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Inverse system of a symbolic power III: thin algebras and fat points

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Abstract. We state a conjectural upper bound for the Hilbert function of the ideal $\mathfrak{I}_P^{(a)}$ of functions vanishing to order at least 'a' at a set P of s generic points of $\mathbb{P}^n$, and verify the bound in some cases. We show that if $3 \leqslant n, s < 2^n$, and a is sufficiently large, then $\mathfrak{I}_P^{(a)}$ is never in μ-generic position (Theorem 1). R. Fröberg has given conjectural lower bounds on the Hilbert function of ideals generated by generic homogeneous polynomials, and thus also for ideals of powers of linear forms; our method is to translate these bounds to the vanishing problem, using Macaulay's inverse systems.

We give an application to bounding the dimensions of spline functions for certain polyhedra in $\mathbb{R}^n$, using a result of L. Rose relating these dimensions to the number of syzygies of power algebras.

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Table of Contents

Introduction. The Hilbert function of vanishing ideals

We first give an overview of the paper, and of our conjectures concerning the Hilbert function of higher order vanishing ideals at points of projective space. Throughout the paper we fix a field k that we assume is algebraically closed, except in Section 3; we usually omit explicit mention of k. Since the conjectures we discuss may depend on characteristic, we will assume that the characteristic is zero, or is larger than any degree being considered. Let $\mathbb{P}^n, n = r - 1$ be projective n-space, and denote by $R = k[x_1, \ldots, x_r]$ its homogeneous coordinate ring. Recall that the Hilbert function $H(M)$ of a graded R-module is the sequence $(h_0(M), \ldots, h_i(M), \ldots), h_i(M) = \dim_k M_i$.

If $P = \{P_1, \ldots, P_s\}$ is a set of s points in $\mathbb{P}^{r-1}$, and $A = (a_1, \ldots, a_s)$ we denote by $\mathfrak{I}_P^{(A)}$ (or $\mathfrak{I}_P^{(a)}$ in the equal vanishing order case) the intersection, $m_{P_1}^{a_1} \cap \cdots \cap m_{P_s}^{a_s}$, and we denote by $Z_{P,A}$ (or $Z_{P,a}$) the associated subscheme $Z_{P,A} = \mathrm{Spec}(R/\mathfrak{I}_P^{(A)})$ of $\mathbb{P}^n$. Thus, $\mathfrak{I}_P^{(A)}$ is the ideal in R of functions vanishing to order at least a_i at each point P_i of P, and the ideal $\mathfrak{I}_P^{(a)}$ is the a-th symbolic power of $\mathfrak{I}_P$. It is well-known that there is a sequence of nonnegative integers HPOINTS (s, A, r) (or HPOINTS (s, a, r) in the equal vanishing order case), such that if P is a generic set of s points in P^{r-1}, then the Hilbert function $H(R/\mathfrak{I}_P^{(A)})$ satisfies

$$H(R/\mathfrak{I}_P^{(A)}) = \text{HPOINTS } (s, A, r). \tag{1}$$

Many authors, including J. Alexander, M. V. Catalisano, K. Chandler, A. Geramita, A. Gimigliano, H. Esnault, E. Viehweg, B. Harbourne, A. Hirschowitz, P. Maroscia, M. Nagata, F. Oreccia, N. V. Trung, and G. Valla and others, have studied the regularity and Hilbert functions of the ideals $\mathfrak{I}_P^{(A)}$, sometimes under particular restrictions for the points P, either as a natural geometric problem, or because of a connection to number theory or the study of field extensions. See [A], [AH1], [AH2], [AH3], [Ch1], [Ch2], [Ch3], [CTV], [EsV], [Gi1], [Gi2], [GO1], [GO2], [GM], [Ha1], [Ha2], [H1], [H2], [N], [T], and [TV].

Our goal here is to make accurate conjectures concerning HPOINTS (s, A, r) and to give evidence for them. We hope this will make the problem of determing HPOINTS (s, A, r) more accessible. In order to state our conjectures for HPOINTS, we at first discuss an apparently unrelated problem. Let $L\colon L_1, \ldots, L_s$, be a set of s linear homogeneous elements of a second polynomial ring $\mathfrak{R} = k[X_1, \ldots, X_r]$, and suppose that $J = (j_1, \ldots, j_s)$ is a fixed sequence of s positive integers. It is easy to see that there is a sequence HPOWLIN (s, J, r) and an open dense subset $U = \mathrm{UPL}(s, J, r)$ of $\hat{\mathbb{P}}^n \times \cdots \times \hat{\mathbb{P}}^n$, such that if the sequence $\langle L_1 \rangle, \ldots, \langle L_s \rangle$ of one-dimensional vector spaces in $\mathbb{P}(\mathfrak{R}_1)$ is in UPL (s, J, r), then the Hilbert function $H(\mathfrak{R}/(L_1^{j_1}, \ldots, L_s^{j_s}))$ satisfies (see Lemma 1.4.2)

$$H(\mathfrak{R}/(L_1^{j_1}, \ldots, L_s^{j_s})) = \text{HPOWLIN } (s, J, r). \tag{2}$$

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Likewise, there is a sequence HGEN (s, J, r) such that if $f_1, \ldots, f_s$ are a generic set of homogeneous polynomials of degrees $j_1, \ldots, j_r$, then (see Lemma 1.4),

$$H(\mathfrak{R}/(f_1, \ldots, f_s) = \text{HGEN}\,(s, J, r). \tag{3}$$

DEFINITION 0.1 An algebra $B = \mathfrak{R}/(f_1, \ldots, f_s)$, for which there is equality in (3), is called a *thin algebra* (see [I2], [An]).

We define a power series $F'(s, J, r, \mathrm{Z})$ with coefficients $F'(s, J, r, \mathrm{Z})_i$ by

$$F'(s, J, r, \mathrm{Z}) = \sum F'(s, J, r)_i \mathrm{Z}^i = (1 - \mathrm{Z})^{-r} \prod_{1 \leqslant u \leqslant s} (1 - \mathrm{Z}^{j_v}). \tag{4}$$

In the equidegree case $j_1 = \cdots = j_s = j$, we denote $F'(s, J, r)$ by $F'(s, j, r)$. Then

$$F'(s, j, r)_i = \dim_k R_i + \sum_{1 \leqslant t \leqslant \min(\lfloor i/j \rfloor, r, s)} (-1)^t (\dim_k R_{i-tj}) \cdot \binom{s}{t}. \tag{4a}$$

The Fröberg sequence $F(s, J, r)$ is

$$F(s, J, r)_i = \begin{cases} F'(s, J, r)_i, & \text{if } F'(s, J, r)_u > 0 \quad \text{for all } u \leqslant i. \\ 0 \text{ otherwise.} \end{cases} \tag{5}$$

R. Fröberg proposed in [F] the first of the following conjectures

Strong Fröberg Conjecture (SFC):

$$\text{HGEN}\,(s, J, r) = F(s, J, r). \tag{6}$$

Weak Fröberg Conjecture (WFC):

$$\text{HGEN}\,(s, J, r) \geqslant F(s, J, r). \tag{6a}$$

Let $B = \mathfrak{R}/(f_1, \ldots, f_s)$ be a generic thin algebra with $\deg(f_i) = j_i$. Consider a minimal $\mathfrak{R}$-free resolution $\mathbb{F}_B$ of B whose maps are of degree-zero, and whose uth term $\mathbb{F}_B(-u)$ is a direct sum of copies of $\mathfrak{R}$ shifted negatively. The exactness of $\mathbb{F}_B$ is equivalent to the exactness of all the homogeneous pieces $\mathbb{F}_{B,i}$ for $i \geqslant 0$. Let $\mathbb{K}_{B,F}$ denote the (not necessarily exact) Koszul resolution of B constructed using $F = (f_1, \ldots, f_r)$, similarly graded in negative degrees. Let $N(s, J, r)$ be the largest integer i such that $F(s, J, r)_{i+1} \neq F'(s, J, r)_{i+1}$. R. Fröberg has shown that the Strong Fröberg Conjecture is equivalent to the following apparently stronger

Thin Algebra Resolution Conjecture (TARC). If B is a generic thin algebra, then $\dim_k(\mathbb{F}_{B,i}(u)) = \dim_k(K_{B,i}(u))$ for every pair (i, u) such that $i \leqslant N(s, J, r)$.

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DEFINITION 0.2 Fröberg error functions. We let FER (s, J, r) (or FER (s, j, r) in the equal degree case), denote the function from the natural numbers N to $\mathbb{Z}$, with value FER $(s, J, r)_i$ for $i \in N$

$$\text{FER}\,(s, J, r) = \text{HGEN}\,(s, J, r) - F(s, J, r). \tag{7a}$$

We let the linear Fröberg Error Function LFER (s, J, r) (or LFER (s, j, r) in the equal degree case), denote the sequence

$$\text{LFER}\,(s, J, r) = \text{HPOWLIN}\,(s, J, r) - F(s, J, r). \tag{7b}$$

We let the linear defect LD (s, J, r) (or LD (s, j, r) in the equal degree case), denote the sequence

$$\text{LD}\,(s, J, r) = \text{HPOWLIN}\,(s, J, r) - \text{HGEN}\,(s, J, r), \tag{7c}$$

the amount by which s generic linear powers fail to calculate the generic Hilbert function HGEN (s, J, r).

