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A REMARK ON THE LOEWY-SERIES OF CERTAIN HOPF ALGEBRAS

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ABSTRACT. An easy proof will be given to show that for finite dimensional Hopf-algebras with nilpotent augmentation ideal over the field of p elements, the upper and lower Loewy-series coincide. In particular, this holds for the restricted universal envelope of nilpotent Lie-p-algebras with nilpotent p-map.

1. **Introduction.** For a finite p-group G, it is well-known that the upper and lower Loewy-series of the group ring KG, K denoting a field of characteristic p, coincide (see [3], p. 157 for the notion of Loewy-series, and [2] for the original statement due to Hill).

In this note, we will generalize this result to finite dimensional Hopf algebras with nilpotent augmentation ideal (since there are no nilpotent ideals other than 0 in the characteristic 0 case, we only consider Hopf algebras over fields of characteristic p). In particular, this implies that equality of the Loewy-series holds for restricted universal envelopes of nilpotent, finite dimensional Lie-p-algebras with nilpotent p-map.

2. **Exposition.** Throughout this section let H denote a finite dimensional Hopf algebra over a field K, i.e. a K-bialgebra with an antipode (see [7] for a detailed definition), whose augmentation ideal $\bar{H}$ is nilpotent.

Under these circumstances, H is filtered by the powers of $\bar{H}$ and

$$gr\, H := \bigoplus_{i \geq 0} \bar{H}^i / \bar{H}^{i+1}$$

becomes a K-algebra by extending $(x + \bar{H}^{i+1})(y + \bar{H}^{j+1}) := xy + \bar{H}^{i+j+1}$ linearly to all of $gr\, H$. In particular, $gr\, H$ can be interpreted as a left gr H-module.

But more is true: Since "gr" is functorial and commutes with "$\otimes$" (see [6], p. 269), $gr\, H$ is a bialgebra. And if S denotes the antipode of H, then S respects the filtration of H, since it is an algebra antimorphism. Hence, grS is an antipode of $gr\, H$, so

LEMMA 2.1. ([7], p. 238):
If H is as above, then $gr\, H$ is a Hopf algebra.

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For the following proposition, we need some terminology. If A is a nilpotent K-algebra, say $A^{n+1} = 0 \neq A^n$, define the right annihilator series of A by

$$\alpha_{r,0}(A) := 0, \alpha_{r,i+1}(A) := \{a \in A | Aa \subset \alpha_{r,i}(A)\},$$

or to make this definition comparable with the powers of A:

$$A_{r,i}(A) := \alpha_{r,n+1-i}(A).$$

An easy induction shows that $\alpha_{r,i}(A)$ is the right annihilator of A^i in A (so that $A_{r,i}(A)$ is the right annihilator of A^{n+1-i} in A). Hence, if we want to show that the upper and lower Loewy-series of A coincide, we only have to show that

$$A^i = A_{r,i}(A).$$

The $A_{r,i}(A)$ form a descending chain of two-sided ideals with the further properties

(2.2) $$A^i \subset A_{r,i}(A)$$

(2.3) $$A^i A_{r,j}(A) \subset A_{r,i+j}(A) \text{ for all } i, j.$$

Hence, if H is a Hopf algebra as above, then by means of (2.3),

$$\hat{H} := K \oplus \bigoplus_{i \geq 1} A_{r,i}(\bar{H})/A_{r,i+1}(\bar{H})$$

can be made a left $gr\,H$-module by expanding $(x + \bar{H}^{i+1})(y + A_{r,j+1}(\bar{H})) := xy + A_{r,i+j+1}(\bar{H})$ for $x \in \bar{H}^i, y \in A_{r,j}(\bar{H})$ to all of grH, resp. $\hat{H}$.

Note that in general, $\hat{H}$ is not a K-algebra.

The inclusions (2.2) applied to H induce a homomorphism of left $gr\,H$-modules

$$\sigma : gr\,H \longrightarrow \hat{H}.$$

PROPOSITION 2.4. Let H be a finite dimensional Hopf algebra with nilpotent augmentation ideal $\bar{H}$. Then $\bar{H}^i = A_{r,i}(\bar{H})$ for $i = 1, \ldots, n$.

