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LYAPUNOV EXPONENTS ON METRIC SPACES

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Abstract

We use the pointwise Lipschitz constant to define an upper Lyapunov exponent for maps on metric spaces different to that given by Kifer [‘Characteristic exponents of dynamical systems in metric spaces’, *Ergodic Theory Dynam. Systems* **3**(1) (1983), 119–127]. We prove that this exponent reduces to that of Bessa and Silva on Riemannian manifolds and is not larger than that of Kifer at stable points. We also prove that it is invariant along orbits in the case of (topological) diffeomorphisms and under topological conjugacy. Moreover, the periodic orbits where this exponent is negative are asymptotically stable. Finally, we estimate this exponent for certain hyperbolic homeomorphisms.

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1. Introduction

The *upper Lyapunov exponent* at $x \in M$ of a differentiable map $f : M \to M$ of a Riemannian manifold M is defined by

$$\Lambda_f(x) = \limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)\|. \tag{1.1}$$

Here Dh is the derivative of a differentiable map $h : M \to M$ and $\|\cdot\|$ is the Riemannian metric. A similar exponent for continuous maps on metric spaces has been considered by Kifer [8] followed by Barreira, Bessa and Silva [2, 3]. In particular, the paper [3] introduced such an exponent for continuous maps on Riemannian manifolds, reducing to (1.1) in the differentiable case. However, as these authors noted, the substitute for the derivative in their exponent is not a cocycle and so it is unclear how the exponent relates to the abstract theory [4].

In this paper we use the subcocycle for the pointwise Lipschitz constant [5] to define an upper Lyapunov exponent on metric spaces. We prove that this exponent reduces to that of [3] for continuous maps on Riemannian manifolds. Moreover, it is

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not larger than Kifer's exponent at the stable points. We also prove that this exponent satisfies some basic properties of the abstract theory such as invariance along orbits for (topological) diffeomorphisms and under topological conjugacy. We also prove that every periodic orbit with negative upper Lyapunov exponent is asymptotically stable. Finally, we estimate the exponent for certain hyperbolic homeomorphisms.

2. Definition and examples

Let (X,d) and (Y,ρ) be metric spaces and $f : X \to Y$ be a map. As in [6], the *pointwise Lipschitz constant* $f'(x)$ of f at $x \in X$ is defined as 0 if x is isolated and

$$f'(x) = \limsup_{y \to x} \frac{\rho(f(x), f(y))}{d(x,y)}$$

otherwise. (The notation $\mathrm{Lip} f$ is used in [6].) This constant is closely related to the absolute derivative in [5]. We use it to introduce the main definition of this paper.

Definition 2.1. The *upper Lyapunov exponent* of a map $f : X \to X$ at $x \in X$ is defined by

$$\chi_f(x) = \limsup_{n \to \infty} \frac{1}{n} \log (f^n)'(x). \tag{2.1}$$

The map assigning the extended number $\chi_f(x) \in [-\infty, \infty]$ to each $x \in X$ will be denoted by χ_f. Sometimes we write $\chi_f^d(x)$ to emphasise the role of the metric d.

This work is devoted to the study of this exponent. First, we note the similarity to the definition of the classical exponent (1.1). Next, we turn to some examples.

Example 2.2. If $f : X \to X$ is a constant map, then $\chi_f = -\infty$.

Example 2.3. If $f : X \to X$ is a *contraction*, that is, there is a constant c with $0 < c < 1$ such that $d(f(x), f(y)) \leq c d(x,y)$ for every $x, y \in X$, then $\chi_f \leq \log c < 0$. To see this, fix $n \in \mathbb{N}^+$. Clearly, $d(f^n(x), f^n(y)) \leq c^n d(x,y)$ for all $x, y \in X$, so $(f^n)'(x) \leq c^n$ for all $x \in X$ and $\chi_f \leq \log c$.

