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Anosov foliations and cohomology

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Abstract. The cohomology action of an Anosov diffeomorphism on a nilmanifold resembles that of a Cartesian product map. Corresponding results hold for infranilmanifolds, giving an invariant bigrading of the cohomology and a fourfold symmetry that extends Poincaré duality. Holonomy invariant cocycles are applied to the action on first cohomology.

0. *Introduction*

The paradigm of stable dynamics is the transitive Anosov diffeomorphism. We will study its behaviour on cohomology and relate it to its Anosov foliations.

Recall that a diffeomorphism $f: M \circlearrowleft$ of an oriented closed connected smooth manifold M is *Anosov* if the tangent bundle decomposes as a continuous invariant direct sum $TM = E^u \oplus E^s$ where Tf contracts the stable bundle E^s and Tf^{-1} contracts the unstable bundle E^u in some convenient Riemannian metric. One says f is transitive if it has a dense orbit. We will also assume that the bundles E^u, E^s are orientable. We let ε_u be $+1$ if f preserves the orientation of E^u, -1 otherwise and we define ε_s accordingly. We let $s = \dim E^s$, $u = \dim E^u$.

At this level of generality, the current knowledge about the real cohomology action $f^*: H^*(M) \circlearrowleft$ is as follows. (Unless otherwise specified, cohomology will always be taken with real coefficients.) Let $h(f) > 0$ be the topological entropy of f. Let $\lambda = e^{h(f)} > 1$. Then λ is the spectral radius of f^*. More precisely the number $\lambda\varepsilon_u$ is in the spectrum of f^* and its generalized eigenspace is a one dimensional subspace of $H^u(M)$. All other eigenvalues have smaller absolute value.

By Poincaré duality $\varepsilon_s\lambda^{-1}$ is in the spectrum of f^* as an eigenvalue on $H^s(M)$. The corresponding eigenvectors $\xi_u \in H^u$, $\xi_s \in H^s$ have non-zero cup product. These facts follow algebraically from those of the previous paragraph, but they also have a geometric proof [**8**], [**10**], [**2**]. One calls ξ_u, ξ_s the Ruelle–Sullivan classes of f: clearly they are unique up to a scalar factor.

We recall the geometric construction of ξ_u. The bundle E^s can be integrated to form a stable foliation W^s. On W^s there is a transverse holonomy invariant measure, unique up to scalar multiple. Since W^s is orientable, one can define a current c that integrates forms supported in a coordinate patch over the stable leaves and then integrates transversally with the transverse measure. This current is closed (by invariance) and so defines a homology class $[c] \in H_s(M; \mathbb{R})$ that represents the 'average homology class of a stable leaf'. ξ_u is the Poincaré dual to $[c]$. That ξ_u is an eigenclass follows from the fact that the transverse measure scales up under f by a factor of λ while the transverse orientation scales by ε_u. ξ_s can be constructed similarly and the transversality of the stable and unstable foliations furnishes the geometric proof that $\xi_u \cup \xi_s \neq 0$.

We will develop in this paper the idea that the cohomology action of f resembles that of a Cartesian product map. Suppose N_i are closed oriented connected smooth d_i-manifolds, $i = 1, 2$, and let $N = N_1 \times N_2$, $d = d_1 + d_2$. Let $g_i: N_i \circlearrowleft$ be smooth maps of degree δ_i and let $g = g_1 \times g_2: N_1 \times N_2 \circlearrowleft$ be their Cartesian product. Note that N carries two transverse foliations, $\mathcal{F}_2$ with leaves $N_1 \times (\text{point})$ and $\mathcal{F}_1$ with leaves $(\text{point}) \times N_2$. Taking any transverse measures, the corresponding currents give the homology classes of these factor manifolds in $H_{d_i}(N)$, up to a scalar multiple. Taking Poincaré duals gives classes $\mathcal{N}_i$ in $H^{d_i}(N)$ that scale by δ_i under f^* and that have non-zero cup product. So in this analogy the Ruelle–Sullivan classes correspond to the fundamental classes $\mathcal{N}_i$ that pull back from the projections $N \to N_i$. But this projection is just the projection of N to the leaf space of the foliation $\mathcal{F}_i$. This suggests that the eigenvalue $\lambda\varepsilon_u$ be viewed as the 'degree' of the map induced by f on the leaf space of the stable foliation.