We let $\underline{u} = (u, \ldots, u)$; the sequence $J = \underline{i+1} - A$ is $j_k = i + 1 - a_k$, $1 \leqslant k \leqslant s$. The main result of J. Emsalem and the author in [EmI] – see Lemma E in Section 1.3 below – implies

LEMMA A. *If A is a sequence of s natural numbers, i is an integer greater than any a_u, and $J = \underline{i+1} - A$, then*

$$\text{HPOINTS}\,(s, A, r)_i = \dim_k \mathfrak{R}_i - \text{HPOWLIN}\,(s, J, r)_i. \tag{8}$$

DEFINITION 0.3. Points Fröberg error. If A is a sequence of s natural numbers, we define a new function $G(s, A, r)$ by

$$G(s, A, r)_i = r_i - F(s, \underline{i+1} - A, r)_i, \tag{9a}$$

and we define the *points Fröberg error sequence* by

$$\text{PFER}\,(s, A, r) = G(s, A, r) - \text{HPOINTS}\,(s, A, r). \tag{9b}$$

LEMMA 0.4 *If $J = \underline{i+1} - A$, we have*

$$\text{PFER}\,(s, A, r)_i = \text{FER}\,(s, J, r)_i + \text{LD}\,(s, J, r)_i. \tag{9c}$$

If FER$(s, J, r)_i \geqslant 0$ *then* PFER $(s, A, r)_i \geqslant 0$. *The Weak Froberg Conjecture implies that $G(s, A, r)$ is an upper bound for* HPOINTS (s, A, r).

Proof. Formula (9c) is immediate from (7a), (7b), (7c), (9a) and (9b). This and LD $(s, J, r) \geqslant 0$ imply the last statement.

We denote the Hilbert function of the ideal determined by μ generic points in $\mathbb{P}^{r-1}$ by HGP(μ, r): it satisfies

$$\text{HGP } (\mu, r)_i = \min(\mu, \dim_k R_i). \tag{9d}$$

A punctual subscheme Z of $\mathbb{P}^n$ determined by the ideal $\mathfrak{J}$ in R is said to be in 'μ-generic position' if $H(R/\mathfrak{J}) = \text{HGP } (\mu, R)$. The ideal $\mathfrak{J}_P^{(A)}$ has multiplicity $\mu = \mu(A) = \Sigma_u \dim_k R_{a_u - 1}$. Subschemes of the form $Z_{P,A} = \text{Spec}(R/\mathfrak{J}_P^{(A)})$ are smoothable, since they are defined locally at each point P_i by a monomial ideal. Thus we have HPOINTS $(s, A, r) \leqslant \text{HGP}(\mu(A), r)$. When are they different? Here for simplicity we consider the case $A = \underline{a}$ of equal vanishing orders. It is easy to see that

LEMMA 0.5 *If* $s \geqslant 2^{r-1}$, *then* $G(s, a, r) = \text{HGP } (\mu, r)$.

Thus, the conjectured upper bound $G(s, a, r)$ for HPOINTS (s, a, r) is of interest on $\mathbb{P}^n$ only for $s < 2^n, n = r - 1$. Since HPOWLIN (s, j, r) is known when $s \leqslant n + 2$ (see Remark 1.0), we usually will omit that case henceforth. Outside of these known cases, $n = 3$ is the lowest embedding dimension where $G(s, a, r) \neq \text{HGP } (\mu, r)$. When $n = 3$, and $s = 6$ or 7 then HPOINTS $(s, A, r) < \text{HGP } (\mu(A), r)$ for most integers 'a'. If P is 6 points in general position on $\mathbb{P}^3$, the upper bound $G(6, 10, 4)$ for HPOINTS $(6, 10, 4)$ prevents $Z_{P,10}$ from being in μ-generic position (See Example 1.5.2.). On $\mathbb{P}^4$ we have $G(s, a, 5)$ is different from HGP $(\mu, 5)$ when $7 \leqslant s < 16$, for most integers a.

Remark. All the evidence we've seen points to FER $(s, J, r)_i$ being zero, if char $(k) = 0$ or char $(k) > i$, but this is known only in a limited set of cases (see Lemma B in Section 1.1.). Calculation with the computer algebra program 'Macaulay' suggests the following Conjecture about HPOINTS (s, a, r) for equal vanishing orders

MAIN CONJECTURE 0.6 Assume that char $(k) = 0$, or is larger than any degree i considered. Then the points Fröberg error PFER $(s, a, r)(= G(s, a, r) -$ HPOINTS $(s, a, r))$ is zero unless $s = r + 2$ or $r + 3$, or (s, r) =(7, 3), (8, 3), (9, 4), or (14, 5).

When $r = 3$, and $s \leqslant 9$, HPOINTS (s, a, r) is known (see [Ha2], [H2]). We thank B. Harbourne for pointing out the exceptional pairs (7, 3) and (8, 3) for points in $\mathbb{P}^2$.

When $r \geqslant 4$ and $s = r + 2$ or $r + 3$, or in the cases (s, r) =(9, 4) or (14, 5), there is ample evidence that PFER (s, a, r) and LD (s, j, r) are *nonzero* in general: see Examples 1.5.2, 1.8, 1.9A, B, 2.6B and the $\delta \neq 0$ entries of Tables III, IV, and V, corresponding to (s, r) =(8,6), (6,4) or (7,4) and (11,9). In a sequel paper we will make specific conjectures for the nonzero values of PFER (s, a, r) and LD (s, j, r), based on extensive calculation [I5].

In using 'PFER (s, A, r)' and 'points Fröberg error' to denote the difference between HPOINTS (s, A, r) and the function $G(s, A, r)$, we do not mean to imply that R. Fröberg suggested that PFER or LD should be zero. I am thankful to R. Fröberg and J. Hollman for making many of the calculations contributing to the Main Conjecture.

Outline of results. We prove the Weak Fröberg Conjecture for HPOWLIN $(s, j, r)_i$ if $i \leqslant 2j$ and for some other cases, using known results on the Strong Fröberg Conjecture. (Theorem 1.1A). Our work implies the following Theorem (see Sect. 1.4.). Let $r_i = \dim_k R_i$, and recall that $\mu(aP) = s \cdot \dim_k R_{a-1} =$ degree $(\mathfrak{I}_P^{(a)})$. If $s < 2^{r-1}$ will say that a is *sufficiently large* for (s, r) if both

$$sr_{a-2} < r_{2a-3}, \quad \text{and} \quad s \cdot r_{a-1} - \binom{s}{2} < r_{2a-2}. \tag{10}$$

THEOREM I. *If $s < 2^{r-1}$ and a is sufficiently large for (s, r), if P is a set of s points on $\mathbb{P}^{r-1}$ and $\mu = \mu\ (aP)$, then $\mathfrak{I}_P^{(a)}$ is never in μ-generic position. In particular, for such triples (s, a, r),*

$$\text{HPOINTS } (s, a, r)_{2a-2} \leqslant \mu(\mathfrak{I}_P^{(a)}) - \binom{s}{2} < \text{HGP } (\mu, r)_{2a-2}. \tag{11}$$

Also, $\mathfrak{I}_P^{(a)}$ is not $(2a - 1)$ regular.

EXAMPLE 0.7. Let P be a set of nine generic points on $\mathbb{P}^4$. Theorem I implies that when $a \geqslant 8$, then $\mathfrak{I}_P^{(a)}$ is not in $\mu(aP)$-generic position, since HPOINTS $(9, a, 5)_{2a-2} < \mu(\mathfrak{I}_P^{(a)})$. However, if $i \geqslant 2a - 1, G(9, a, 5)_i = \mu(\mathfrak{I}_P^{(a)})$. Here, when $a = 8, \mu = 9(330)$.

We have the following simple calculation of the putative upper bound $G(s, a, r)$ for HPOINTS, when s satisfies

$$(3/2)^{r-1} \leqslant s < 2^{r-1}. \tag{12}$$

Let $r_i = \dim_k R_i = \binom{r+i-1}{r-1}$, interpreted as 0 if $i < 0$.

LEMMA 0.8 *If s satisfies* (12), *then for all i*

$$G(s, a, r)_i = \text{Min}\left(r_i, sr_{a-1} - \binom{s}{2} r_{2(a-1)-i}\right). \tag{13}$$

Furthermore, if s satisfies (12), *if PFER $(s, a, r)_i \geqslant 0$, and if P is any set of s points of $\mathbb{P}^{r-1}$, then*

$$\dim_k(\mathfrak{I}_P^{(a)})_i \geqslant \text{Max}\left(0, r_i - sr_{a-1} + \binom{s}{2} r_{2(a-1)-i}\right). \tag{14}$$

Proof. This is a special case of Theorem 2.2.

EXAMPLE 0.9. When (s, r) =(9, 5), and $a = 20$, and P is a generic set of 9 points in $\mathbb{P}^4$, calculation in 'Macaulay' shows $H(R/\mathfrak{I}_P^{(20)}) = G(9, 20, 5)$, so PFER $(9, 20, 5) = 0$. By (14) and (14a),

$$\nu(\mathfrak{I}_P^{(20)}) = 35, \qquad \mathrm{reg}(\mathfrak{I}_P^{(20)}) = 40, \qquad \mu = \deg(\mathfrak{I}_P^{(20)}) = 9r_{19} = 95634,$$

$$\begin{aligned} G(9, 20, 5)_{35,\ldots,39} &= (95634 - 35(36), 95634 - 15(36), 95634 - 5(36), \\ &\qquad 95634 - 36, 95634), \end{aligned}$$

$$\begin{aligned} &\mathrm{HGP}\ (\mu, 5)_{35,\ldots 39} - \mathrm{HPOINTS}\ (9, 20, 5)_{35,\ldots,39} \\ &\quad = (35 \cdot 36, 15 \cdot 36, 5 \cdot 36, 36, 0), \end{aligned}$$

$$\dim_k(\mathfrak{I}_P^{(20)})_{(35,\ldots,39)} = (3816, 12235, 21755, 32271, 43715).$$

In Section 2.1 we study further the properties of $G(s, a, r)$, the putative upper bound for HPOINTS (s, a, r). We show that, given only a, the degrees i fall into Koszul intervals S_a; in the region S_a the bound $G(s, a, r)_i$ is polynomial in i (Theorem 2.2, Corollary 2.3).