Example 2.4. If $f : X \to X$ is *bi-Lipschitz*, that is, there exists $A \geq 1$ such that $A^{-1} d(x,y) \leq d(f(x), f(y)) \leq A d(x,y)$ for all $x, y \in X$, then $A^{-n} \leq (f^n)'(x) \leq A^n$ for all $n \in \mathbb{N}$ and $x \in X$. Thus, $-\log A \leq \chi_f \leq \log A$. In particular, if f is an *isometric embedding* (that is, $d(f(x), f(y)) = d(x,y)$ for all $x, y \in X$), then $\chi_f = 0$. Hence, $\chi_{\mathrm{Id}_X} = 0$, where Id_X is the identity of X.

Example 2.5. Let $f : X \to X$ be a continuous map of a compact metric space X. If $f : X \to X$ *expands small distances*, that is, there are $\epsilon_0 > 0$ and $\lambda > 1$ such that $d(f(x), f(y)) \geq \lambda d(x,y)$ whenever $x, y \in X$ satisfy $d(x,y) \leq \epsilon_0$ (cf. [13]), then $\chi_f \geq \log \lambda > 0$.

For the proof, fix $n \in \mathbb{N}^+$. Since f is continuous and X compact, there is $\epsilon_n > 0$ such that $d(f^i(x), f^i(y)) \leq \epsilon_0$ for all $0 \leq i \leq n-1$ whenever $x, y \in X$ satisfy $d(x,y) < \epsilon_n$. From this we obtain recursively that $d(f^n(x), f^n(y)) \geq \lambda^n d(x,y)$ whenever $d(x,y) \leq \epsilon_n$. Then $(f^n)'(x) \geq \lambda^n$ for all $x \in X$ and so $\chi_f(x) \geq \log \lambda$ for all $x \in X$.

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EXAMPLE 2.6. We say that two metrics d and ρ of the same space X are *equivalent* if there is $A \geq 1$ such that $A^{-1}d(x,y) \leq \rho(x,y) \leq Ad(x,y)$ for every $x,y \in X$. It is easy to see that if d and ρ are equivalent metrics, then $\chi_f^d = \chi_f^\rho$. This may be false if the metrics d and ρ are only *compatible*, that is, they define the same topology.

3. Comparisons with the other exponents

In this section we will compare the upper Lyapunov exponent (2.1) with the existing ones in the literature. We start with the top exponent introduced by Bessa and Silva [3].

Let M denote a compact, connected, boundaryless smooth Riemannian manifold. By compactness and Darboux's theorem [1], there is a finite atlas $A = \{\phi_i : U_i \to \mathbb{R}^m\}$, where $m = \dim(M)$ and each $U_i \subset M$ is an open set. Assume that, for any $x \in M$, we choose univocally $i(x) := \min\{i \in 1,\dots,k : x \in U_i\}$. The Riemannian metric fixed in the beginning will not be used, but instead we consider the metric in T_xM defined by $\|v\| := \|D\phi_i(x)\cdot v\|$. Thus, the computations below are performed in the Euclidean space via the fixed charts.

Let $f : M \to M$ be a continuous map and, for each $x \in M$, $\delta > 0$ and $n \in \mathbb{N}$, consider the set

$$B_x(\delta,n) = \{y \in M : d(f^i(x), f^i(y)) < \delta \text{ for all } i = 0,\dots,n\}. \tag{3.1}$$

Define

$$\Delta(f,n,x,y) = \frac{d(f^n(x), f^n(y))}{d(x,y)},$$

where d is the distance generated by the Riemannian metric $\|\cdot\|$. For n fixed, the map $\delta \mapsto (1/n)\log \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y)$ is nondecreasing, so the limit

$$\lim_{\delta\to 0} \frac{1}{n} \log \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y) = \inf_{\delta>0} \frac{1}{n} \log \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y)$$

exists. In this way, Bessa and Silva [3] defined the top exponent

$$\chi_N^+(f,x) = \limsup_{n\to\infty} \lim_{\delta\to 0} \frac{1}{n} \log \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y) \quad \text{for all } x \in M.$$

This exponent reduces to our exponent for continuous maps on Riemannian manifolds. More precisely, we have the following result.

