dots,x_n]$ on generators x_i of degree 2.*

Furthermore, in [**2**, Theorem 1.2] they show that an algebra in R of finite transcendence degree is a ring of invariants in its algebraic closure if and only if it satisfies two conditions, which can be interpreted as an integral closure condition and an inseparable closure condition. Based on this work, Goerss *et al*. identified sufficient conditions for an unstable algebra over the Steenrod algebra to be a ring of invariants. Before stating their result we need some preparation.

Let $\mathcal{A}$ denote the mod p Steenrod algebra and let $\mathcal{A}'$ be the subalgebra generated by the power operations P^i, $i \geqslant 0$. We then have a splitting of A'-modules,

$$\mathcal{A} = \mathcal{A}' \oplus \mathcal{A}'',$$

where $\mathcal{A}''$ is the $\mathbb{F}_p$-vector subspace of $\mathcal{A}$ generated by admissible sequences involving the Bockstein operation. Let $\mathcal{U}$ denote the category of unstable $\mathcal{A}$-modules and let $\mathcal{K}$ denote the category of unstable $\mathcal{A}$-algebras. In both cases, morphisms are of degree zero. Let $\mathcal{U}'$ and $\mathcal{K}'$ denote the corresponding full subcategories whose objects are evenly graded.

By [**17**], the forgetful functor $\theta : \mathcal{U}' \to \mathcal{U}$ has a right adjoint $\tilde{\theta} : \mathcal{U} \to \mathcal{U}'$, which sends an unstable $\mathcal{A}$-module M to the submodule of elements x of even degree satisfying $\alpha(x) = 0$ for all $\alpha \in \mathcal{A}''$, and morphisms to restrictions to these submodules. For unstable $\mathcal{A}$-algebras, the same construction gives a right adjoint $\tilde{\theta} : \mathcal{K} \to \mathcal{K}'$ to the forgetful functor $\theta : \mathcal{K}' \to \mathcal{K}$, and we have a commutative diagram of functors

$$\begin{tikzcd}
\mathcal{K}' \arrow[r, "\theta"] \arrow[d] & \mathcal{K} \arrow[r, "\tilde{\theta}"] \arrow[d] & \mathcal{K}' \arrow[d] \\
\mathcal{U}' \arrow[r, "\theta"] & \mathcal{U} \arrow[r, "\tilde{\theta}"] & \mathcal{U}'
\end{tikzcd}$$

where the vertical functors are forgetful functors. As a consequence we obtain the following lemma.

Lemma 4.6. *Properties (C1)–(C4) are preserved by* $\tilde{\theta}$.

Proof. This is mostly self-evident. The functor diagram is only needed to make sense of (C2). □

We need to recall some things about reduced $\mathcal{U}$-injectives.

Definition 4.7. An $\mathcal{A}$-module M is a *reduced* $\mathcal{U}$*-injective* if it is an injective object in the category $\mathcal{U}$, and

$$\hom_{\mathcal{U}}(\Sigma N, M) = 0,$$

for every $\mathcal{A}$-module N, where Σ denotes the suspension functor. An unstable $\mathcal{A}$-algebra R is a reduced $\mathcal{U}$-injective if it is a reduced $\mathcal{U}$-injective when regarded as an $\mathcal{A}$-module.

For an elementary abelian p-group V, the cohomology ring $H^*(BV)$ is a reduced $\mathcal{U}$-injective by [**17**]. If (f, t, X) is a retractive transfer triple over V, then $H^*(X)$ is a direct summand of $H^*(BV)$ as $\mathcal{A}$-modules, and hence $H^*(X)$ is also a reduced $\mathcal{U}$-injective. This allows us to apply the following theorem to show that $H^*(X)$ is a ring of invariants in $H^*(BV)$.

Theorem 4.8 (Goerss *et al.* [12, Theorem 1.3]). *Let R be an unstable $\mathcal{A}$-algebra that is a reduced $\mathcal{U}$-injective satisfying the following:*

(i) *$\tilde{\theta}R$ is a Noetherian integral domain;*

(ii) *$\tilde{\theta}R$ is integrally closed in its field of fractions.*

There then exists an integer n and a subgroup $W \leqslant GL(n, \mathbb{Z}/p)$ such that R is isomorphic to the ring of invariants $H^*(B(\mathbb{Z}/p)^n; \mathbb{F}_p)^W$. Furthermore, W has order prime to p.

Remark 4.9. Looking closely at the proof of the theorem in [**12**] and the tools from [**2**] used therein, one sees that $\tilde{\theta}H^*(B(\mathbb{Z}/p)^n;\mathbb{F}_p)$ is in fact the algebraic closure of $\tilde{\theta}R^*$ in $\mathcal{AW}$ and W is the group of automorphisms of $(\mathbb{Z}/p)^n$ acting trivially on R. The point is that, in their proof, Goerss *et al*. apply [**2**, Theorem 1.2], which, as mentioned above, really gives necessary and sufficient conditions for when the embedding of an algebra of finite transcendence degree into its algebraic closure in $\mathcal{AW}$ is a Galois extension. This is made clear in the introduction of [**2**], although the authors chose to make the statement of the theorem less technical.