The Main Conjecture 0.6 implies

CONJECTURE N. Suppose $2 \leqslant n$, and let $P_1, \ldots, P_s$ be independent generic points of $\mathbb{P}^n$. Suppose $s \geqslant \max(n + 5, 2^n)$, and $(s, n) \neq (7, 2), (8, 2), (9, 3)$, or (14, 4). If a hypersurface of degree d passes through each of the points with multiplicity $a (> 0)$, then d/a is greater than $\sqrt[n]{s}$.

M. Nagata made this conjecture in the case $n = 2$ and showed it when s is a perfect square; he applied the result in his counterexample to Hilbert's 14th problem [N]. Note that Conjecture N concerns only the order ORD(HPOINTS (s, a, r)), the smallest degree d for which $(\mathfrak{I}_P^{(a)})_d \neq 0$, for P a generic set of s points of $\mathbb{P}^n$. Because of the inversion $a \to j = i + 1 - a$ in (8), roughly speaking the order of HPOINTS (s, a, r) is the integer d that is the *socle degree* of HPOWLIN (s, j, r), $j \cong d + 1 - a$. Here, the *socle degree* of HPOWLIN (s, j, r) is the largest degree i for with HPOWLIN $(s, j, r) \neq 0$.

In Section 2.2 we first define $b_{s,r} = \lim_{j \to \infty} \mathrm{SOCDEG}(F(s, j, r))/j$, and determine $b_{s,r}$ (Proposition 2.8). We then determine the asymptotic ratio $c_{s,r}$ for ORD (HPOINTS $(s, a, r)) = c_{s,r}a$ (Theorem 2.11). When $s < 2^n$, then $c_{s,r} < \sqrt[n]{s}, n = r - 1$. If the Weak Fröberg Conjecture is satisfied, $n > 2$, and $s < 2^n$ the conclusion of Conjecture N must be replaced by '$d/a \geqslant c_{s,r}$', where $c_{s,r}$ is strictly smaller than $\sqrt[n]{s}$ (see Conjecture N' and Example 2.13).

EXAMPLE 0.9. If PFER $(9, a, 5) = 0$, then the order $\nu(\mathfrak{I}_P^{(a)})$ is asymptotic to $c_{9,5}a$, where $c_{9,5} \approx 1.721541987$. Here $c_{9,5} = b_{9,5}/(b_{9,5} - 1)$, where $b_{9,5} \approx 2.385920733$

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is the real root of $x^4 - 9(x-1)^4 + 36(x-2)^4 = 0$ between 2 and 3. The estimate, order $(\mathfrak{I}_P^{(a)}) = \lceil 1.721541987a \rceil$, is in fact very accurate, predicting for a =7, 8, 10, 20, 40, the actual orders 13, 14, 18, 35, and 69, respectively, obtained by computer calculation.

We also show,

COROLLARY 2.14 *Assuming WFC, if P is any set of s points in* $\mathbb{P}^n$, *the degrees* i *for which* $H(R/\mathfrak{I}_P^{(a)})_i < \mathrm{HGP}(\mu, r)_i = \min(\mu, r_i)$, *includes an interval asymptotic, for large a, to*

$$c_{s,r}a \leqslant i \leqslant 2a - 2. \tag{15}$$

In Section 3, we apply our results to obtain lower bounds on the dimension of certain families of spline functions, on suitable polyhedrons containing the origin of Euclidean space.

Some of the main results of this article were announced in [I4]. It may be read independently of [EmI], on which it depends; it does not depend on [I3], its immediate predecessor in a series of articles using Macaulay's inverse systems.

1. Bounds for Hilbert functions of vanishing ideals

1.1. Known results on the Fröberg Conjectures

The Strong Fröberg Conjecture states that FER $(s, J, r)_i$ is zero. R. Stanley, R. Fröberg, D. Anick, M. Hochster, D. Laksov, F. Fröberg, J. Hollman, and M. Aubry have shown partial results, the most extensive of which are those of D. Anick, when r = 3 and, M. Aubry for arbitrary r and special values of i.

LEMMA B. *The Strong Fröberg Conjecture FER* $(s, J, r) = 0$ *is known in the following cases*

$s \leqslant r$ *(Obvious as* $(f_1, \dots, f_s)$ *is a complete intersection);*
$s = r + 1$ *(R. Stanley,* 1984 *(reported p.* 367 *of* [*I2*]));
$r = 2$ *(R. Fröberg,* 1985, *Section* 5 *of* [*F*]);*
$r = 3$ *(D. Anick,* 1986, [*An*]);

The equal degree Strong Fröberg Conjecture FER $(s, j, r)_i = 0$ *is known in the following additional cases*

$i = j + 1$ *(M. Hochster and D. Laksov,* 1987, [*HL*]);
$r \leqslant 11$ *and* $j = 2$ *(R. Fröberg and J. Hollman,* 1993, [*FH*]);
$r \leqslant 8$ *and* $j = 3$ (" " [*FH*]); *and*

* R. Fröberg shows in [*F*] that suitable monomial ideals $M = (f_1, \dots, f_s)$ in $k[x, y]$ satisfy $H(R/M) = F(s, J, 2)$. The author in Theorem 4.3 and Propositions 4.6, 4.7 of [I1] characterized the possible Hilbert functions $H(R/(V))$ of graded ideals in $R = k[x, y]$ having s generators of degree j. This result also shows that $\mathrm{HGEN}(s, j, 2) = F(s, j, 2)$.

$i = j + \delta$, *satisfying* (1.1), $\delta \leqslant j$, $r \geqslant 4$ (*M. Aubry*, 1994 [*Au*]):

$$j \geqslant 2\delta \frac{r-1}{\sqrt[r-1]{(r-1)!}} - \delta + \delta^2 \frac{1}{\sqrt[r-2]{(r-2)!}} + \frac{(r-1)^2}{\sqrt[r-1]{(r-1)!}} - r + 5. \quad (1.1)$$

REMARK 1.0. In several cases LFER (s, J, r) is known to be zero. First, J. Alexander and A. Hirschowitz have shown that PFER $(s, 2, r) = 0$ except for the four classically known exceptions (s, a, r) =(5, 2, 3), (9, 2, 4), (14, 2, 5), and (7, 2, 5), where the value is one (see [A], [AH1], [AH2]; they gave a shorter proof for $i \geqslant 5$ in [AH3]; later, K. Chandler gave a still shorter proof for $i \geqslant 4$ [Ch1]). By a classically known case of Lemma A (see [T], [I3]), their result shows that LFER $(s, j, r)_{j+1} = 0$ except for $(s, j+1, r)$ =(5, 4, 3), (9, 4, 4), (14, 4, 5), and (7, 3, 5), for which LFER $(s, j+1, r) = 1$. This and a simple calculation in the exceptional cases implies that FER $(s, j, r)_{j+1} = 0$, for all triples (s, j, r). K. Chandler has recently shown that PFER $(s, 3, r)_i = 0$ if $i \geqslant 6$ except for the exceptional cases corresponding to those of Conjecture 0.4 [Ch3]. Likewise, a result of R. Stanley shows that LFER $(s, J, r) = 0$ if $s = r + 1$:

LEMMA C. (R. Stanley). *If* $s \leq r + 1$ *then* FER $(s, J, r) =$ LFER $(s, J, r) = 0$.

Proof. The known Hilbert function of complete intersections $(L_1^{j_1}, \dots, L_s^{j_s})$ handles the case $s \leqslant r$. When $s = r + 1$ R. Stanley's proof concerning thin algebras, quoted p. 367 of [I2], applies also to thin power algebras: the strong Lefschetz theorem on the cohomology ring $\beta = H^*(\mathbb{P}) = \mathcal{R}/(X_1^{j_1}, \dots, X_r^{j_r})$ of a product $\mathbb{P} = \mathbb{P}^{j_1 - 1} \times \cdots \times \mathbb{P}^{j_r - 1}$ of projective spaces, shows that the Hilbert function of the Artin algebra $A = B/(L_{r+1}^{j_{r+1}})$ is the Fröberg function $F(r + 1, J, r)$.

REMARK. The Weak Fröberg Conjecture that FER $(s, j, r)_i \geqslant 0$ is easily shown to be true in degrees $i \leqslant 2j - 1$, and is of course true under the hypotheses of Lemma B. K. Chandler shows LFER$(s, z, r) \leqslant 0$ for many s in [Ch2].

R. Fröberg showed in [F]

LEMMA D. *If the triple* (s, a, r) *is fixed, then* FER $(s, a, r)_i \geqslant 0$ *for* $i = \min\{k \mid \text{FER } (s, a, r)_k \neq 0\}$.

Fröberg thus proved a *lexicographic inequality* $F(s, j, r) \geqslant^l$ HGEN (s, j, r).* But the WFC, surprisingly, remains open.

1.2. STRONG FRÖBERG IMPLIES WEAK FRÖBERG

Recall that if $V \subset R_j$, then $R_i V$ denotes the vector space span $\langle gh \mid g \in R_i, h \in V\rangle$; also (V) denotes the ideal generated by V. We denote the Hilbert function

* I am indebted to R. Fröberg and G. Valla for informing me that D. Anick's assertion in [A], that the weak Fröberg conjecture was a theorem due to R. Fröberg, resulted from his misreading of R. Fröberg's result in [F].