PROOF. Since the dimension of $gr\,H$ and $\hat{H}$ are the same and finite, it suffices by (2.2) to show that σ is injective.

Obviously, σ restricted to $gr_n H = \bar{H}^n$ is injective. On the other handside, $\ker\sigma$ is in our module structure already a left-ideal contained in

$$\triangle := \bigoplus_{i \geq 1} \bar{H}^i/\bar{H}^{i+1}.$$

If $\ker\sigma \neq 0$, there would exist an element different from 0, simultaneously being contained in $\ker\sigma$ and in $\alpha_{r,1}(\triangle) = A_{r,n}(\triangle)$ (with $x \in \ker\sigma$ and not being in $\alpha_{r,1}(\triangle)$,

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there exists y_1 such that $y_1x \neq 0$. But $y_1x \in \ker\sigma$; and if $y_1x \notin \alpha_{r,1}(\triangle)$, one could continue in this way. Since $\triangle$ is nilpotent, this process has to stop after a finite number of steps).

So, if we could show that $A_{r,n}(\triangle) = gr_nH$, we would be through. Because of $\triangle^n = gr_nH \subset A_{r,n}(\triangle)$, sufficient would be, for example, to show dim $A_{r,n}(\triangle) = 1$.

Now, $gr\,H$ being a finite dimensional Hopf algebra, is Frobenius (see [5]). By [3], Thm.61.3, there exists a linear map $gr\,H \longrightarrow K$, the kernel of which contains neither a left- nor a right-ideal different from 0. But since the intersection of any subspace of $gr\,H$ with $A_{r,n}(\triangle)$ is a left-ideal, this implies by reason of dimension, dim $A_{r,n}(\triangle) = 1$. □

It may be worthwhile stating a small observation coming out of this last dimension argument:

Let A be a finite dimensional, nilpotent K-algebra. $K \oplus A$ is a Frobenius algebra if and only if the left- as well as the right-annihilator of A is 1-dimensional.

Furthermore, it is clear that the left-sided version of (2.4), too, is true. Thus, left-, right- and two-sided annihilators of $\bar{H}$ coincide with $\bar{H}^i$.

A corresponding statement for Frobenius or symmetric algebras fails to hold: By our above-mentioned observation,

$$B := K[x,y]/(x^2 - y^2 - y^3, xy - y^2, y^4),$$

where x and y are commuting indeterminates, provides a counterexample (Let $\bar{B}$ denote the ideal of B generated by the images $\bar{x}$ (resp. $\bar{y}$) in B of x (resp. y). Then $\bar{B}^4 = 0$, and $\bar{y}^3$ is a base of $\bar{B}^3$, $\bar{x}^2 + \bar{B}^3, \bar{y}^2 + \bar{B}^3$ is a base of $\bar{B}^2/\bar{B}^3$ and $\bar{x} + \bar{B}^2, \bar{y} + \bar{B}^2$ is one of $\bar{B}/\bar{B}^2$. The annihilator of $gr\bar{B}$ then contains the two linearly independent elements $\bar{x} - \bar{y} + \bar{B}^2$ and $\bar{y}^3$. Thus, grB is not a Frobenius algebra).

We finish this paper by noting two special cases of (2.4), which are of particular interest:

Let G be a finite p-group and K a field of characteristic p. Then the group ring KG satisfies the hypothesis of (2.4). Thus

COROLLARY 2.5 ([2]). *The upper and lower Loewy-series of KG coincide.*

The finite dimensional, nilpotent Lie-p-algebras L with nilpotent p-map are known to be precisely those Lie-p-algebras, which generate a nilpotent ideal u_0L in their universal envelope uL (see [4]). Since it is well-known that uL is a Hopf algebra, we obtain

COROLLARY 2.6. *The upper and lower Loewy-series of uL coincide.*

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