It is unclear how far this analogy can be pushed. We hope in a future work to use the symbolic dynamics of f, including the incidence relation on elements of a Markov partition, to study this question. This tool is suggested by the construction of transverse measures for the foliations W^u, W^s from a Markov partition [**8**] where the incidence data was not needed, but for lower dimensional cohomology this data is essential. In this paper, we will present those results we have obtained that don't use symbolic methods. One may regard these as guideposts for some subsequent deeper analysis.

We begin with the study of known examples. These are topologically conjugate to certain automorphisms of nilmanifolds and quotients of these by a finite symmetry group. For nilmanifolds we will see in § 1 that the analogy with a Cartesian product is extreme: the cohomology algebra factors as in Kunneth's theorem so that the Ruelle–Sullivan classes correspond to the fundamental classes of the factor manifolds. When the finite symmetry group is introduced, this factorization breaks down but certain features persist. These are the existence of an invariant bigrading and a biduality analogous to the 4-fold symmetry of the cohomology of a Cartesian product. These matters are discussed in § 2.

In § 3 we consider Alexander cochains. We show that holonomy invariant cocycles account for the hyperbolic part of the action of f on first cohomology. This is related to the cohomology of the leaf spaces of the Anosov foliations in the guise of holonomy invariant cocycles.

1. *Anosov automorphisms*

We recall some terminology. Let N be a simply connected nilpotent Lie group and Γ a discrete subgroup. If the collection of right cosets $X = \Gamma\backslash N$ is compact we call X a nilmanifold. When N is abelian, X is just a torus. If $\alpha: N \to N$ is a Lie group automorphism that induces a 1-1 map $A: X \to X$ we call A an automorphism of X. When N is abelian, X is a group and A is an automorphism in the usual sense. Let $\mathfrak{n} = T_e N$ be the Lie algebra of N and let $\alpha_*: \mathfrak{n} \circlearrowleft$ be the automorphism of $\mathfrak{n}$ induced by α, $\alpha_* = T_e \alpha$. If α_* is hyperbolic as a linear map then α is called hyperbolic and A is called hyperbolic or Anosov.

Indeed, such an A is an Anosov diffeomorphism as defined above. The invariant splitting $TX = E^u \oplus E^s$ is found by taking the α_* invariant splitting $\mathfrak{n} = U \oplus S$, where U corresponds to eigenvectors outside the unit disc and S to those in its interior, extending this to a splitting of TN by left translation and factoring by Γ. If we give an inner product for which $\|\alpha_* | S\| < 1$, $\|\alpha_*^{-1} | U\| < 1$ then one can extend it to a left-invariant Riemannian metric on N and factor by Γ to obtain a Riemannian metric on X. The splitting and metric on X are clearly such that stable (unstable) vectors shrink under $A(A^{-1})$, as required.

The unstable and stable foliations on X relate to the group structure as follows. S is a subalgebra of $\mathfrak{n}$ and the left cosets $n \cdot \exp(S)$, $n \in N$, foliate N and induce the foliation W^s on X. Likewise the left cosets of the subgroup $\exp(U)$ determine W^u.

Clearly these foliations are orientable. Also A is transitive so the Ruelle–Sullivan classes ξ_u, ξ_s are defined. We show that A splits the cohomology of X as though it were a Cartesian product.

THEOREM 1. *Let $A: X \circlearrowleft$ be an Anosov automorphism of a nilmanifold X. Then there are subspaces $\mathcal{S}_0, \ldots, \mathcal{S}_u$ and $\mathcal{U}_0, \ldots, \mathcal{U}_s$, $\mathcal{U}_i$, $\mathcal{S}_i \subset H^i(X; \mathbb{R})$, such that*

(1) $\mathcal{S}_0 = \mathcal{U}_0 = H^0(X; \mathbb{R})$.