$H(R/(V))$ by $T(V)$. We use the lexicographic order on degree-d monomials of $\mathfrak{R}$: thus

$$X_1^d \geqslant X_1^{d-1}X_2 \geqslant X_1^{d-1}X_3 \geqslant \cdots \geqslant X_1^{d-1}X_n \geqslant X_1^{d-2}X_2^2 \geqslant \cdots \geqslant X_n^d.$$

We let $\mathrm{IN}(f)$ denote the initial monomial of an element $f \in \mathfrak{R}_j$. If $V \subset \mathfrak{R}_j$ is a vector subspace, we let $IN(V) = \langle\{\mathrm{IN}(f) \mid f \in V\}\rangle$. Let IN $(s,j,r) = \langle\mu_1,\ldots,\mu_s\rangle$ be the vector space span of the first s monomials $\mu_1,\ldots,\mu_s$ of degree j. We let $\mathrm{LAST}(s,j,r) = \langle\mu_{s+1},\ldots,\mu_N\rangle$, where $N = r_j = \dim_k\mathfrak{R}_j$; $\mathrm{LAST}(s,j,r)$ is the span of the last $N-s$ degree-j monomials, and is a complementary space to IN (s,j,r).

The following Theorem allows us to use any known case of the Strong Fröberg Conjecture FER $(s-1,j,r)_i = 0$, when $i < 2j$ to show the Weak Fröberg Conjecture FER $(s,j,r)_{i+j} \geqslant 0$ in the higher degree $i+j$. Strangely enough, this is helpful for us. In the proof we isolate the case $i = 2j$ as we use only this case in Theorem I, and the proof is simpler there than for the general case.

THEOREM 1.1A. *If* $2j \leqslant i < 3j$, *then* $\mathrm{FER}(s-1,j,r)_{i-j} = 0$ *implies* $\mathrm{FER}(s,j,r)_i \geqslant 0$. *If* $i \leq 2j+1$, *we have* $\mathrm{FER}(s,j,r)_i \geqslant 0$.

THEOREM 1.1B. *If* $J = (j_1,\ldots,j_s), j_1 \leqslant \cdots \leqslant j_s$, *and* $2j_1 \leqslant i \leqslant 3j_1$, *then* $\mathrm{FER}(s-1,J-j_1,r)_{i-j_1} = 0$ *implies* $\mathrm{FER}(s,J,r)_i \geqslant 0$.

Proof of A. Suppose that $V = (f_1,\ldots,f_s)$ is a general subspace of R_j (parametrized by a point in a suitable open dense subset of the Grassmannian). The homomorphism $\phi_{i-j,V}$

$$\phi_{i-j,V}\colon \mathfrak{R}_{i-j} \otimes_k V \to \mathfrak{R}_{i-j}V = (V) \cap \mathfrak{R}_i.$$

is evidently surjective. It follows that

$$\dim_k\mathfrak{R}_{i-j}V \leqslant \min\,(s(\dim_k\mathfrak{R}_{i-j}), \dim_k\mathfrak{R}_i). \tag{1.2a}$$

Case (i). When $j \leqslant i < 2j$, we have

$$F(s,j,r)_i = \dim_k\mathfrak{R}_i - \min(s\cdot\dim_k\mathfrak{R}_{i-j}, \dim_k\mathfrak{R}_i)$$

so (1.2a) implies FER $(s,j,r)_i \geqslant 0$ for these values of i.

Case (ii). When $i = 2j$, possibly after deforming V we may assume WOLOG that IN $(V) =$ IN (s,j,r); then, after a change of basis for V we may assume that for $1 \leqslant u \leqslant s, f_u - \mu_u \subset$ LAST (s,j,r). For $1 \leqslant u \leqslant s$ we let $V_u = \langle f_1,\ldots,f_u\rangle$. Consider

$$W_V = (V_1 \otimes_k f_1 \oplus \cdots \oplus V_s \otimes_k f_s) \oplus \big(V \otimes_k \mathrm{LAST}(s,j,r)\big) \subset \mathfrak{R}_j \otimes_k V, \tag{1.2b}$$

a space of dimension

$$\dim_k W_V = (1 + 2 + \cdots + s) + s(r_j - s) = s \cdot r_j - s(s-1)/2.$$

Denote by $\Lambda^2 V$ the exterior power, and consider the sequence

$$\begin{aligned} &\Lambda^2(V) \xrightarrow{\theta} \mathfrak{R}_j \otimes_k V \xrightarrow{m} \mathfrak{R}_j V \to 0, \\ &\theta\colon f_u \wedge f_v \to f_u \otimes f_v - f_v \otimes f_u; m = \phi_{j,v}. \end{aligned} \tag{1.2c}$$

Clearly, the image of θ is in the kernel of the multiplication map m, and W_V is a complementary space to the image of θ, in the sense that $\theta(\Lambda^2 V) + W_V = \mathfrak{R}_j \otimes V$. Thus, the dimension of $\mathfrak{R}_j V$ satisfies

$$\dim_k \mathfrak{R}_j V \leqslant \dim_k W_V = r_j \cdot s - s(s-1)/2. \tag{1.2d}$$

This proves both statements of the Theorem when $i = 2j$.

Case (iii). Suppose that $2j \leqslant i < 3j$, and that $\mathrm{FER}(s-1, j, r)_{i-j} = 0$. Let $\delta = i - 2j$, and suppose that a generic sequence $B = (f_1, \ldots, f_s)$ spans a subspace V of $\mathfrak{R}_j$; we denote by $S_u = S_u(B)$ the span of $(f_1, \ldots, f_{u-1}, f_{u+1}, \ldots, f_s)$. Each length- $(s-1)$ subsequence is generic since the projections are surjective, hence we have

$$T(S_u) = \mathrm{HGEN}\ (s-1, j, r) \quad \text{for each } u \in \{1, \ldots, s\}. \tag{*}$$

If $\mathfrak{R}_\delta \cdot S_u = \mathfrak{R}_{i-j}$ it is trivial to see that $T(V)_\tau = F(s, j, r)_\tau$ for $\tau \geqslant i - j$. So we may assume $\mathfrak{R}_\delta \cdot S_u \neq \mathfrak{R}_{i-j}$ for a pair (V, B) satisfying $(*)$. This and the Strong Fröberg assumption implies that

$$\mathrm{HGEN}(s-1, j, r)_{i-j} = F(s-1, j, r)_{i-j} = \mathfrak{R}_{i-j} - \mathfrak{R}_\delta \cdot (s-1),$$

so for each u, $1 \leqslant u \leqslant s$,

$$\dim_k \mathfrak{R}_\delta \cdot S_u = r_\delta \cdot (s-1) < r_{i-j}, \tag{1.2e}$$

and the multiplication map $\phi_{\delta, S_u} : \mathfrak{R}_\delta \otimes S_u \to \mathfrak{R}_\delta \cdot S_u$ is an injection for each u. Now consider the sequence

$$\begin{aligned} &\mathfrak{R}_\delta \otimes_k \Lambda^2(V) \xrightarrow{\theta} \mathfrak{R}_{i-j} \otimes_k V \xrightarrow{m} \mathfrak{R}_{i-j} V \to 0, \\ &\theta\colon h \otimes f_u \wedge f_v \to h \cdot f_u \otimes f_v - h \cdot f_v \otimes f_u; m = \phi_{i-j,V}. \end{aligned} \tag{1.2f}$$

As in Case (ii), the image of θ is in the kernel of m. We claim that θ is injective. Suppose, by way of contradiction, that

$$\theta\left(\sum_{u<v} h_{uv} \otimes f_u \wedge f_v\right) = 0, \quad \text{with } h_{uv} \in \mathfrak{R}_\delta.$$

Collecting coefficients of f_v we have

$$\sum_v \left(\sum_{u<v} h_{uv} f_u - \sum_{u>v} h_{vu} f_u\right) \otimes f_v = 0 \quad \text{in } \mathfrak{R}_{i-j} \otimes V,$$

thus for each v,

$$\sum_{u<v} h_{uv} f_u - \sum_{u>v} h_{vu} f_u = 0.$$

By the injectivity of $\mathfrak{R}_\delta \otimes S_v \to \mathfrak{R}_{i-j}$, each coefficient $h_{uv} = 0$. This completes the proof of the injectivity of θ in (1.2f). As $\phi_{i-j,V}$ is surjective, it follows that

$$\dim_k \mathfrak{R}_{i-j} V \leqslant r_{i-j} s - (r_\delta) s(s-1)/2 = r_i - F(s,j,r)_i. \tag{1.2g}$$

This shows that $\dim_k(\mathfrak{R}_i/\mathfrak{R}_{i-j}V) \geqslant F(s,j,r)_i$, hence that $\mathrm{FER}(s,j,r)_i \geqslant 0$. This completes the proof of Theorem 1.1A.

The proof – which we omit – of Theorem 1.1B is entirely similar, but requires a more complex notation.

As a consequence of Theorem 1.1A we have

COROLLARY 1.2. *Cases for which the weak Fröberg conjecture is known. Suppose* $r \geqslant 4$. *We have* $\mathrm{FER}(s,j,r)_i \geqslant 0$

(A) *When* $i \leqslant 2j+1$, *or*
(B) *When both of the following conditions are satisfied*
  (i) $2j+1 \leqslant i < 3j$ *and*
  (ii) *The integers* $\delta = i - 2j$ *and* j *satisfy* (1.1).

Proof. WFC for $i \leqslant 2j-1$ is obvious; WFC for $i = 2j, 2j+1$, and in case B above, are immediate from Theorem 1.1 and Lemma B.