THEOREM 3.1. *Let $f : M \to M$ be a continuous map of a Riemannian manifold M. For every $x \in M$, if $\chi_f(x)$ is computed with the distance generated by the Riemannian metric, then $\chi_f(x) = \chi_N^+(f,x)$.*

PROOF. Since the map $t \mapsto \log t$ is continuous, we can rewrite $\chi_N^+(f,x)$ as

$$\chi_N^+(f,x) = \limsup_{n\to\infty} \frac{1}{n} \log \lim_{\delta\to 0} \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y).$$

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On the other hand, using Darboux charts, we can rewrite the derivative $(f^n)'(x)$ for $n \in \mathbb{N}$ as

$$(f^n)'(x) = \lim_{\delta\to 0} \sup_{y\in B_x(\delta,0)\setminus\{x\}} \Delta(f,n,x,y) = \inf_{\delta>0} \sup_{y\in B_x(\delta,0)\setminus\{x\}} \Delta(f,n,x,y).$$

Since $B_x(\delta,n) \subset B_x(\delta,0)$,

$$\sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y) \le \sup_{y\in B_x(\delta,0)\setminus\{x\}} \Delta(f,n,x,y),$$

yielding

$$\lim_{\delta\to 0} \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y) \le (f^n)'(x).$$

Taking the log, dividing by n and taking the lim sup as $n \to \infty$ gives

$$\chi_N^+(f,x) \le \chi_f(x).$$

Conversely, fix $\delta > 0$. Since f is continuous, given $n \in \mathbb{N}^+$ there is a $\gamma > 0$ such that $d(f^i(x), f^i(y)) < \delta$ for $i = 0, \dots, n$ whenever $y \in M$ satisfies $d(x,y) < \gamma$. This amounts to $B_x(\gamma,0) \subset B_x(\delta,n)$ and yields

$$\sup_{y\in B_x(\gamma,0)\setminus\{x\}} \Delta(f,n,x,y) \le \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y).$$

It follows that

$$(f^n)'(x) \le \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y) \quad \text{for all } \delta > 0$$

and so

$$(f^n)'(x) \le \lim_{\delta\to 0} \sup_{y\in B_x(\delta,n)\setminus\{x\}} \Delta(f,n,x,y).$$

Again by taking the log, dividing by n and taking the lim sup as $n \to \infty$,

$$\chi_f(x) \le \chi_N^+(f,x).$$

This completes the proof. □

Next, we compare our exponent with Kifer's exponent [8]. Consider a metric space without isolated points X and a map $f : X \to X$. In [8], the upper exponent is defined for all $x \in X$ by

$$\Delta^+(f,x) = \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{y\in B_x(\delta,n)\setminus\{x\}} \frac{d(f^n(x), f^n(y))}{d(x,y)},$$

where $B_x(\delta,n)$ is defined as in (3.1) but with X instead of M. Apart from the domain of the maps, the main difference between this exponent and χ_f is the role played by the limits $\lim_{\delta\to 0}$ and $\limsup_{n\to\infty}$, which makes the comparison difficult. However, such a comparison can be done under certain conditions.

We say that $x \in X$ is a *stable point* of $f : X \to X$ if for every $\gamma > 0$ there is $\delta > 0$ such that $d(f^n(x), f^n(y)) < \gamma$ for every $n \in \mathbb{N}$ whenever $y \in X$ satisfies $d(x,y) < \delta$. With this definition, we have following result.

THEOREM 3.2. *Let $f : X \to X$ be a map on a metric space X without isolated points. Then $\chi_f(x) \leq \Delta^+(f, x)$ for every stable point x.*

PROOF. As in the proof of Theorem 3.1,

$$(f^n)'(x) = \inf_{\delta \to 0} \sup_{y \in B_x(\delta,0)\setminus\{x\}} \frac{d(f^n(x), f^n(y))}{d(x,y)} \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Fix $\gamma > 0$. Since x is stable, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(f^n(x), f^n(y)) < \gamma$ for all $n \in \mathbb{N}$. Then $B_x(\delta, 0) \subset B_x(\gamma, n)$ for every $n \in \mathbb{N}$, so

$$\sup_{y \in B_x(\delta,0)\setminus\{x\}} \frac{d(f^n(x), f^n(y))}{d(x,y)} \leq \sup_{y \in B_x(\gamma,n)\setminus\{x\}} \frac{d(f^n(x), f^n(y))}{d(x,y)} \quad \text{for all } n \in \mathbb{N}.$$