(2) *$\mathcal{S}_u$ is spanned by ξ_u, $\mathcal{U}_s$ is spanned by ξ_s.*

(3) *$\mathcal{U} = \bigoplus \mathcal{U}_i$ and $\mathcal{S} = \bigoplus \mathcal{S}_i$ are closed under cup-product.*

(4) *The cup product maps $\mathcal{U}_i \otimes \mathcal{U}_{s-i} \to \mathcal{U}_s$ and $\mathcal{S}_i \otimes \mathcal{S}_{u-i} \to \mathcal{S}_u$ are perfect pairings.*

(5) *Each $\mathcal{U}_i$, $\mathcal{S}_i$ is f^*-invariant.*

(6) *For $i > 0$, $\mathcal{S}_i$ is expanded by f^* and $\mathcal{U}_i$ is contracted.*

(7) *The cup product map $\mathcal{U} \otimes \mathcal{S} \to H^*(X; \mathbb{R})$ is an isomorphism of graded algebras.*

Before proving this theorem, we consider the simple case when N is abelian. Then $\mathcal{S}_i$ is just the cohomology classes of the i-forms on X with constant coefficients that annihilate the stable foliation. This suggests our choice of notation. In a similar way $\mathcal{U}_j$ is determined by the unstable foliation.

Proof. Let $\mathcal{A}$ be the algebra of left-invariant differential forms on N. Clearly $\mathcal{A}$ is isomorphic to the exterior algebra $E(\mathfrak{n}^*)$, since a left invariant form on N is determined by its value at e. Clearly α acts on $\mathcal{A}$. Since d commutes with maps, $d\mathcal{A} \subset \mathcal{A}$.

A left-invariant form on N is in particular Γ-invariant. This gives an injection $\mathcal{A} \to \Omega^*(X)$ of $\mathcal{A}$ into the de Rham complex of X. By a theorem of Nomizu this natural map gives a cohomology isomorphism $H^*(\mathcal{A}, d) \cong H^*(X)$, see [7].

The theorem will be shown by decomposing $\mathcal{A}$ according to the splitting $\mathfrak{n} = S \otimes U$. We identify S^*, the vector space dual of S, with the elements of $\mathfrak{n}^*$ that annihilate

U and likewise, we embed U^* in $\mathfrak{n}^*$. Then $\mathfrak{n}^* = U^* \otimes S^*$. This gives inclusions $E(U^*) \subset E(\mathfrak{n}^*)$ and $E(S^*) \subset E(\mathfrak{n}^*)$ such that $E(\mathfrak{n}^*) = E(U^*) \otimes E(S^*)$.

We must show that d preserves these factors. Note that $E(U^*)$ consists of those forms in $\mathcal{A}$ that annihilate the left cosets of exp (S). By the converse of Frobenius' theorem, the exterior derivative $d\omega$ of a form ω that annihilates a foliation must also annihilate that foliation. So $dE(U^*) \subset E(U^*)$ and symmetrically, $dE(S^*) \subset E(S^*)$.

We have, then, that $E(U^*)$, $E(S^*)$ are differential algebras. So by the algebraic Kunneth theorem, $H^*(E(\mathfrak{n}^*)) = H^*(E(U^*)) \otimes H^*(E(S^*))$. This defines a factorization $H^*(X; \mathbb{R}) = \mathcal{S} \otimes \mathcal{U}$ using the Nomizu isomorphism. Clearly $\mathcal{S}$ and $\mathcal{U}$ are homogeneous subalgebras. So (1), (3), (5), (6) and (7) are obvious.