1.3. The Macaulay duality: power ideals and fat points

In this section we review the Macaulay duality behind Lemma A of the Introduction. If P is any set of s points of $\mathbb{P}^n, n = r-1$, then the Hilbert function $H(R/\mathfrak{I}_P^{(a)})$ of the a-th symbolic power of $\mathfrak{I}_P$ may be calculated from the Hilbert functions $H(\mathfrak{R}/(L_1^j, \ldots, L_s^j))$, where $L_1, \ldots, L_s$ is the corresponding set of linear forms. Recall that $\mathfrak{R} = k[X_1, \ldots, X_r]$ denotes the polynomial ring over an infinite field k and that $R = k[x_1, \ldots, x_r]$ denotes a second polynomial ring. Here R acts on $\mathfrak{R}$ as a ring of partial differential operators, giving a variant of the Macaulay or Matlis duality [Mac]. If $h \in R, f \in \mathfrak{R}$, we have $h \circ f = h(\partial \cdot / \partial X_1, \ldots, \partial \cdot / \partial X_r) \circ f$. We assume henceforth for simplicity that the characteristic of k is zero, or is larger than any degree i being considered, and that $r \geqslant 2$. * Recall that the *power*

* If the characteristic were less than the degree i, we would need to replace $\mathfrak{R}$ by the divided power ring $\mathfrak{D}$, and use the contraction action of R on $\mathfrak{D}$ in Lemma E below. For further discussion see [EmI].

ideal $(L_1^{j_1}, \dots, L_s^{j_s})$ in $\mathfrak{R}$ is generated by powers of a set of s linear homogeneous elements of $\mathfrak{R}$; the vanishing ideal $\mathfrak{I}_P^{(A)}$, $A = (a_1, \dots, a_s)$ in R is defined by the condition that $h \in \mathfrak{I}_P^{(A)}$ if and only if h vanishes to order at least a at each point of the set $P = (P_1, \dots, P_s)$ of s distinct points in $\mathbb{P}^{r-1}$. Such vanishing ideals are called *fat point ideals* by A. Geramita et al.

The point $P = (p_1 : \cdots : p_r) \in \mathbb{P}^n$ corresponds to the one dimensional vector space $\langle L \rangle = \langle p_1 X_1 + \cdots + p_r X^r \rangle$: we say that P corresponds to the linear form L. If $A = (a_1, \dots, a_s)$, we let $J = \underline{i+1} - a = (i+1-a_1, \dots, i+1-a_s)$. If $V \subset R_i$ we denote by Ann (V) its annihilator in $\mathfrak{R}_i$: Ann $(V) = \{f \in \mathfrak{R}_i \mid V \circ f = 0\}$. We denote by $L_i^J = (L_1^{j_1}, \dots, L_S^{j_s})_i$ the span of $(\mathfrak{R}_{a_1-1} L_1^{i+1-a_1}, \dots, \mathfrak{R}_{a_s-1} L_1^{i+1-a_s})$ in $\mathfrak{R}_i$. J. Emsalem and the author showed in [EmI],

LEMMA E. *If the points P_i correspond to the linear forms L_i, then the ith graded piece $L_i^J = (L_1^{j_1}, \dots, L_s^{j_s})_i$, $J = \underline{i+1} - a$, satisfies*

$$(L_1^{i+1-a_1}, \dots, L_s^{i+1-a_s})_i = \text{Ann}\,((\mathfrak{I}_P^{(A)})_i) \cap \mathfrak{R}_i. \tag{*}$$

Lemma E implies Lemma A of Section 0, *that HPOINTS* $(s, A, r)_i = \dim_k R_i -$ HPOWLIN (s, J, r).

EXAMPLE 1.3. If $P_1 = (1,0,0), P_2 = (0,1,0), P_3 = (0,0,1), P_4 = (1,2,3)$ in $\mathbb{P}^2 = \text{Proj}\,(k[x,y,z])$, then $L = (L_1, \dots, L_4) = (X, Y, Z, X + 2Y + 3Z)$. Taking $A = (3,3,3,3)$, we have

$$(L^4)_6 = (X^4, Y^4, Z^4, (X + 2Y + 3Z)^4)_6 = \text{Ann}(m_{p_1}^3 \cap \cdots \cap m_{p_4}^3) \cap \mathfrak{R}_6.$$

We next show that HGEN (s, J, r) is attained. Fixing r, we let $n_i = \dim_k \mathfrak{R}_i - 1$.

LEMMA 1.4. Generically chosen functions determine a thin algebra. *There is an open dense subset* $\mathbf{TA}(s, J, r) \subset \hat{\mathbb{P}} = \hat{\mathbb{P}}^{n_1} \times \cdots \times \hat{\mathbb{P}}^{n_s}$ *such that if the sequence* $\langle f_1 \rangle, \dots, \langle f_s \rangle \in \mathbf{TA}\ (s, J, r)$, *then the ideal* $F = (f_1, \dots, f_s)$, *then the ideal* $F = (f_1, \dots, f_s)$ *satisfies* $H(R/F) =$ HGEN (s, J, r).

Proof. By the minimality of HGEN (s, J, r), *the equality* $\dim_k(R/F)_i =$ HGEN$(s, j, r)_i$ *is equivalent to the inequality*

$$\dim_k(R/F)_i < \text{HGEN}\,(s, J, r)_i + 1, \tag{1.2}$$

which defines an open dense subset $U_i(s, J, r)$ of the irreducible variety $\hat{\mathbb{P}}$. It is well known that, given (s, J, r) only a finite number of sequences occur as Hilbert functions for the ideals F of generator degrees J (see [Be]). It follows that there is a finite collection $\{i_1, \dots, i_t\}$ of indices such that

$$H(R/V) = \text{HGEN}\,(s, J, r) \leftrightarrow F \in U_{i_1}(s, j, r) \cap \cdots \cap U_{i_t}(s, j, r).$$

Since $\hat{\mathbb{P}}$ is irreducible the Lemma follows.

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REMARK 1.4.1. D. Berman showed in [Be] that there are a finite number of 'complete Hilbert functions' possible for a vector space V of degree-j forms. The complete Hilbert function includes the dimension of any vector space, constructed beginning from V by any sequence of operations of the form W goes to $R_i W$ or W goes to $W : R_i = \{f \mid R_i f \subset V\}$. It is a finer invariant than the Hilbert function of $R/(V)$. The equal degree case of the proof of Lemma 1.4 could be refined to show that there is an 'extremal complete Hilbert function' CH (s, j, r) that is attained for vector spaces V in a dense open subset TCH $(s, j, r) \subset \mathbf{GRASS}(\mathfrak{R}_j, s)$.

The following Lemmas for powers of linear forms are readily shown.

LEMMA 1.4.2 *Given* (s, J, r), *there is an open dense subset UPL* (s, J, r) *of* $\hat{\mathbb{P}}^n \times \cdots \times \hat{\mathbb{P}}^n$ *such that if the sequence* $\langle L_1 \rangle, \ldots, \langle L_s \rangle$ *of one-dimensional vector spaces in* $\mathbb{P}(\mathfrak{R}_1)$ *is in UPL* (s, J, r), *and* $L^J = \langle L_1^{j_1}, \ldots, L_s^{j_s} \rangle$, *then* $H(R/L^J) =$ HPOWLIN (s, J, r).

LEMMA 1.4.3. *If L is any set of s linear elements of* $\mathfrak{R}$, $\dim_k \mathfrak{R}_u L^{j+1} \geqslant \dim_k \mathfrak{R}_u L^j$. *If* $\dim_k \mathfrak{R}_u L^j = s \cdot \dim_k \mathfrak{R}_u$, *then* $\dim_k \mathfrak{R}_u L^j = s \cdot \dim_k \mathfrak{R}_u$.

Proof. Set $a = u + 1$ and let P be the set of s points in $\mathbb{P}^n$ corresponding to L. $R/\mathfrak{J}_P^{(a)}$ is Cohen–Macaulay of dimension one, so $H(R/\mathfrak{J}_P^{(a)})$ is nondecreasing, and stabilizes at the value $\mu = s \cdot \dim_k \mathfrak{R}_u$ (see [GM]). This with Lemma E implies Lemma 1.4.3.

1.4. Higher Order Vanishing ideals are not in μ-generic position

Recall the notation $r_i = \dim_k R_i$, and $\mu(aP) = s \cdot \dim_k R_{a-1} = \text{degree}\ (\mathfrak{J}_P^{(a)})$. If $s < 2^{r-1}$ recall that a is *sufficiently large* for (s, r) if both

$$s r_{a-2} < r_{2a-3}, \quad \text{and} \quad s \cdot r_{a-1} - \binom{s}{2} < r_{2a-2}. \tag{1.3}$$

LEMMA 1.5.1. *If* $s < r_{a-1}, r \geqslant 4$, *and* $a \geqslant 3$, *the second inequality of* (1.3) *implies the first. If* (s, r) *satisfy* $r \geqslant 4$, *and* $s < 2^{r-1}$, *there is an integer* $N(s, r)$ *such that* $a \geqslant N(s, r)$ *implies* (s, a, r) *satisfies* (1.3).

Proof. Assume $s < r_{a-1}$. The first statement follows from solving $s \cdot r_{a-1} - s(s-1)/2 = r_{2a-2}$ for $s_0 = r_{2a-2}/r_{a-1} - \varepsilon, \varepsilon > 0$; the inequality $s_0 r_{a-2} < r_{2a-3}$ is implied by the inequality $r_{2a-2} \cdot r_{a-2} < r_{2a-3} \cdot r_{a-1}$, which is well known.

Since $r_i = \binom{r+i-1}{i} = i^{r-1}/(r-1)! + O(i^{r-2})$ the leading terms of the inequality $s \cdot r_{a-1} < r_{2a-2}$ can be written, taking $c = 1/(r-1)!$,

$$cs \cdot (a-1)^{r-1} < c(2a-2)^{r-1} \bmod O(a^{r-2}). \tag{1.3a}$$

Since $s < 2^{r-1}$ it follows that there is an integer $N(s, r)$ such that (1.3) is satisfied for $a > N(s, r)$.