Then (3.2) implies that

$$(f^n)'(x) \leq \sup_{y \in B_x(\gamma,n)\setminus\{x\}} \frac{d(f^n(x), f^n(y))}{d(x,y)} \quad \text{for all } n \in \mathbb{N}.$$

Since $\gamma > 0$ is arbitrary,

$$\frac{1}{n} \log(f^n)'(x) \leq \frac{1}{n} \log \sup_{y \in B_x(\gamma,n)\setminus\{x\}} \frac{d(f^n(x), f^n(y))}{d(x,y)} \quad \text{for all } n \in \mathbb{N}^+, \gamma > 0. \tag{3.3}$$

If we take the lim sup as $n \to \infty$ in (3.3),

$$\chi_f(x) \leq \limsup_{n \to \infty} \frac{1}{n} \log \sup_{y \in B_x(\gamma,n)\setminus\{x\}} \frac{d(f^n(x), f^n(y))}{d(x,y)} \quad \text{for all } \gamma > 0.$$

Taking the limit as $\gamma \to 0$ gives $\chi_f(x) \leq \Delta^+(x)$. □

REMARK 3.3. It would be nice to find an example where the inequality in Theorem 3.2 is strict, but we have not been able to do so.

4. Some properties of χ_f

Next, we present some properties of the upper Lyapunov exponent χ_f. To begin, we compute the pointwise Lipschitz constant in the differentiable case.

LEMMA 4.1. *Let $h : M \to M$ be a map of a Riemannian manifold M. If h is differentiable at $x \in M$, then $h'(x) = \|Dh(x)\|$.*

PROOF. By using exponential charts we can assume that $M = \mathbb{R}^n$, where $n = \dim(M)$. Then

$$\begin{aligned} h'(x) &= \limsup_{y \to x} \frac{d(h(x), h(y))}{d(x,y)} = \limsup_{y \to x} \frac{\|h(y) - h(x)\|}{\|y - x\|} \\ &= \limsup_{y \to x} \left\| \frac{h(y) - h(x) - Dh(x)(y - x)}{\|y - x\|} + Dh(x)\left(\frac{y - x}{\|y - x\|}\right) \right\| \\ &\leq \|Dh(x)\|. \end{aligned}$$

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On the other hand, if we fix a unitary vector u such that $\|Dh(x)u\| = \|Dh(x)\|$ and define $y_n = x + (1/n)u$, then $y_n \to x$ and

$$\begin{aligned}
&\limsup_{y\to x}\left\|\frac{h(y)-h(x)-Dh(x)(y-x)}{\|y-x\|}+Dh(x)\Big(\frac{y-x}{\|y-x\|}\Big)\right\| \\
&\quad\geq \lim_{n\to\infty}\left\|\frac{h(y_n)-h(x)-Dh(x)(y_n-x)}{\|y_n-x\|}+Dh(x)\Big(\frac{y_n-x}{\|y_n-x\|}\Big)\right\| \\
&\quad= \|Dh(x)u\| = \|Dh(x)\|.
\end{aligned}$$

Thus, $h'(x) = \|Dh(x)\|$, completing the proof. □

Although the result below follows from Theorem 3.1 and [3, Theorem 2.3], we can prove it quickly by applying Lemma 4.1 to $h = f^n$ with $n \in \mathbb{N}$.

Theorem 4.2. *Let $f : M \to M$ be a differentiable map of a Riemannian manifold M. If d is the Riemannian distance of M, then $\chi_f^d(x) = \Lambda_f(x)$.*

We will need the following chain rule for the pointwise Lipschitz constant f'.

Lemma 4.3 (Upper Chain Rule). *For every pair of maps $f, g : X \to X$ of a metric space X, if $x \in X$ and $g'(x) < \infty$, then*

$$(f\circ g)'(x) \leq f'(g(x))\cdot g'(x).$$

Proof. We can assume that $(f\circ g)'(x) \neq 0$ for otherwise we are done. Fix a sequence $y_n \to x$ such that