The following technical result is needed.

Lemma 4.10. *Let R and H be unstable $\mathcal{A}$-algebras that are reduced $\mathcal{U}$-injectives and suppose that $f : R \to H$ is a morphism of $\mathcal{A}$-algebras making H finitely generated over R. Then $\tilde{\theta}f$ makes $\tilde{\theta}H$ finitely generated over $\tilde{\theta}R$.*

Proof. Recall from [**12**] that there are unique $\mathcal{A}'$-algebra homomorphisms

$$\pi_R : R \to \tilde{\theta}R \quad \text{and} \quad \pi_H : H \to \tilde{\theta}H$$

such that

$$\pi_R \circ i_R = \mathrm{id}_{\tilde{\theta}R} \quad \text{and} \quad \pi_H \circ i_H = \mathrm{id}_{\tilde{\theta}H},$$

where

$$i_R : \tilde{\theta}R \to R \quad \text{and} \quad i_H : \tilde{\theta}H \to H$$

denote the natural inclusions. (Strictly speaking this is an abuse of notation and we should replace $\tilde{\theta}H$ and $\tilde{\theta}R$ by $\theta\tilde{\theta}H$ and $\theta\tilde{\theta}R$, respectively.)

Now, suppose that $\{h_1,\dots,h_n\}$ is a set of generators for H over R. Let $h \in \tilde{\theta}H$. Then we can write

$$i_H(h) = \sum_{j=1}^{n} f(r_j)h_j,$$

for some $r_j \in R$. Consequently,

$$\begin{aligned} h = \pi_H \circ i_H(h) &= \pi_H\left(\sum_{j=1}^{n} f(r_j)h_j\right) \\ &= \sum_{j=1}^{n} \pi_H(f(r_j))\pi_H(h_j). \end{aligned}$$

Therefore, if we can show that

$$\pi_H \circ f = \tilde{\theta}f \circ \pi_R,$$

then we can deduce that $\{\pi_H(h_1),\dots,\pi_H(h_n)\}$ is a set of generators for $\tilde{\theta}H$ over $\tilde{\theta}R$ and we are done.

To prove this we first observe that by [**12**, Corollary 3.3 (i)] there is a unique morphism of unstable $\mathcal{A}$-algebras $g : R \to H$ such that

$$\pi_H \circ g = \tilde{\theta}f \circ \pi_R.$$

Next we note that by construction of $\tilde{\theta}$ we have

$$i_H \circ \tilde{\theta}g = g \circ i_R,$$

from which it follows that

$$\tilde{\theta}g = \pi_H \circ i_H \circ \tilde{\theta}g = \pi_H \circ g \circ i_R = \tilde{\theta}f \circ \pi_R \circ i_R = \tilde{\theta}f.$$

But, by [**12**, Corollary 3.3 (ii)], the map

$$\tilde{\theta} : \hom_{\mathcal{K}}(R, H) \to \hom_{\mathcal{K}'}(\tilde{\theta}R, \tilde{\theta}H)$$

is a bijection, so $g = f$, and hence

$$\pi_H \circ f = \pi_H \circ g = \tilde{\theta}f \circ \pi_R.$$

□

Proposition 4.11. *Let (f, t, X) be a retractive transfer triple over a finite elementary abelian p-group V, and let $W \leqslant \operatorname{aut}(V)$ be the subgroup of automorphisms of V acting trivially on $f^*(H^*(X))$. The map induced by f in cohomology is a split monomorphism,*

$$f^* : H^*(X) \hookrightarrow H^*(BV),$$

with image the ring of invariants $H^*(BV)^W$. Furthermore, W has order prime to p.

Proof. We already know that f^* is a split monomorphism. Let us show that $H^*(X)$ satisfies the conditions of Theorem 4.8 above. By the remark after Definition 4.7, $H^*(X)$ is a reduced $\mathcal{U}$-injective. By [**17**, **29**] we have

$$\tilde{\theta}H^*(BV) \cong \mathbb{F}_p[x_1, \dots, x_n], \tag{4.1}$$

where n is the rank of V. In particular, $\tilde{\theta}H^*(BV)$ is a Noetherian integral domain. A similar argument to the proof of Lemma 4.1 shows that $\tilde{\theta}H^*(X)$ is also a Noetherian integral domain. It remains only to show that $\tilde{\theta}H^*(X)$ is integrally closed in its field of fractions.