THEOREM I. If $s < 2^{r-1}$ and a is sufficiently large for (s, r), if P is a set of s points on $\mathbb{P}^{r-1}$ and $\mu = \mu(aP)$ then $\mathfrak{I}_P^{(a)}$ is never in μ-generic position. In particular, for such triples (s, a, r), we have

$$\text{HPOINTS }(s, a, r)_{2a-2} \leqslant \mu(\mathfrak{I}_P^{(a)}) - \binom{s}{2} < \text{HGP}(\mu, r)_{2a-2}. \tag{1.4}$$

Also, $\mathfrak{I}_P^{(a)}$ is not $(2a-1)$ regular.

Proof. By Lemma A and (7b),(7c)

$$\begin{aligned}
&\text{HPOINTS }(s, a, r)_{2a-2}\\
&\quad = \dim_k R_{2a-2} - \text{HPOWLIN }(s, a-1, r)_{2a-2}\\
&\quad = \dim_k R_{2a-2} - F(s, a-1, r)_{2a-2} - \text{LFER }(s, a-1, r)_{2a-2}\\
&\quad \leqslant \dim_k R_{2a-2} - F(s, a-1, r)_{2a-2} - \text{FER }(s, a-1, r)_{2a-2}.
\end{aligned}$$

By the definition of the Fröberg function (0.6), and by (1.3), we have $F(s, a-1, r)_{2a-2} = F'(s, a-1, r)_{2a-2}$, hence

$$\dim_k R_{2a-2} - F(s, a-1, r)_{2a-2} = \mu(\mathfrak{I}_P^{(a)}) - \binom{s}{2}. \tag{1.6}$$

By Theorem 1.1A, FER $(s, a-1, r)_{2a-2} \geqslant 0$, so (1.5) and (1.6) imply (1.4). Since $\mu(\mathfrak{I}_P^{(a)}) = s \cdot r_{a-1}$ it follows that whenever a satisfies (1.3) then the ideal $\mathfrak{I}_P^{(a)}$ is not in μ-generic position. An ideal $\mathfrak{I}$ in R of dimension one, arising from a length μ zero-dimensional scheme on $\mathbb{P}^n$ is i-regular if $\dim_k(R_{i-1}/I_{i-1}) = \mu$. Hence $\mathfrak{I}_P^{(a)}$ is also not $(2a-1)$-regular.

REMARK It follows from Theorem I that if (s, a, r) satisfy $s < 2^{r-1}$ and a is sufficiently large, then given a general set P of s points in $\mathbb{P}^{r-1}$ (lying in a suitable open set of a parameter space,) there is an interval of degrees d (including the value $d = 2a-2$), for which there are degree-d hypersurfaces that vanish to order at least a at each point of P, but for which such vanishing fails to cut out $\mu(\mathfrak{I}_P^{(a)})$ conditions on the vector space $\mathcal{O}_{\mathbb{P}}^{r-1}(d) \cong R_d$, of all degree d hypersurfaces in $\mathbb{P}^{r-1}$. Corollary 2.14 gives a lower bound for the asymptotic length of this interval, assuming WFC.

EXAMPLE 1.5.2 A vanishing ideal at six general points of $\mathbb{P}^3$ that is not in μ-generic position. If $(s, a, r) = (6, 10, 4)$, then the multiplicity $\mu(\mathfrak{I}_P^{(10)}) = 6\dim_k R_9 = 1320$. In degree $i = 18$, $\dim_k \mathfrak{R}_{18} = 1330$. By Theorem I, we have

$$\text{HPOINTS }(6, 10, 4)_{18} \leqslant G(6, 10, 4)_{18} = 1320 - \binom{6}{2} = 1305.$$

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Thus, the ideal $\mathfrak{I}_P^{(10)}$ is not in μ-generic position, nor is it 19-regular. Using 'random' points and the 'Macaulay' symbolic algebra program we calculated HPOWLIN $(6,9,4)_{18} = 60$. By definition,

$$F(6,9,4)_{18} = 1330 - 6\dim_k \mathfrak{R}_9 + \binom{6}{2} = 25.$$

Thus, we have

$$\begin{aligned}
&\text{PFER } (6,10,4)_{18} = \text{LFER } (6,9,4)_{18} \\
&\quad = \text{HPOWLIN } (6,9,4) - F(6,9,4) = 35, \quad \text{and} \\
&\quad \text{HPOINTS } (6,10,4)_{18} = G(6,10,4)_{18} - \text{PFER } (6,10,4)_{18} = 1270.
\end{aligned}$$

REMARK. When $s = r+2$ or $r+3$, computer calculations of many examples indicate that LFER $(s, a-1, r)_{2a-2} =$ PFER $(s, a, r)_{2a-2}$ and is usually nonzero. (See the Main Conjecture 0.6).

EXAMPLE 1.5.3. Twenty-four fat points in $\mathbb{P}^9$ not in μ-generic position, defect zero. Let $(s, a, r) = (24, 4, 10), i = 6$, and let $j = i + 1 - a = 3$. Consider the Hilbert function $H(R/\mathfrak{I}_P^{(4)})$, where P consists of 24 general enough points of $\mathbb{P}^9$. The degree of $\mathfrak{I}_P^{(4)}$ is $\mu = (24)(\dim_k R_3) = 24(220) = 5280$. By Theorem I we have

$$\text{HPOINTS } (24,4,10)_6 \leqslant 5280 - \binom{24}{2} = 5004.$$

Since $r_6 = 5005$ it follows that $\mathfrak{I}_P^{(4)}$ is not in μ-generic position. A calculation in 'Macaulay' (done in characteristic 17), verifies that HPOINTS $(24,4,10)_6 = 5004$. In other degrees $i \neq 6$, HPOINTS $(24,4,10)_i =$ HGP $(5280,10)_i$.

1.5. Upper Bounds for HPOINTS (s, A, r) in Special Cases

Recall that the sequence $G(s,A,r)$ is defined from the Fröberg bounds by $G(s,A,r)_i = \dim_k R_i - F(s, \underline{i+1} - A, r)_i$; in the equal vanishing order case we denote $G(s,A,r)$ by $G(s,a,r)$. In the statement of Theorem 1.6 we list after each case, the authors of the corresponding case of the Strong Fröberg Conjecture needed for the result (see Section 1A for the actual references).

THEOREM 1.6. *Upper bound for the Hilbert function of vanishing ideals. Assume that the field k is algebraically closed of characteristic zero, or of characteristic* $p > j = i + 1 - a$, *and assume* $a \geqslant 2$. *If P is any set of s points of* $\mathbb{P}^{r-1}$, *then the algebra* $R/\mathfrak{I}_P^{(A)}$ *satisfies*

$$H(R/\mathfrak{I}_P^{(A)})_i \leqslant \text{HPOINTS } (s,A,r)_i \leqslant G(s,A,r)_i, \tag{1.7}$$

provided any of the following seven conditions holds

(i) $r \leqslant 3$, (*Fröberg* $r = 2$, *D. Anick*, $r = 3$);
(ii) $s \leqslant r + 1$, (*R. Stanley*).

For the next conditions we assume equal vanishing orders $A = (\underline{a})$.

(iii) $i \geqslant 2a - 3$, (*Hochster–Laksov*);
(iv) $r \geqslant 4, (3a/2) - 1 < i \leqslant 2a - 3$ *and the integers* $\delta = 2a - i - 2$ *and* $j = i + 1 - a$ *satisfy* (1.1). (*M. Aubry*);
(v) $r \leqslant 11$ *and* $i = a + 1; r \leqslant 8$ *and* $i = a + 2$ (*R.Fröberg and J. Hollman*).
(vi) $a \leqslant 4$; *or* $a = 5$ *and* $r \leqslant 11$; *or* $a = 6$ *and* $r \leqslant 8$.
(vii) $i \leqslant a$ (*obvious, as* $j = i + 1 - a \leqslant 1$).

When $s \leqslant r$ *or* $i \leqslant \min\{a_u\}$, *or* $r = 2$ *there is equality in all of* (1.7); *when* $s = r + 1$ *there is equality on the right of* (1.7).

Proof. The first two cases follow from Theorem 1.1B and the first four cases of Lemma B. The third case follows fromTheorem 1.1A and Lemma B in the cases $i \leqslant 2j + 1$ (taking $j = i + 1 - a$). The fourth case follows similarly from (1.1). The fifth is directly from the verification by R. Fröberg and J. Hollman of Strong Fröberg for $j = 2$ when $r \leqslant 11$, or $j = 3$ when $r \leqslant 8$ (without using Theorem 2.1). The sixth is a consequence of cases (i), (iii), and (v). The statements concerning equality in (1.7) arise from the CI case, and Stanley's Lemma C.