$$(f\circ g)'(x) = \lim_{n\to\infty}\frac{d(f(g(x)), f(g(y_n)))}{d(x,y_n)}.$$

Since $(f\circ g)'(x) \neq 0$, we can assume that $g(x) \neq g(y_n)$ for every $n \in \mathbb{N}$. Therefore, $d(g(x), g(y_n)) > 0$ for all $n \in \mathbb{N}$. Since $g'(x) < \infty$, g is pointwise Lipschitz (hence continuous) at x. Thus, $g(y_n) \to g(x)$ and so

$$\begin{aligned}
(f\circ g)'(x) &= \lim_{n\to\infty}\frac{d(f(g(x)), f(g(y_n)))}{d(x,y_n)} \\
&= \lim_{n\to\infty}\Big[\frac{d(f(g(x)), f(g(y_n)))}{d(g(x),g(y_n))}\cdot\frac{d(g(x),g(y_n))}{d(x,y_n)}\Big] \\
&\leq \limsup_{z\to g(x)}\frac{f(g(x)), f(z))}{d(g(x),z)}\cdot\limsup_{y\to x}\frac{d(g(x),g(y))}{d(x,y)} \\
&= f'(g(x))\cdot g'(x),
\end{aligned}$$

proving the result. □

We say that $f : X \to X$ is *differentiable at* $x \in X$ if $0 < f'(x) < \infty$, and we say that f is *differentiable* if it is differentiable at every $x \in X$. Given a map $\phi : X \to \mathbb{R}\cup\{\infty\}$, we define $\|\phi\|_\infty = \sup_{x\in X}|\phi(x)|$. A homeomorphism $h : X \to X$ is called a *diffeomorphism* if $\|k'\|_\infty < \infty$ for $k \in \{h, h^{-1}\}$.

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THEOREM 4.4. *The following properties hold for every map $f : X \to X$ of a metric space X without isolated points.*

(1) *If $k \in \mathbb{N}$, then $\chi_{f^k} \le k \cdot \chi_f$.*
(2) *If f is differentiable at $x \in X$, then $\chi_f(f(x)) \ge \chi_f(x)$. In particular, if f is a diffeomorphism, then $\chi_f \circ f = \chi_f$.*
(3) *If $h : X \to X$ is a diffeomorphism, then $\chi_{h\circ f\circ h^{-1}} = \chi_f \circ h^{-1}$.*
(4) *$\chi_f \le \log \|f'\|_\infty$ and, if f is a homeomorphism, $\chi_{f^{-1}} \ge -\log \|f'\|_\infty$.*

PROOF. For (1), observe that

$$\begin{aligned}\chi_{f^k}(x) &= \limsup_{n\to\infty} \frac{1}{n} \log((f^k)^n)'(x) = \limsup_{n\to\infty} \frac{1}{n} \log(f^{n\cdot k})'(x) \\ &= k \cdot \limsup_{n\to\infty} \frac{1}{n\cdot k} \log(f^{n\cdot k})'(x) \\ &\le k \cdot \chi_f(x).\end{aligned}$$

For (2), observe that $(f^n)'(f(x)) \ge (f^{n+1})'(x)/f'(x)$ by the upper chain rule, so

$$\begin{aligned}\chi_f(f(x)) &= \limsup_{n\to\infty} \frac{1}{n} \log(f^n)'(f(x)) \\ &\ge \limsup_{n\to\infty} \frac{1}{n}[\log(f^{n+1})'(x) - \log f'(x)] \\ &= \limsup_{n\to\infty} \frac{1}{n} \log(f^{n+1})'(x) \\ &= \chi_f(x).\end{aligned}$$

If f is a diffeomorphism, applying this argument to f^{-1} yields $\chi_f(f^{-1}(x)) \ge \chi_f(x)$. These inequalities together prove (2).

For (3), first observe by the upper chain rule that

$$\begin{aligned}\chi_{h\circ f\circ h^{-1}}(x) &= \limsup_{n\to\infty} \frac{1}{n} \log((h\circ f\circ h^{-1})^n)'(x) = \limsup_{n\to\infty} \frac{1}{n} \log(h \circ f^n \circ h^{-1})'(x) \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log(h'(f^n(h^{-1}(x))) \cdot (f^n)'(h^{-1}(x)) \cdot (h^{-1})'(x)) \\ &= \limsup_{n\to\infty} \frac{1}{n}[\log(h'(f^n(h^{-1}(x))) + \log(f^n)'(h^{-1}(x)) + \log(h^{-1})'(x))] \\ &= \chi_f(h^{-1}(x)).\end{aligned}$$

Therefore, $\chi_f = \chi_{h^{-1}\circ(h\circ f\circ h^{-1})\circ h} \le \chi_{h\circ f\circ h^{-1}} \circ h$ and so $\chi_f \circ h^{-1} \le \chi_{h\circ f\circ h^{-1}}$. Taken together, these inequalities prove (3).