For this we first recall from [**2**] that $\tilde{\theta}H^*(BV)$ is integrally closed in its field of fractions. Now, let x be in the field of fractions of $\tilde{\theta}H^*(X)$ and suppose that x is integral over $\tilde{\theta}H^*(X)$. Write $x = a/b$, with $a, b \in \tilde{\theta}H^*(X)$. Then $\tilde{\theta}f^*(x) = \tilde{\theta}f^*(a)/\tilde{\theta}f^*(b)$ is also integral over $\tilde{\theta}H^*(BV)$, and since $\tilde{\theta}H^*(BV)$ is integrally closed, this implies that $\tilde{\theta}f^*(x) \in \tilde{\theta}H^*(BV)$. We now have the equation $\tilde{\theta}f^*(a) = \tilde{\theta}f^*(b)\tilde{\theta}f^*(x)$ in $\tilde{\theta}H^*(BV)$. Applying $\tilde{\theta}t^*$ and using $\tilde{\theta}H^*(X)$-linearity (C2), we get

$$a = \tilde{\theta}t^*(\tilde{\theta}f^*(a)) = \tilde{\theta}t^*(\tilde{\theta}f^*(b)\tilde{\theta}f^*(x)) = b\tilde{\theta}t^*(\tilde{\theta}f^*(x)).$$

Since $\tilde{\theta}H^*(BV)$ is an integral domain, this implies that

$$x = a/b = \tilde{\theta}t^*(\tilde{\theta}f^*(x)) \in \tilde{\theta}H^*(X).$$

Before applying Theorem 4.8, we note that since $H^*(BV)$ is finitely generated over $H^*(X)$, Lemma 4.10 applies, showing that $\tilde{\theta}H^*(BV)$ is finitely generated over $\tilde{\theta}H^*(X)$, and hence $\tilde{\theta}H^*(BV)$ is an algebraic extension of $\tilde{\theta}H^*(X)$. Since $\tilde{\theta}H^*(BV)$ is algebraically closed in $\mathcal{AW}$, we conclude that $\tilde{\theta}H^*(BV)$ is the algebraic closure of $\tilde{\theta}H^*(X)$ in $\mathcal{AW}$.

Applying Theorem 4.8 along with Remark 4.9 and the observation in the preceding paragraph now completes the proof. □

As an immediate corollary, a retractive transfer triple over an elementary abelian p-group V has the cohomology type of a p-local finite group over V.

Corollary 4.12. *Let $(f, t, X), V$ and W be as in the previous proposition. Let G be the semi-direct product $G := W \ltimes V$, and let i be the inclusion $i : V \hookrightarrow G$. There is an isomorphism of unstable $\mathcal{A}$-algebras*

$$h^* : H^*(X) \xrightarrow{\cong} H^*(BG)$$

making the following diagram commute:

$$\begin{tikzcd} H^*(X) \arrow[dd, "\cong"', "h^*"] \arrow[dr, "f^*"] & \\ & H^*(BV) \\ H^*(BG) \arrow[ur, "Bi^*"] & \end{tikzcd}$$

Proof. By a well-known transfer argument, Bi^* is a split monomorphism with image $H^*(BV)^W$. A $H^*(BV)^W$-linear splitting map is given by $(1/|W|)\,\mathrm{tr}_V^*$, where tr_V is the transfer associated with the $|W|$-fold covering map Bi. For the map h^* one can take the composite $(1/|W|)\,\mathrm{tr}_V^* \circ f^*$. □

5. Homotopy type of retractive transfer triples

Having identified the cohomology type of a retractive transfer triple over an elementary abelian p-group V as that of a p-local finite group over V in the preceding section, in this section we carry that result over to homotopy. More precisely, we construct a map of spaces realizing the map h^* of Corollary 4.12. We follow the approach taken by Dwyer *et al*. in [**10**], using Lannes technology [**16**] to pass from cohomology to homotopy.

Throughout this section, let (f, t, X) be a fixed retractive transfer triple over an elementary abelian group V, and let W be the group of automorphisms of V acting trivially on $f^*H^*(X) \subseteq H^*(BV)$. Recall from Proposition 4.11 that W has order prime to p. Let $G := W \ltimes V$ be the semi-direct product, and let $i : V \hookrightarrow G$ be the inclusion.

Proposition 5.1. *There exists a map $h : BG \to X$ making the following diagram commute up to pointed homotopy:*

$$\begin{tikzcd} & BG \arrow[dd, "h"] \\ BV \arrow[ur, "Bi"] \arrow[dr, "f"'] & \\ & X. \end{tikzcd}$$

The rest of this section is dedicated to constructing the map h.

Replace $BV \xrightarrow{f} X$ with a homotopy equivalent fibration

$$B\mathcal{V} \xrightarrow{\tilde{f}} X.$$

This can be done in such a way (by just using the standard construction) that there is a homotopy equivalence $s : BV \to B\mathcal{V}$ such that $f = \tilde{f} \circ s$ and an (abusively denoted) homotopy inverse $s^{-1} : B\mathcal{V} \to BV$ such that $s^{-1} \circ s = \mathrm{id}_{BV}$ and $s \circ s^{-1} \simeq \mathrm{id}_{B\mathcal{V}}$.