We now single out the case related to the Strong Fröberg result of Hochster–Laksov. First, we need

LEMMA 1.6.1. *If* $a \geqslant 3$ *the following inequalities are equivalent to* $F(s, a - 2, r)_{2a-3} = F'(s, a - 2, r)_{2a-3}$

$$s \cdot r_{a-3} < r_{2a-5}, \quad s \cdot r_{a-2} - \binom{s}{r} < r_{2a-4}, \quad s \cdot r_{a-1} - r \cdot \binom{s}{2} < r_{2a-3}. \tag{1.9}$$

Furthermore, if (s, a, r) *satisfy* $r \geqslant 4, a \geqslant 4, s < r_{a-2}$ *then the last inequality of* (1.9) *implies the first two.*

Proof. By (5), $F'(s, a - 2, r)_{2a-3} = F(s, a, r)_{2a-3}$ if for all integers $i, 0 \leqslant i < 2a - 3$, we have $F'(s, a - 2, r)_i > 0$, and $F'(s, a - 2, r)_{2a-3} \geqslant 0$. This condition is empty for $i < a - 2$. If $a - 2 \leqslant i \leqslant 2a - 5$ then

$$F'(s, a - 2, r)_i \leqslant 0 \;\Leftrightarrow\; sr_{i-(a-2)} \geqslant r_i \Rightarrow sr_{a-3} \geqslant r_{2a-5},$$

since for $t = a - 2 > 0, r_u/r_{u+t} \leqslant (r_{u+1}/r_{u+t+1})$. Hence $F'(s, a - 2, r)_{2a-3} = F(s, a - 2, r)_{2a-3} \Leftrightarrow sr_{a-3} \leqslant r_{2a-5}$. Equality in the last formula of (1.9) gives, as in the proof of Corollary 1.5, $s_1 = 0.5 + r_{a-1}/r - \in$; the second inequality for $s = s_1$ is implied by $r_{2a-3}.r_{a-2} < r_{2a-4}.r_{a-1}$ and $a \geqslant 4$. By Corollary 1.5 this implies $sr_{a-3} \leqslant r_{2a-5}$.

COROLLARY 1.6.2. *Upper bound* $G(s,a,r)_i$ *for* $i = 2a-3$. *If* (s,a,r) *satisfies* (1.9) *then*

$$\text{HPOINTS } (s,a,r)_{2a-3} \leqslant G(s,a,r)_{2a-3} = \mu(aP) - r \cdot \binom{s}{2}. \qquad (1.10)$$

Proof. By Lemma 1.6.1 the hypotheses imply $F(s,a-2,r)_{2a-3} = F'(s,a-2,r)_{2a-3}$, thus $F(s,a-2,r)_{2a-3} = r_{2a-3} - s \cdot r_{a-1} + r \cdot \binom{s}{2}$. This and Theorem 1.6 imply (1.10).

EXAMPLE 1.7A. Two fat points not in μ-generic position. Two fat points are rarely in μ-generic position; we illustrate this in a special case $(s,a,r) = (2,3,3)$. By Theorem 1.6, when $P = (p_1, p_2)$ are arbitrary in $\mathbb{P}^2$, we have $\mu(\mathfrak{I}_P^{(3)}) = 2(\dim_k \mathfrak{R}_2) = 12$, but

$$H(R/\mathfrak{I}_P^{(3)}) = G(2,3,3) = (1,3,6,9,11,12,12,\ldots).$$

The values 9 and 11 for $G(2,3,3)$ are given by (1.10) and (1.6). We now use Lemma E to understand these two values for $H(R/\mathfrak{I}_P^{(3)})$. When $\mathbb{P} = ((1,0,0),(0,1,0))$ we have $\mathfrak{I}_P^{(3)} = (y,z)^3 \cap (x,z)^3$. By Lemma E the inverse system

$$(\mathfrak{I}_P^{(3)})_i^{\perp} = (X^{i+1-a}, Y^{i+1-a}) \cap \mathfrak{R}_i \quad \text{so we have}$$

$$(\mathfrak{I}_P^{(3)})_3^{\perp} = (X,Y) \cap \mathfrak{R}_3, \quad \text{of dimension } 9 = G(2,3,3)_3;$$

$$(\mathfrak{I}_P^{(3)})_4^{\perp} = (X^2,Y^2) \cap \mathfrak{R}_4, \quad \text{of dimension } 11 = G(2,3,3)_4.$$

The corresponding homogeneous summends of $\mathfrak{I}_P^{(3)}$ are

$$\mathfrak{I}_P^{(3)}{}_3 = \langle z^3 \rangle, \quad \text{and} \quad \mathfrak{I}_P^{(3)}{}_4 = \langle xz^3, yz^3, xyz^2, z^4 \rangle.$$

EXAMPLE 1.7B. Eight fat points in $\mathbb{P}^5$, $a = 7$. If P is a set of 8 points in $\mathbb{P}^5$, then $\mu(Z_{P,7}) = 8r_6 = 3696$. By (1.10) and (1.6) we have

$$\text{HPOINTS } (8,7,6)_{11,12} \leqslant G(8,7,6)_{11,12} = (3528, 3668),$$

and $G(8,7,6)_i = \text{HGP } (3696,6)_i$ for $i \neq 11, 12$. A computer calculation shows that HPOINTS $(8,7,6) = G(8,7,6)$ except for $i = 10$, where HPOINTS $(8,7,6)_{10} = 2090 < G(8,7,6)_{10} = 3003$, so PFER $(8,7,6)_{10} = 13$.

EXAMPLE 1.7C. Ten fat points in P^5, $a = 7$. Let $(s,a,r) = (10,7,6)$. Consider the scheme $Z_{P,7}$ of order 7 neighborhoods at a set P of 10 points in $\mathbb{P}^5$; here the multiplicity $\mu(Z_{P,7}) = 10r_6 = 4620$. By (1.10) and (1.6) we have

$$\text{HPOINTS } (10,7,6)_{11,12} \leqslant (4350, 4575),$$

$$\text{but HGP } (4620,6)_{11,12} = (4368, 4620).$$

$G(10,7,6)_i = \text{HGP}\ (4620,6)_i$ for $i \neq 11,12$. The Main Conjecture predicts that HPOINTS $(10,7,6) = G(10,7,6)$.

REMARK. As s decreases from 2^{r-1}, the difference HGP $(\mu, r) - G(s,a,r)$ becomes proportionally greater, can be positive for smaller values of a, and is positive for more values of i (see Corollary 2.14).

REMARK 1.7.1. Every power algebra is thin when $r = 2$. For any set of distinct points $P = (P_1, \ldots, P_s)$ in $\mathbb{P}^1$, and set of orders $(a_1, \ldots, a_s)$ the ideal $\mathfrak{I}_P^{(A)}$ is principal, with generator $g_{P,A}$ of degree $\mu(\mathfrak{I}_P^{(A)})$. It is easy to see that $\mathfrak{I}_P^{(A)}$ is in μ-generic position, and $R/\mathfrak{I}_P^{(A)}$ is a thin algebra. Thus, we have LD $(s,j,2) =$ FER $(s,j,2) =$ LFER $(s,j,2) = 0$. If $n = \sum a_k$ satisfies $n \leqslant i+1 = \dim_k \mathfrak{R}_i$, and if $j_k = i+1-a_k$, then the vector subspace of $\mathfrak{R}_i$, $\mathfrak{R}_{a_1-1}L_1^{j_1} \oplus \cdots \oplus \mathfrak{R}_{a_s-1}L_s^{j_s}$ is a direct sum. This statement is an avatar of a classical 'Jordan Lemma' (Appendix III of [GY]).

1.6. THE LINEAR DEFECT, AND HPOINTS (s,a,r)

We now give some examples where HPOINTS $(s,a,r) \neq G(s,a,r)$. We will consider this topic further in a sequel.

EXAMPLE 1.8 (J. Alexander, A. Hirschowitz [A], [AH1], [AH2], [AH3], [H]; see also the recent proof by K. Chandler [Ch1]). Suppose that k is an infinite field, $a = 2$, and we consider $P = s$ generic points in $\mathbb{P}^n$, so $\mu(2P) = sr$. If $i \geqslant a$ then

$$\text{HPOINTS}\ (s,2,r)_i = \min\ (sr, \dim_k R_i),$$

with four exceptional cases $(s,r;i) = (5,3;4), (9,4;4), (14,5,4), (7,5;3)$ for which HPOINTS $(s,2,r)_i = sr - 1$.

In other words, if P is a general enough set of s distinct points of $\mathbb{P}^{r-1}$ then the subscheme $Z_{P,2}$ is in sr-generic position, with four exceptions. For the exceptional triples, PFER $(s,2,r)_i = 1$.

EXAMPLE 1.9A. Nonzero defect: thin power algebras. When $(s,a,r) = (5,8,3)$, $\mathfrak{R} = k[X,Y,Z]$, then $V = \langle X^8, Y^8, Z^8, (X+Y+Z)^8, (X+13Y+7Z)^8 \rangle$, appears to be general enough so $H(\mathfrak{R}/(V)) =$ HPOWLIN $(5,8,3)$. Using 'Macaulay' [BSE] we found

$$\text{HPOWLIN}\ (5,8,3)_{(8,\ldots,14)} = (40,40,36,28,16,\mathbf{6},\mathbf{1}),$$

$$\text{LFER}\ (5,8,3)_{(8,\ldots,14)} = (0,0,0,0,0,\mathbf{6},\mathbf{1}).$$

Since FER $(s,j,3) = 0$, by Anick's result [An], we have

$$\text{LD}\ (5,8,3)_{12,13,14} = (0,6,1).$$

We also calculated using 'Macaulay',

$$\text{HPOWLIN } (5,7,3)_{10,11,12} = \text{HPOWLIN } (5,8,3)_{12,13,14}$$
$$= \text{HPOWLIN } (5,9,3)_{14,15,16} = \cdots = (\mathbf{16},\mathbf{6},\mathbf{1}) \tag{1.11}$$

When $j = 20$, the stable ending sequence of HPOWLIN $(5,8,3)_{\ldots,38}$ has grown to $(\ldots,106,76,51,31,16,6,1)$ with 1 in the socle degree $\sigma = 38$ (see [I5]).