Let $d = \dim X = s + u$. Then $H^d(X; \mathbb{R})$ is one-dimensional. As $E(U^*)$, $E(S^*)$ are zero above dimensions u, s respectively, we must have $\mathcal{S}_u$ and $\mathcal{U}_s$ one dimensional. By (6), any summand $\mathcal{S}_i \otimes \mathcal{U}_j$ with $(i, j) \neq (u, 0)$ has a smaller spectral radius than $\mathcal{S}_u$. This proves (2).

To see (4), let $u_i \in \mathcal{U}_i$. Then $u_i \cup \xi_u \in H^{u+i}(X)$ and by Poincaré duality on X there is a $c_j \in H^j(X)$ with $c_j \cup (u_i \cup \xi_u) \neq 0$, where $j + u + i = d$. Thus $j = s - i$. Decomposing $c_j = c_j' + c_j''$ with $c_j' \in \mathcal{U}_j$, $c_j'' \in \mathcal{U} \otimes (\bigoplus_{j>0} \mathcal{S}_j)$, we see $c_j'' \cup (u_i \cup \xi_u) = 0$ for dimensional reasons. Thus $c_j' \cup u_i \cup \xi_u \neq 0$, so $c_j' \cup u_i \neq 0$. This proves that $\mathcal{U}_i \to \mathcal{U}_{s-i}^*$ is 1-1. Reversing the roles of i and $s - i$ proves $\mathcal{U}_i \otimes \mathcal{U}_{s-i} \to \mathcal{U}_s$ is a perfect pairing. Reasoning similarly on $\mathcal{S}$ gives (4). □

Suppose $\omega \in E(U^*) \subset \mathcal{A}$ is also invariant under *right* translations. Then as a form on X, ω is a holonomy invariant form for the foliation W^s, since holonomy can be defined using right multiplication along a path in S and forms in $E(U^*)$ are really defined on the normal bundle of W^s. This applies in particular to the closed 1-forms in $E(U^*)$, since these pull back from the fibration of X over a torus, corresponding to the map $\pi_1 X \to H_1(X; \mathbb{Z})/\text{torsion}$.

2. *Bigrading and biduality*

We now turn to the automorphisms of infranilmanifolds and show that part of theorem 1 remains true for these.

Suppose N is a simply connected nilpotent Lie group. By combining the natural left action of N on itself with the left action of the automorphism group Aut N on N, we obtain a left action of the group G of affine automorphisms of N. An element $g \in G$ is a pair $(n, \beta) \in N \times \text{Aut } N$ that acts on N by $g(x) = n \cdot \beta(x)$. We define the group structure by the embedding of G in the diffeomorphisms of N. It is immediate that G is the semidirect product of the normal subgroup N by the subgroup Aut N. In case N is abelian, G is the affine automorphisms of the vector space N.

Suppose $\pi \subset G$ is a discrete group that acts freely and properly discontinuously on N with compact quotient Y. If $\pi \cap N = \Gamma$ has finite index in π then we call Y an infranilmanifold. This implies that $X = \Gamma \backslash N$ is compact. Thus Y is the quotient of the nilmanifold X by the finite group $H = \pi/\Gamma$ so Y is below (infra) X.

Suppose $\alpha: N \to N$ is a hyperbolic automorphism that normalizes π in the sense that there is an automorphism $\phi \circ \pi \to \pi$ with $\alpha \circ g = \phi(g) \circ \alpha$ for all $g \in \pi$. Then α induces a map $B: Y \to Y$ that is called an Anosov automorphism of Y. We note that the corresponding automorphism of X, $A: X \to X$, is a lift of B relative to the covering $p: X \to Y$. Again B is an Anosov diffeomorphism. Every known Anosov diffeomorphism is topologically conjugate to such a B.