We have a fibration

$$\mathrm{Map}(B\mathcal{V}, B\mathcal{V})_{\tilde{f}} \xrightarrow{\tilde{f}\circ\cdot} \mathrm{Map}(B\mathcal{V}, X)_{\tilde{f}},$$

where $\mathrm{Map}(B\mathcal{V}, X)_{\tilde{f}}$ is the connected component of $\mathrm{Map}(B\mathcal{V}, X)$ containing $\tilde{f}$ and $\mathrm{Map}(B\mathcal{V}, B\mathcal{V})_{\tilde{f}}$ is the subspace of $\mathrm{Map}(B\mathcal{V}, B\mathcal{V})$ consisting of those components that map to $\mathrm{Map}(B\mathcal{V}, X)_{\tilde{f}}$. We denote the fibre over $\tilde{f}$ by $\mathcal{W}$. This is the space of self-maps g of $B\mathcal{V}$ such that $\tilde{f} \circ g = \tilde{f}$. Such a map g is necessarily a homotopy equivalence since $\tilde{f}$ is a homotopy monomorphism.

The next lemma can be interpreted as saying that $BV \xrightarrow{f} X$ is centric.

Lemma 5.2. *For $g \in \mathcal{W}$, the map*

$$\mathrm{Map}(B\mathcal{V}, B\mathcal{V})_g \xrightarrow{\tilde{f}\circ\cdot} \mathrm{Map}(B\mathcal{V}, X)_{\tilde{f}} \tag{5.1}$$

is a homotopy equivalence.

Proof. The map in question is adjoint to the bottom row of the commutative diagram

$$\begin{tikzcd} \mathrm{Map}(BV, BV)_{\mathrm{id}} \times BV \arrow[r, "ev"] \arrow[d, "(g_* \circ c_s)\times s", "\simeq"'] & BV \arrow[r, "Bi"] \arrow[d, "g \circ s", "\simeq"'] \arrow[dr, "f"] & BG \\ \mathrm{Map}(B\mathcal{V}, B\mathcal{V})_g \times B\mathcal{V} \arrow[r, "ev"] & B\mathcal{V} \arrow[r, "\tilde{f}"] & X \end{tikzcd}$$

where c_s is the map sending a self-map $u \in \mathrm{Map}(BV, BV)_{\mathrm{id}}$ to its 'conjugate' $s \circ u \circ s^{-1} \in \mathrm{Map}(B\mathcal{V}, B\mathcal{V})_{\mathrm{id}}$, and g_* is composition with g. Applying the cohomology functor, we

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obtain the commutative diagram

$$\begin{tikzcd}
H^*(X) \arrow[r, "\tilde{f}^*"] \arrow[d, "\cong"', "h^*"] \arrow[dr, "f^*"] & H^*(B\mathcal{V}) \arrow[r, "ev^*"] \arrow[d, "\cong"', "(g\circ s)^*"] & H^*(\mathrm{Map}(B\mathcal{V}, B\mathcal{V})_g) \otimes H^*(B\mathcal{V}) \arrow[d, "\cong"', "(g_*\circ c_s)^*\otimes s^*"] \\
H^*(BG) \arrow[r, "Bi^*"] & H^*(BV) \arrow[r, "ev^*"] & H^*(\mathrm{Map}(BV, BV)_{\mathrm{id}}) \otimes H^*(BV)
\end{tikzcd}$$

where h^* is the isomorphism from Corollary 4.12. Taking adjoints and restricting to components, we obtain the commutative diagram

$$\begin{tikzcd}
T^{\mathcal{V}}_{\tilde{f}}(H^*(X)) \arrow[r, "T^{\mathcal{V}}_g(\tilde{f}^*)"] \arrow[d, "\cong"', "\eta"] & T^{\mathcal{V}}_g(H^*(B\mathcal{V})) \arrow[r, "\lambda_g"] \arrow[d, "\cong"', "\eta"] & H^*(\mathrm{Map}(B\mathcal{V}, B\mathcal{V})_g) \arrow[dd, "\cong"', "(g_*\circ c_s)^*"] \\
T^V_f(H^*(X)) \arrow[r, "T^V_{g\circ s}(\tilde{f}^*)"] \arrow[d, "\cong"', "T^V_{Bi}(h^*)"] \arrow[dr, "T^V_{\mathrm{id}}(f^*)"] & T^V_{g\circ s}(H^*(B\mathcal{V})) \arrow[d, "\cong"', "T^V_{\mathrm{id}}((g\circ s)^*)"] & \\
T^V_{Bi}(H^*(BG)) \arrow[r, "T^V_{\mathrm{id}}(Bi^*)"] & T^V_{\mathrm{id}}(H^*(BV)) \arrow[r, "\lambda_{\mathrm{id}}"] & H^*(\mathrm{Map}(BV, BV)_{\mathrm{id}})
\end{tikzcd} \tag{5.2}$$

where the Lannes functors T^V and $T^{\mathcal{V}}$ are the left adjoints to the functors $-\otimes H^*(BV)$ and $-\otimes H^*(B\mathcal{V})$, respectively, and η is the natural isomorphism of functors induced by the isomorphism s^*. The subscript notation $T^V_{\tilde{f}}$ denotes the component of T^V corresponding to the map $\tilde{f}^*$, and the maps λ_{id} and λ_g are appropriate restrictions of the adjoint to the evaluation map. The reader is referred to [**10**, § 3] for further explanation of the notation, to [**25**] for a good exposition on Lannes functor technology, and to [**16**], the original paper by Lannes.