Finally, for (4), we first observe that

$$(f^n)'(x) \le \prod_{i=0}^{n-1} f'(f^i(x))$$

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by the upper chain rule. Thus, $(f^n)'(x) \le \|f'\|_\infty^n$ for every $x \in X$. By taking the logarithm, dividing by n and taking the lim sup, we obtain $\chi_f \le \log \|f'\|_\infty$. If f is a homeomorphism, $1 = (f^n \circ f^{-n})'(x) \le (f^n)'(f^{-n}(x)) \cdot (f^{-n})'(x)$ by the upper chain rule. This gives $(f^{-n})'(x) \ge 1/(f^n)'(f^{-n}(x))$ and so

$$\begin{aligned}
\chi_{f^{-1}}(x) &= \limsup_{n\to\infty} \frac{1}{n} \log(f^{-n})'(x) \\
&\ge \limsup_{n\to\infty} \frac{1}{n} [-\log(f^n)'(f^{-n}(x))] \\
&= -\liminf_{n\to\infty} \frac{1}{n} \log(f^n)'(f^{-n}(x)) \\
&\ge -\liminf_{n\to\infty} \frac{1}{n} \log \|f'\|_\infty^n \\
&= -\log \|f'\|_\infty.
\end{aligned}$$

□

5. The upper Lyapunov exponent, stability and hyperbolicity

Let $f : X \to X$ be a map of a metric space X. We say that $x \in X$ is *periodic* if there is a minimal positive integer n_x (called the *period*) such that $f^{n_x}(x) = x$. Also, x is *asymptotically stable* if there is $\delta > 0$ such that

$$\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{for all } y \in B(x, \delta). \tag{5.1}$$

The following result shows how the upper exponent can be used to detect asymptotically stable periodic points.

Theorem 5.1. *Every periodic point of a continuous map of a metric space with negative upper Lyapunov exponent is asymptotically stable.*

Proof. Let $f : X \to X$ be a continuous map of a metric space X and $x \in X$ be a point with $\chi_f(x) < 0$. We can assume that x is not isolated for otherwise the result is trivial. Let n_x be the period of x. Then $x \in \mathrm{Fix}(f^{n_x})$, where $\mathrm{Fix}(h)$ denotes the fixed point set of h. Since $\chi_f(x) < 0$, we obtain $\chi_{f^{n_x}}(x) < 0$ from Theorem 4.4(1). Thus, we can assume that $x \in \mathrm{Fix}(f)$ by replacing f by f^{n_x} if necessary.

Set $\Delta = -\frac{1}{2}\chi_f(x)$. Then $\Delta < 0$ and there is $N \in \mathbb{N}^+$ such that $(f^N)'(x) < e^{N\Delta}$. Since $x \in \mathrm{Fix}(f)$, the definition of χ_f implies that there is $\delta_0 > 0$ such that $d(x, f^N(y)) < e^{N\Delta} d(x, y)$ for all $y \in B(x, \delta_0)$. From this, $d(x, f^{kN}(y)) < \delta_0$ and $d(x, f^{kN}(y)) < (e^{N\Delta})^k d(x, y)$ for every $k \in \mathbb{N}^+$ and $y \in B(x, \delta_0)$. Since $e^{N\Delta} < 1$,

$$\lim_{k\to\infty} f^{kN}(y) = x \quad \text{for all } y \in B(x, \delta_0).$$

By choosing $0 < \delta < \delta_0$ such that $f^i(y) \in B(x, \delta_0)$ whenever $y \in B(x, \delta)$ and $0 \le i \le N$, we obtain (5.1). □

There are several definitions of hyperbolicity for a homeomorphism of a metric space. These include Mañé's hyperbolicity [10], $\mathcal{L}$-hyperbolicity [14] and the standard definition [9] (that is, expansivity with the pseudo-orbit tracing property). It is known that Mañé's definition and the standard one are equivalent up to some compatible metric (see, for example, [11, 12]). Here we use the following definition, which is implicit in [7].