We next recall the behaviour of cohomology under a finite cover. The map p has a multivalued inverse map that acts on forms in a well-defined way. If $k = |H|$ is the number of sheets, and if $x_1, \ldots, x_k \in X$ are the inverse images of $y \in Y$ then there are natural isomorphisms $\psi_j: T_{x_j}X \to T_y Y$ induced by Tp. In this way an i-form ω on X determines an i-form $\tau(\omega)$ on Y by

$$\tau(\omega)(v_1, \ldots, v_i) = \sum_{j=1}^{k} \omega(\psi_j^{-1} v_1, \ldots, \psi_j^{-1} v_i).$$

This transfer map τ commutes with d and so induces a map $\tau^*: H^*(X) \to H^*(Y)$. Then action of p on cohomology is related to τ^* by $(1/k)\tau^* \circ p^* = \mathrm{id}: H^*(Y) \circlearrowleft$. This shows p^* is 1-1, so $H^*Y \subset H^*X$.

To identify the image, note that pulled-back forms are H-invariant and so determine H-invariant cohomology classes. Using the notation V_H to denote the vectors fixed by a linear action of H on a vector space V, we have $H^*Y \subset (H^*X)_H$. Actually equality holds here: one may take a form ω whose class $c = [\omega]$ is invariant and average ω under H to obtain an H-invariant representative $\omega' = (1/k) \sum_{h \in H} h^*\omega$ for c. But an H-invariant form is a pullback from Y, so $\omega' = p^*\omega_Y$. This gives $c = p^*[\omega_Y]$ and shows

$$(H^*X)_H = p^*(H^*Y) \cong H^*Y.$$

So H^*Y can be computed in X.

Now we consider the action of H on the Anosov foliations of X. We have that the action of A is equivariant with respect to the automorphism $\psi: H \circlearrowleft$ induced by $\phi: \pi \circlearrowleft$, i.e. $A \circ h = \psi(h) \circ A$. This implies

$$A^n \circ h = \psi^n(h) \circ A^n.$$

Choose n so $\psi^n(h) = h$. One sees that h commutes with A^n so h permutes the stable and unstable sets of A^n. Thus h preserves the Anosov foliations on X. Thus π preserves the lifts of these foliations to N, i.e. π permutes the left cosets of $\exp(S)$ and $\exp(U)$.

Now we recall the constructions of theorem 1. The splitting $\mathfrak{n} = S \oplus U$ is definable using these left cosets alone, so π preserves this direct sum (here we identify a Lie algebra element with a left-invariant vector field). It follows that π preserves the splitting $\mathscr{A} = E(S^*) \otimes E(U^*)$. Also since Γ acts trivially on $\mathscr{A}$, this is really an action of H on $\mathscr{A}$: it is clearly that which arises by viewing H as an automorphism group of $\mathfrak{n}$ under the map $\pi \to G \to \mathrm{Aut}\, N \to \mathrm{Aut}\, \mathfrak{n}$ which has kernel Γ.

Since the injection $\mathscr{A} \to \Omega^*X$ is H equivariant, it induces an H-equivariant map of cohomology. Using Nomizu's theorem and the formula $H^*Y \cong (H^*X)_H$, we see that $H^*Y \cong (H^*\mathscr{A})_H$. Thus $H^*Y \cong (\mathscr{U} \otimes \mathscr{S})_H$.

We have $\mathcal{U}_H \otimes \mathcal{S}_H \subset (\mathcal{U} \otimes \mathcal{S})_H$ but the inclusion may be strict. Thus the factorization of H^*X breaks down for H^*Y. On the other hand, the action of H on $\mathcal{U} \otimes \mathcal{S}$ preserves the bigrading given by $(\mathcal{U} \otimes \mathcal{S})_{ij} = \mathcal{U}_i \otimes \mathcal{S}_j$, $i = 0, \ldots, s$, $j = 0, \ldots, u$. Thus $H^*Y = (\mathcal{U} \otimes \mathcal{S})_H$ carries a bigrading.

We now recall the standing assumption that our Anosov diffeomorphisms have orientable foliations. This means here that the action of H on S and U preserves orientation and so H fixes $\mathcal{U}_s$ and $\mathcal{S}_u$. This implies that the duality pairing of $\mathcal{U}_i$ and $\mathcal{U}_{s-i}$ defined by $c_i \cup c_{s-i} = \langle c_i, c_{s-i} \rangle \xi_s$ is H-invariant, i.e. $\langle hc_i, hc_{s-i} \rangle = \langle c_i, c_{s-i} \rangle$. Similarly the pairing of $\mathcal{S}_j$ and $\mathcal{S}_{u-j}$ is H-invariant.