By Lannes's comparison theorem, [**16**, Theorem 3.3.2], the map (5.1) is a homotopy equivalence if and only if the composite of the top row in (5.2) is an isomorphism. (We have used the fact that the spaces X and $\mathrm{Map}(B\mathcal{V}, B\mathcal{V})_g \simeq BV$ are already p-complete here.) The proof is completed by recalling from [**16**] that the maps in the bottom row are both isomorphisms. □

The next proposition corresponds to [**10**, Theorem 2.9].

Proposition 5.3. *The space $\mathcal{W}$ is homotopy discrete, and the cohomology functor induces a natural monomorphism*

$$H^*(-) : \pi_0\mathcal{W} \to \mathrm{aut}_{\mathcal{K}}(H^*(B\mathcal{V}))$$

with image W under the identification

$$\mathrm{aut}_{\mathcal{K}}(H^*(B\mathcal{V})) \xrightarrow{\cong} \mathrm{aut}_{\mathcal{K}}(H^*(BV)) \cong \mathrm{aut}(V)$$

induced by $s : BV \xrightarrow{\simeq} B\mathcal{V}$.

Proof. The first claim follows from Lemma 5.2. Also from Lemma 5.2 and the long exact sequence in homotopy induced by the fibre sequence

$$\mathcal{W} \xrightarrow{\text{incl}} \mathrm{Map}(B\mathcal{V}, B\mathcal{V})_{\tilde{f}} \xrightarrow{\tilde{f}\circ\cdot} \mathrm{Map}(B\mathcal{V}, X)_{\tilde{f}}$$

we deduce that the map

$$\pi_0\mathcal{W} \xrightarrow{\text{incl}} \pi_0 \mathrm{Map}(B\mathcal{V}, B\mathcal{V})_{\tilde{f}}$$

is a bijection. In the commutative diagram

$$\begin{tikzcd}
{[B\mathcal{V}, B\mathcal{V}]} \arrow[r, "\tilde{f}\circ\cdot-"] \arrow[d, "H^*(-)"] & {[B\mathcal{V}, X]} \arrow[d, "H^*(-)"] \\
\mathrm{aut}_{\mathcal{K}}(H^*(B\mathcal{V})) \arrow[r, "\cdot\circ\tilde{f}^*"] & \hom_{\mathcal{K}}(H^*(X), H^*(B\mathcal{V}))
\end{tikzcd}$$

the left-hand vertical arrow is an isomorphism by Miller's theorem [**20**] and the right-hand vertical arrow is an isomorphism by Lannes's theorem [**16**]. Consequently, a map $g : B\mathcal{V} \to B\mathcal{V}$ belongs to $\mathrm{Map}(B\mathcal{V}, B\mathcal{V})_{\tilde{f}}$ if and only if $g^* \circ \tilde{f}^* = \tilde{f}^*$ in cohomology or, equivalently, if and only if $g^* \in W$ (under the identification above). $\square$

Observing that $\mathcal{W}$ is a group-like topological monoid, we have a contractible CW-complex $E\mathcal{W}$ on which $\mathcal{W}$ acts freely, which allows us to form the classifying space $B\mathcal{W} = E\mathcal{W}/\mathcal{W}$ and the Borel construction $E\mathcal{W} \times_{\mathcal{W}} B\mathcal{V}$. By construction, $\tilde{f}$ induces a map $\tilde{h} : E\mathcal{W}\times_{\mathcal{W}}B\mathcal{V} \to X$ fitting into a commutative diagram

$$\begin{tikzcd}
 & E\mathcal{W} \times_{\mathcal{W}} B\mathcal{V} \arrow[dd, "\tilde{h}"] \\
B\mathcal{V} \arrow[ur, "B\iota"] \arrow[dr, "f"'] & \\
 & X
\end{tikzcd}$$

where $B\iota$ is the obvious map. Proposition 5.3 implies that $E\mathcal{W} \times_{\mathcal{W}} B\mathcal{V}$ is homotopy equivalent to BG. Using the fact that these are classifying spaces of finite groups, one sees that this homotopy equivalence can in fact be realized by a map $k : BG \to E\mathcal{W} \times_{\mathcal{W}} B\mathcal{V}$, making the top rectangle in the following diagram commute up to pointed homotopy:

$$\begin{tikzcd}
BV \arrow[r, "Bi"] \arrow[d, "\simeq"', "s"] & BG \arrow[d, "\simeq"', "k"] \\
B\mathcal{V} \arrow[r, "B\iota"] \arrow[dr, "\tilde{f}"] & E\mathcal{W} \times_{\mathcal{W}} B\mathcal{V} \arrow[d, "\tilde{h}"] \\
 & X
\end{tikzcd}$$

Since $\tilde{f} \circ s = f$, the composite $\tilde{h} \circ k$ gives the desired map h in Proposition 5.1.

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6. p-local finite groups induce retractive transfer triples

In § 3 the notion of a retractive transfer triple over a finite p-group S was introduced and in §§ 4 and 5 it was shown that, in the case where S is elementary abelian, such a triple induces a p-local finite group. In this section we consider the reverse implication and prove the following theorem.