Definition 5.2. A bijective map $f : X \to X$ is *hyperbolic* if there are constants $C > 0$, $\lambda > 1$ and a sequence $\epsilon_n > 0$ such that, for all $n \in \mathbb{N}$ and all $x, y \in X$,

$$\max\{d(f^n(x), f^n(y)), d(f^{-n}(x), f^{-n}(y))\} \geq C\lambda^n d(x, y) \quad \text{if } d(x, y) < \epsilon_n. \tag{5.2}$$

We now estimate the upper Lyapunov exponent of such hyperbolic maps.

Theorem 5.3. *Let $f : X \to X$ be a hyperbolic bijection of a metric space X without isolated points. Then $\inf_{x\in X} \max\{\chi_f(x), \chi_{f^{-1}}(x)\} > 0$.*

Proof. Let C, λ and ϵ_n be as in the definition of a hyperbolic bijection. Fix $x \in X$ and $n \in \mathbb{N}$. Since X has no isolated points, there is a sequence $y_k \in X \setminus \{x\}$ converging to x and we can assume that $d(x, y_k) < \epsilon_n$ for all k. If $d(f^n(x), f^n(y_k)) < C\lambda^n d(x, y_k)$ for large k, then $d(f^{-n}(x), f^{-n}(y_k)) \geq C\lambda^n d(x, y_k)$ for k large by (5.2), so

$$(f^{-n})'(x) \geq \limsup_{k\to\infty} \frac{d(f^{-n}(x), f^{-n}(y_k))}{d(x, y_k)} \geq C\lambda^n.$$

Otherwise, $d(f^n(x), f^n(y_k)) \geq C\lambda^n d(x, y_k)$ for large k and then $(f^n)'(x) \geq C\lambda^n$ as before. We conclude that

$$\max\{(f^n)'(x), (f^{-n})'(x)\} \geq C\lambda^n \quad \text{for all } x \in X, n \in \mathbb{N}. \tag{5.3}$$

Now fix $x \in X$. If $\chi_f(x) < \log\lambda$, then $(f^n)'(x) < C\lambda^n$ for large n, so $(f^{-n})'(x) \geq C\lambda^n$ for large n (by (5.3)) and thus $\chi_{f^{-1}}(x) \geq \log\lambda$. Hence, $\max\{\chi_f(x), \chi_{f^{-1}}(x)\} \geq \log\lambda$. □

A direct consequence of this result follows. A homeomorphism $f : X \to X$ of a metric space X is *expansive* if there is $\delta > 0$ such that $x = y$ whenever $x, y \in X$ satisfy $d(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$.

Corollary 5.4. *For every expansive homeomorphism $f : X \to X$ of a compact metric space X without isolated points there is a compatible metric d such that*

$$\inf_{x\in X} \max\{\chi^d_f(x), \chi^d_{f^{-1}}(x)\} > 0.$$

Proof. Fathi proved in [7] that there are $\lambda > 1$, $\epsilon_0 > 0$ and a compatible metric d satisfying

$$\max\{d(f(x), f(y), d(f^{-1}(x), f^{-1}(y))\} \geq \min\{\lambda d(x, y), \epsilon_0\} \quad \text{for all } x, y \in X.$$

Fix $n \in \mathbb{N}$. Since f is a homeomorphism and X compact, there exists $\epsilon_n > 0$ such that

$$\max_{|i|<n} d(f^i(x), f^i(y)) < \frac{\epsilon_0}{\lambda} \quad \text{whenever } d(x, y) < \epsilon_n.$$

Then, the proof of Theorem 5.4 in [7] implies that

$$\max\{d(f^n(x), f^n(y)), d(f^{-n}(x), f^{-n}(y))\} \geq \lambda^n d(x, y) \quad \text{if } d(x, y) < \epsilon_n$$

and so f is hyperbolic with respect to d. Now apply Theorem 5.3. □

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