We prove

THEOREM 2. *Let $B: Y \rightleftarrows$ be an Anosov automorphism of the infranilmanifold Y with oriented foliations. The grading of H^*Y by dimension can be refined to a bigrading $H^{i,j}$, $0 \leq i \leq s$, $0 \leq j \leq u$, so that*

(1) $H^k(Y) = \bigoplus_{i+j=k} H^{i,j}$.
(2) *B preserves bigrading, i.e.* $B^* H^{i,j} \subset H^{i,j}$.
(3) *Under cup product* $H^{i,j} \cdot H^{k,l} \subset H^{i+k,j+l}$.
(4) *ξ_s spans $H^{s,0}$, ξ_u spans $H^{0,u}$.*
(5) *Biduality: the summands $H^{i,j}$, $H^{s-i,j}$, $H^{s-i,u-j}$ and $H^{i,u-j}$ have equal dimension.*

Proof. Let $H^{i,j} = (\mathcal{U}_i \otimes \mathcal{S}_j)_H$. (1), (2), (3) and (4) are clear from theorem 1. To prove biduality, we need an algebraic lemma.

LEMMA 1. *Suppose H is a finite group and $\psi: H \rightleftarrows$ an automorphism. Let P, Q, W be finite dimensional real vector spaces with a linear action of H. Suppose P, Q are dual by a perfect pairing $P \oplus Q \to \mathbb{R}$, denoted $\langle p, q \rangle$, and that $\langle hp, hq \rangle = \langle p, q \rangle$ for all $h \in H$, $p \in P$, $q \in Q$. Then*

$$\dim (P \otimes W)_H = \dim (Q \otimes W)_H.$$

Granted this lemma, we take $P = \mathcal{U}_i$, $Q = \mathcal{U}_{s-i}$, $W = \mathcal{S}_j$ and $\langle \ , \ \rangle$ as above. The lemma shows $H^{i,j}$ and $H^{s-i,j}$ have equal dimension. Reversing the roles of $\mathcal{S}$, $\mathcal{U}$ gives biduality.

Proof of lemma. We need

SUBLEMMA. $\dim W_H = |H|^{-1} \sum_h \operatorname{Trace} h_W$, *where h_W denotes the action of h on W.*

Proof. The map $\delta(w) = |H|^{-1} \sum_h h_W(w)$ projects W onto W_H. Take its trace.

Now we apply the sublemma to $P \otimes W$ and $Q \otimes W$ to get

$$\dim (P \otimes W)_H = |H|^{-1} \sum_h \operatorname{Trace} h_P \cdot \operatorname{Trace} h_W$$

and

$$\dim (Q \otimes W)_H = |H|^{-1} \sum_h \operatorname{Trace} h_Q \cdot \operatorname{Trace} h_W.$$

Now we identify Q with P^* and give P an H-invariant inner product. This identifies h_P with an orthogonal transformation and h_Q with its inverse transpose, so they have the same trace for all $h \in H$. Thus all terms in the above sums are equal.

Note that the structure furnished by the theorem is dependent only on the Anosov foliations, not on B itself. This structure is topological and so it exists also on any known Anosov diffeomorphism with oriented foliations.

3. *Holonomy invariant cocycles*

We now turn to a general transitive Anosov diffeomorphism $f: M \to M$. We will study its behaviour on H^1M and characterize the subspaces of H^1M that are expanded and contracted by f^* in terms of the foliations of f. This reduces the problem whether $f^*: H^1M \circlearrowleft$ is hyperbolic to a question about the foliation pair [1], [4], [5], [6].