Theorem 6.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then the natural inclusion $\theta : BS \to |\mathcal{L}|^\wedge_p$ has a retractive transfer t, and $(\theta, t, |\mathcal{L}|^\wedge_p)$ is a retractive transfer triple.*

There are two parts to the proof. In § 6.1, which is joint work with Ran Levi, we show that $|\mathcal{L}|^\wedge_p$ and the inclusion $\theta : BS \to |\mathcal{L}|^\wedge_p$ of the Sylow subgroup satisfy the technical conditions of a retractive transfer triple. Most notably, we show that the classifying space of a p-local finite group is both a torsion space and a nilpotent space, a result which is of independent interest.

In § 6.2 we apply results from [**24**] to obtain a stable retraction t of the inclusion $\theta : BS \to |\mathcal{L}|^\wedge_p$, and show that it satisfies Frobenius reciprocity. These results combine to complete the proof of Theorem 6.1.

6.1. Technical conditions

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. In this subsection, which is joint work with Ran Levi, we verify that the space $|\mathcal{L}|^\wedge_p$ and the natural map

$$\theta : BS \to |\mathcal{L}|^\wedge_p$$

satisfy the technical conditions of retractive transfer triples. It has already been shown in [**6**] that $|\mathcal{L}|^\wedge_p$ is p-complete, that the fundamental group of $|\mathcal{L}|^\wedge_p$ is finite and that θ is a homotopy monomorphism. We proceed to show that $|\mathcal{L}|^\wedge_p$ is nilpotent.

Lemma 6.2. *For a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, the groups $H^k(|\mathcal{L}|^\wedge_p; \mathbb{Z})$ and $H_k(|\mathcal{L}|^\wedge_p; \mathbb{Z})$ are finite p-groups for all $k \geqslant 1$.*

Proof. As is shown in [**6**], $|\mathcal{L}|^\wedge_p$ is a stable retract of BS. In particular, $H^*(|\mathcal{L}|^\wedge_p; \mathbb{Z})$ is a subring of $H^*(BS; \mathbb{Z})$. Since $H^k(BS; \mathbb{Z})$ is a finite p-group for $k \geqslant 1$ (a transfer argument shows that $H^k(BS; \mathbb{Z})$ is p-torsion and finite generation is evident from the cell structure of BS), it follows that $H^k(|\mathcal{L}|^\wedge_p; \mathbb{Z})$ is a finite p-group for $k \geqslant 1$.

The same argument works in homology. □

Proposition 6.3. *For a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, the homotopy groups $\pi_k(|\mathcal{L}|^\wedge_p)$ are finite p-groups for all $k \geqslant 1$. In particular, $|\mathcal{L}|^\wedge_p$ is a torsion space.*

Proof. We first reduce this to the case where $|\mathcal{L}|^\wedge_p$ is simply connected. It is shown in [**6**] that $\pi_1(X)$ is a finite p-group. Letting $\tilde{X}$ be a universal cover of $|\mathcal{L}|^\wedge_p$, it therefore suffices to show that the homotopy groups of $\tilde{X}$ are all finite p-groups. However, it is shown in [**7**] that $\tilde{X}$ is again the classifying space of a p-local finite group, so we can reduce this to the simply connected case.

Assume therefore that $|\mathcal{L}|_p^\wedge$ is simply connected. By Lemma 6.2, $H_k(|\mathcal{L}|_p^\wedge;\mathbb{Z})$ is a finite p-group for all $k \geqslant 1$. Since $|\mathcal{L}|_p^\wedge$ is simply connected, we can apply the Hurewicz theorem modulo the class of finite abelian p-groups, and deduce that the homotopy groups of $|\mathcal{L}|_p^\wedge$ are all finite p-groups. □

Corollary 6.4. *The classifying space of a p-local finite group is nilpotent.*

Proof. This follows from Proposition 6.3, the fact that any finite p-group is nilpotent and the fact that the action of any finite p-group on a finite abelian p-group is nilpotent. □

6.2. The retractive transfer of a p-local finite group

In this subsection we show that the natural inclusion of a Sylow subgroup into the classifying space of a p-local finite group has a retractive transfer. We use results from [**24**], which were actually originally developed for this purpose but have turned out to be perhaps more interesting than their intended goal, and are therefore published separately.

Remark 6.5. There is a slight difference between the spectra appearing in this paper and those in [**24**]. The difference arises because here we add a base point to our spaces before forming suspension spectra. The effect at the level of stable homotopy is to add a sphere wedge summand to all spectra in sight. It is easy to check that all the results quoted from [**24**] carry over in the form stated in this section.

Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group, and let $\tilde{\omega}$ be the idempotent of $\Sigma_+^\infty BS$ induced by the characteristic idempotent of $\mathcal{F}$, as defined in [**24**]. We refer to $\tilde{\omega}$ as the *pointed stable idempotent of* $\mathcal{F}$. It has the following properties, which determine $\tilde{\omega}$ uniquely:

(a) $\tilde{\omega}$ is a $\mathbb{Z}_p^\wedge$-linear combination of homotopy classes of maps of the form $\Sigma_+^\infty B\varphi \circ \mathrm{tr}_P$, where P is a non-trivial subgroup of S, $\varphi \in \hom_{\mathcal{F}}(P,S)$ and tr_P denotes the transfer of the inclusion $P \leqslant S$;

(b) for each subgroup $P \leqslant S$ and each $\varphi \in \hom_{\mathcal{F}}(P,S)$, the restrictions $\tilde{\omega}\circ\Sigma_+^\infty Bi_P$ and $\tilde{\omega}\circ\Sigma_+^\infty B\varphi$ are homotopic as maps $\Sigma_+^\infty BP \to \Sigma_+^\infty BS$;

(c) $\tilde{\omega}$ has augmentation 1.

The augmentation in (c) corresponds to an augmentation of the double Burnside ring. As (c) is not used in this paper, the reader is referred to [**24**] for the details. We refer to (b) as *$\mathcal{F}$-stability*.

The *pointed classifying spectrum of* $\mathcal{F}$ is the stable summand $\mathbb{B}_+\mathcal{F}$ of $\Sigma_+^\infty BS$ induced by $\tilde{\omega}$. This is the infinite mapping telescope of $\tilde{\omega}$:

$$\mathbb{B}_+\mathcal{F} = \mathrm{HoColim}(\Sigma_+^\infty BS \xrightarrow{\tilde{\omega}} \Sigma_+^\infty BS \xrightarrow{\tilde{\omega}} \Sigma_+^\infty BS \xrightarrow{\tilde{\omega}} \cdots),$$

and as such it comes equipped with a *pointed structure map*, $\Sigma_+^\infty BS \xrightarrow{\sigma} \mathbb{B}_+\mathcal{F}$, which is the structure map of the homotopy colimit, and a unique (up to homotopy) map $\mathbb{B}_+\mathcal{F} \xrightarrow{t} \Sigma_+^\infty BS$ such that $\sigma\circ t \simeq \mathrm{id}_{\mathbb{B}_+\mathcal{F}}$ and $t\circ\sigma \simeq \tilde{\omega}$.

Just as in [**24**], one can show that the pointed structure map $\Sigma^\infty_+ BS \xrightarrow{\sigma} \mathbb{B}_+\mathcal{F}$ is equivalent to the infinite pointed suspension

$$\Sigma^\infty_+ BS \xrightarrow{\Sigma^\infty_+\theta} \Sigma^\infty_+ |\mathcal{L}|^\wedge_p$$

of the natural inclusion $BS \xrightarrow{\theta} |\mathcal{L}|^\wedge_p$, as objects under $\Sigma^\infty_+ BS$. We may therefore replace $\Sigma^\infty_+ BS \xrightarrow{\sigma} \mathbb{B}_+\mathcal{F}$ by

$$\Sigma^\infty_+ BS \xrightarrow{\Sigma^\infty_+\theta} \Sigma^\infty_+ |\mathcal{L}|^\wedge_p$$

in the discussion above, and obtain a unique (up to homotopy) map

$$t : \Sigma^\infty_+ |\mathcal{L}|^\wedge_p \to \Sigma^\infty_+ BS,$$

such that

$$\Sigma^\infty_+ \theta \circ t \simeq \mathrm{id}_{\Sigma^\infty_+ |\mathcal{L}|^\wedge_p} \quad \text{and} \quad t \circ \Sigma^\infty_+ \theta = \tilde{\omega}.$$

We proceed to show that t satisfies the Frobenius reciprocity relation illustrated in (3.1), and thus t is a retractive transfer for θ.

Proposition 6.6. *The idempotent $\tilde{\omega}$ satisfies the Frobenius reciprocity relation*

$$(\tilde{\omega} \wedge \tilde{\omega}) \circ \Delta \simeq (\tilde{\omega} \wedge 1) \circ \Delta \circ \tilde{\omega},$$

where $\Delta : \Sigma^\infty_+ BS \to \Sigma^\infty_+ BS \wedge \Sigma^\infty_+ BS$ is the image of the diagonal of BS under the infinite suspension functor Σ^∞_+.

Proof. Recall that we can write $\tilde{\omega}$ as a linear combination with $\mathbb{Z}^\wedge_p$-coefficients of maps $\Sigma^\infty_+ B\varphi \circ \mathrm{tr}_P \in \{BS_+, BS_+\}$, where $P \leqslant S$ and $\varphi \in \hom_{\mathcal{F}}(P, S)$. For such a map $\Sigma^\infty_+ B\varphi \circ \mathrm{tr}_P$ we have, by $\mathcal{F}$-stability of $\tilde{\omega}$,

$$\tilde{\omega} \circ \Sigma^\infty_+ B\varphi \simeq \tilde{\omega} \circ \Sigma^\infty_+ Bi_P, \tag{6.1}$$

where i_P is the inclusion $P \leqslant S$. We will take advantage of this and the fact [**1**] that the transfer tr_P of the inclusion i_P satisfies the Frobenius relation

$$(1 \wedge \mathrm{tr}_P) \circ \Delta_S \simeq (\Sigma^\infty_+ Bi_P \wedge 1) \circ \Delta_P \circ \mathrm{tr}_P, \tag{6.2}$$

where Δ_P and Δ_S are the diagonals of $\Sigma^\infty_+ BP$ and $\Sigma^\infty_+ BS$, respectively. We will also use the fact that, since $\Sigma^\infty_+ B\varphi$ has a desuspension, it commutes with the diagonals as follows:

$$\Delta_S \circ \Sigma^\infty_+ B\varphi \simeq (\Sigma^\infty_+ B\varphi \wedge \Sigma^\infty_+ B\varphi) \circ \Delta_P. \tag{6.3}$$