As the foliations of f have no transversal smoothness, we will follow Shub's lead and work with Alexander cohomology, specifically the variant that uses continuous alternating cochains [9]. Recall that a cochain in dimension k is a function $M^{k+1} \to \mathbb{R}$ and two such are identified when they agree on a neighbourhood of the diagonal Δ.

We say a cochain in a foliated manifold is *holonomy invariant* if in each foliation chart it pulls back from a cochain on the local leaf space. Given a foliation $\mathcal{F}$, the holonomy invariant cochains form a subcomplex. Let $Z^k(\mathcal{F})$ be the holonomy invariant cocycles and $A^k(\mathcal{F}) \subset H^k(M)$ the classes with holonomy invariant representations. Clearly $A^*(\mathcal{F})$ is a graded subalgebra of $H^*(M)$, but it may consist only of $H^0(M)$. If $\mathcal{F}$ is a fibration with total space E and base B then $A^*(\mathcal{F})$ is the image of $H^*(B; \mathbb{R})$ in $H^*(E; \mathbb{R})$.

Suppose that $\mathcal{F} = W^s(f)$ is the stable foliation of our Anosov f. Let $c: M^2 \to \mathbb{R}$ be a holonomy invariant 1-cocycle. Then there is a neighbourhood U of $\Delta \subset M^2$ on which

$$c(m_1, m_2) + c(m_2, m_3) = c(m_1, m_3).$$

Taking two points m_1, m_2 that are nearby on the same unstable leaf and iterating f^{-1}, we see that $f^{-n^*}c$ takes ever smaller values on U. This implies $f^{-n^*}[c] \to 0$, i.e. $[c] \in H^1(M; \mathbb{R})$ is expanded by f. We will show

THEOREM 3. *The natural map* $Z^1(W^s(f)) \to H^1(M)$ *is* 1–1 *with image the expanding subspace of* $f^*: H^1(M) \circlearrowleft$.

Given this, one applies it to f and f^{-1} to obtain

COROLLARY. $f^*: H^1(M) \circlearrowleft$ *is hyperbolic* $\Leftrightarrow A^1(W^u) + A^1(W^s) = H^1M$.

As mentioned above, this characterizes the hyperbolicity on H^1 in terms of the foliations alone.

Proof of theorem. Suppose $c \in Z^1(W^s(f))$, as above. If $[c] = 0$ then one can integrate c to obtain a function $c': M \to \mathbb{R}$ with $c = \delta c'$, i.e. $c(m_1, m_2) = c'(m_1) - c'(m_2)$. As c is holonomy invariant, c' is constant on leaves. As f is transitive, stable leaves are dense. Thus c' is constant and $c = 0$. This proves injectivity.

For surjectivity, we use [9] to produce a vector space V of 1-cocycles on M that is f-invariant and contains one representative for each expanded class in $H^1(M; \mathbb{R})$. It is enough to check that V is holonomy invariant.

Let V^* be the dual space of V. We can identify V^* with a quotient space of $H_1(M;\mathbb{R})$, so $\pi_1 M$ acts on V^*. There is an equivariant map $D: \tilde{M} \to V^*$ such that $v \in V$ is identified with $\delta(v \circ D)$, the latter being an equivariant 1-cocycle on $\tilde{M}$.

If p, q are nearby in $\tilde{M}$ and on the same stable leaf, then the lift $\tilde{f}: \tilde{M} \circlearrowleft$ brings p, q nearer, hence (as D is uniformly continuous),

$$\|D\tilde{f}^n p - D\tilde{f}^n q\| \to 0.$$

As $\tilde{f}$ is equivariant with respect to an expanding map of V^*, we must have $Dp = Dq$. This implies that all cocycles pulled back under D are holonomy invariant. □

The relationship between the holonomy invariant cochains and the cohomology of an Anosov map probably goes deeper. Symbolic dynamics is the logical tool to use in a further study. It would be interesting to reproduce the bigrading and biduality results of previous sections in terms that did not involve group structure but only the foliation pair, thereby generalizing them to include any undiscovered Anosov diffeomorphisms.

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