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Now,

$$
\begin{aligned}
(\tilde{\omega}\wedge 1)\circ \Delta_S\circ \Sigma^\infty_+ B\varphi\circ \mathrm{tr}_P &\overset{(6.3)}{\simeq} (\tilde{\omega}\wedge 1)\circ(\Sigma^\infty_+ B\varphi\wedge \Sigma^\infty_+ B\varphi)\circ \Delta_P\circ \mathrm{tr}_P\\
&\simeq ((\tilde{\omega}\circ \Sigma^\infty_+ B\varphi)\wedge \Sigma^\infty_+ B\varphi)\circ \Delta_P\circ \mathrm{tr}_P\\
&\overset{(6.1)}{\simeq} ((\tilde{\omega}\circ \Sigma^\infty_+ Bi_P)\wedge \Sigma^\infty_+ B\varphi)\circ \Delta_P\circ \mathrm{tr}_P\\
&\simeq (\tilde{\omega}\wedge \Sigma^\infty_+ B\varphi)\circ(\Sigma^\infty_+ Bi_P\wedge 1)\circ \Delta_P\circ \mathrm{tr}_P\\
&\overset{(6.2)}{\simeq} (\tilde{\omega}\wedge \Sigma^\infty_+ B\varphi)\circ(1\wedge \mathrm{tr}_P)\circ \Delta_S\\
&\simeq (\tilde{\omega}\wedge (\Sigma^\infty_+ B\varphi\circ \mathrm{tr}_P))\circ \Delta_S.
\end{aligned}
$$

By summing over the different $\Sigma^\infty_+ B\varphi\circ \mathrm{tr}_P$, we get the desired result. □

Corollary 6.7. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and let $t:\Sigma^\infty_+|\mathcal{L}|^\wedge_p\to \Sigma^\infty_+ BS$ be as constructed above. Then t is a retractive transfer of the inclusion $\theta: BS\to |\mathcal{L}|^\wedge_p$.*

Proof. We deduce the Frobenius reciprocity relation

$$(1\wedge t)\circ \Delta_{|\mathcal{L}|^\wedge_p}\simeq (\Sigma^\infty_+\theta\wedge 1)\circ \Delta_S\circ t$$

from

$$(\tilde{\omega}\wedge\tilde{\omega})\circ \Delta_S\simeq (\tilde{\omega}\wedge 1)\circ \Delta_S\circ\tilde{\omega} \tag{6.4}$$

as follows. Applying $(\Sigma^\infty_+\theta\wedge 1)\circ\cdot\circ t$ to the left-hand side of (6.4) and rewriting, we get

$$
\begin{aligned}
(\Sigma^\infty_+\theta\wedge 1)\circ(\tilde{\omega}\wedge\tilde{\omega})\circ \Delta_S\circ t &\simeq (\Sigma^\infty_+\theta\wedge 1)\circ((t\circ \Sigma^\infty_+\theta)\wedge (t\circ \Sigma^\infty_+\theta))\circ \Delta_S\circ t\\
&\simeq ((\Sigma^\infty_+\theta\circ t)\wedge t)\circ(\Sigma^\infty_+\theta\wedge \Sigma^\infty_+\theta)\circ \Delta_S\circ t\\
&\simeq (1\wedge t)\circ \Delta_{|\mathcal{L}|^\wedge_p}\circ \Sigma^\infty_+\theta\circ t\\
&\simeq (1\wedge t)\circ \Delta_{|\mathcal{L}|^\wedge_p}.
\end{aligned}
$$

Doing the same with the right-hand side yields

$$
\begin{aligned}
(\Sigma^\infty_+\theta\wedge 1)\circ(\tilde{\omega}\wedge 1)\circ \Delta_S\circ\tilde{\omega}\circ t &\simeq ((\Sigma^\infty_+\theta\circ t\circ \Sigma^\infty_+\theta)\wedge 1)\circ \Delta_S\circ(t\circ \Sigma^\infty_+\theta\circ t)\\
&\simeq (\Sigma^\infty_+\theta\wedge 1)\circ \Delta_S\circ t.
\end{aligned}
$$

Combining these equivalences, we have

$$(1\wedge t)\circ \Delta_{|\mathcal{L}|^\wedge_p}\simeq (\Sigma^\infty_+\theta\wedge 1)\circ \Delta_S\circ t.$$

□

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