EXAMPLE 1.9B. Nonzero defect, and vanishing ideals. If $(s,a,r) = (5,6,3)$, and P consists of 5 general enough points of $\mathbb{P}^2$, then $\mathfrak{I}_P^{(6)}$ has degree $\mu(\mathfrak{I}_P^{(6)}) = (21)(5) = 105$. Since $r = 3$, Anick's theorem that FER $(s,j,3) = 0$ [An] and Theorem 1.6 imply that HPOINTS $(5,6,3)$ is bounded above by

$$G(5,6,3) = (1,3,\ldots,78,\mathbf{91},\mathbf{105},\mathbf{105},105,\ldots),$$

which is just the Hilbert function of an ideal in 105-generic position. However, using, 'Macaulay' we find

$$\text{HPOINTS } (5,6,3)$$
$$= (1,3,6,10,15,21,28,36,45,55,66,78,\mathbf{90},\mathbf{99},\mathbf{104},105,\ldots).$$

Here $90 = \text{HPOINTS } (5,6,3)_{12} = \dim_k R_{12} - \text{HPOWLIN } (5,12+1-6,3)_{12} = 91 - 1$.

A. Hirschowitz explains this kind of example in [H2]. Here the five points P lie on a conic Y, and $\dim_k(\Gamma(Y,\mathcal{O}(13)) = 105 - 78 = 27$. The condition that a form of degree 13 on Y vanish to order 6 at each of the points would tend to impose $6{\cdot}5 = 30$ conditions, but there are only 27 available: three don't count. This shows that there is a defect, but more careful examination is needed to explain its value. See Section 1–4 of [H2], and also [Ha1], [Ha2], [G].

REMARK 1.9C. Pattern in the defect. The part of the Hilbert function HPOINTS $(5,6,3)$, that varies from the upper bound $G(5,6,3)$ is

$$\text{HPOINTS } (5,6,3)_{12,13,14}$$
$$= H(R)_{12,13,14} - (\text{HPOWLIN } (5,7,3)_{12}), \text{HPOWLIN } (5,8,3)_{13},$$
$$(\text{HPOWLIN } (5,9,3)_{14}) = (91,105,120) - (1,6,16)$$
$$= (90,99,104).$$

The difference

$$G(5,6,3)_{12,13,14} - \text{HPOINTS } (5,6,3)_{12,13,14} = (1,6,16),$$

reflects the stable ending sequence (16,6,1) in the Hilbert functions HPOWLIN $(s,j,r) = H(R/L^j)$ (see (1.11) and [I5]).

1.7. Status of the bounds for HPOINTS (s, a, r)

One of our aims here and in the sequel [I5] is to give an accurate conjecture for HPOINTS (s, a, r). Our hope is that having the right conjecture might aid in finding this extremal function.

A major result of our investigation here and in [I5], is that when the number s of points P in $\mathbb{P}^n$ satisfies $s < 2^n, n = r - 1$, the conjectural formulas for HPOWLIN (s, j, r) are very much simpler than those for HPOINTS (s, a, r), even though the latter can be derived from the former (see §2A below). When $s \geqslant 2^n$, then – with a few exceptions detailed in [I5] – we conjecture PFER $(s, a, r) = 0$, implying HPOINTS $(s, a, r) =$ HGP $(\mu(\mathfrak{I}_P^{(a)}), r)$. Thus, the behavior of HPOINTS (s, a, r) depends greatly on the size of s compared to r. In Table I we summarize what we know or conjecture, concerning the behavior of HPOINTS (s, a, r), according to the size of s compared to r (first row). The second row describes the behavior of HPOINTS $(s, a, r)_i$, assuming the Weak Fröberg Conjecture. The third row describes what is known about HPOINTS (s, a, r) for a general. The next rows describe the case $a = 2$ resolved by J. Alexander and A. Hirschowitz (See Example 1.8), and the case $a = 3$ resolved by K. Chandler in degrees $i \geqslant 6$ [Ch3]. We say that (s, r) is *exceptional* if there is a value a such that HPOINTS $(s, a, r) \neq G(s, a, r)$ (equivalently, if PFER $(s, a, r) \neq 0$).

REMARK 1.10. Since D. Anick proved the Strong Fröberg Conjecture when $r = 3$, the upper bound HPOINTS $(s, a, 3) \leqslant G(s, a, 3)$ is known for $\mathbb{P}^2$. However for $s \geqslant 5$, we have $G(s, a, 3) =$ HGP $(\mu, 3)$, the μ-generic position bound, where $\mu = \mu(\mathfrak{I}_P^{(a)})$; and if $s \leqslant 9$ then HPOINTS $(s, a, 3)$ was already known (see [H2]). Thus, the upper bound $G(s, a, 3)$ gives us nothing new for $\mathbb{P}^2$. That HPOINTS (s, a, r) is known for $s = r + 1$ (for $n + 2$ points on $\mathbb{P}^n$), seems not to have been generally realized by specialists.

We now rephrase the question of determining HPOWLIN (s, j, r). We denote by $\mathfrak{F}_i(L^j)$ the space of degree-i relations among the powers $L_1^j, \dots, L_s^j$. If V is the span of $L_1^j, \dots, L_s^j$, we have

$$\begin{aligned}\mathfrak{F}_i(L^j) &= \langle\{(b_1, \dots, b_s) \mid b_1, \dots, b_s \in \mathfrak{R}_{i-j} \quad \text{and} \quad \sum_{1 \leqslant v \leqslant s} b_v L_v^j = 0\}\rangle \\ &\cong \ker(\phi_{i-j,V}) : \mathfrak{R}_{i-j} \otimes V \to \mathfrak{R}_i. \qquad (1.12a)\end{aligned}$$

QUESTION. Relations for powers of generic linear forms. What is the dimension $d(s, u, r; i)$ of the vector subspace $\mathfrak{R}_u Li - u = \text{Image}(\phi_u, L^{i-u})$ of $\mathfrak{R}_i$, when L is a generic set of s linear elements of $\mathfrak{R}$? Equivalently, what is the dimension of the space $\mathfrak{F}_i(L^{i-u})$ of degree-i relations among the powers $L_1^{i-u}, \dots, L_s^{i-u}$?

Table I. Predicted behavior of HPOINTS $(s, a, r)_i$ when char $(k) = 0$.

(s,r):	$s \leqslant r+1$	$s = r+2, r+3$; $(s,r) = (7,3)$ $(8,3), (9,4)$ $(14,5), \ldots$	$r+4 \leqslant s < 2^{r-1}$ (s,r) not $(14,5), \ldots$	$s \geqslant 2^{r-1}$ except (7,3)(8,3) $(9,4)$
HPOINTS: (Assume WFC)	Known – (Stanley Lemma C)	Exceptional – see [15] ?	$\leqslant G(s,a,r)$ piecewise polynomial, intervals determined by a. (Theorem 2.2) ?	$\leqslant$ HGP ?
HPOINTS (Known)	all cases	Some values calculated, $r \leqslant 10$	$\leqslant G(s,a,r) <$ HGP, if a is large enough (Theorem I)	$r = 3$ $(\mathbb{P}^2)$ $s \leqslant 9$
$a = 2$ (known)		4 classical exceptions $(9,4), (14,5)$ $(5,3), (7,4)$;	= HGP (J. Alexander and A. Hirschowitz, see [A], [AH1], [AH2], [H] or [Ch1])	= HGP (ibid)
$a = 3$ (known)	"	4 exceptions $(9,4), (14,5)$ $(5,3), (9,7)$.	$= G(s,a,r)$ (K. Chandler [Ch3]) $i \geqslant 6$	= HGP [Ch3] $i \geqslant 6$

Evidently, we have

$$\dim_k \mathfrak{F}_i(L^{i-u}) = s \cdot \dim_k \mathfrak{R}_u - d(s,u,r;i), \tag{1.12b}$$

$$d(s,u,r;i) = \dim_k \mathfrak{R}_i - \text{HPOWLIN } (s, i-u, r)_i, \tag{1.12c}$$

$$\dim_k \mathfrak{F}_i(L^{i-u}) = s \cdot \dim_k \mathfrak{R}_u - \dim_k \mathfrak{R}_i + \text{HPOWLIN } (s, i-u, r)_i. \tag{1.12d}$$

As we shall see in Section 3, these integers are related to the dimensions of certain spaces of spline functions.

Geometric viewpoint. Let VER (j, r) denote the Veronese embedding of $\hat{\mathbb{P}}^n$ into $\mathbb{P}(\mathfrak{R}_j)$, via j-th powers. When we restrict to HPOWLIN $(s, j, r)_{j+1}$, the above Question is equivalent to asking for the dimension of the tangent space TAN $(1, s, j, r)$ to the s-secant variety SEC (s, j, r) of the Veronese embedding VER (j, r). This is the classical approach of Terracini-Bronowski to studying a Waring problem for forms (see [T]); this Waring problem is now solved by the results of J. Alexander and A. Hirschowitz concerning HPOINTS $(s, 2, r)$ (See Example 1.8 above and [I3]). In a similar manner, the vector space HPOWLIN $(s, j, r)_{j+u}$ where $u > 1$ is the tangent space to a higher osculating variety

TAN (u, s, j, r) to the s-secant variety. Lemma A shows that the dimensions of these tangent spaces are determined by HPOINTS $(s, u+1, r)$.

2. The Hilbert function of vanishing ideals

In Section 2.1 we study further the function $G(s, a, r)$ (Theorem 2.4ff). In Section 2.2 we study the socle degree of the Fröberg function $F(s, j, r)$. By Lemma A, this gives information concerning the order – initial degree – of vanishing ideals $\mathfrak{I}_P^{(a)}$ in $\mathbb{P}^{r-1}$ having Hilbert function bounded above by $G(s, a, r)$.

2.1. Properties of $G(s, a, r)$

The integers i fall into 'Koszul intervals' S_u which depend only on the order of vanishing a, and not on either r or s. For integers i in the region S_u the function $G(s, a, r)_i$ is governed by min (s, u, r) terms of the Koszul resolution for the corresponding thin algebra R/L^{i+1-a}. Furthermore, if (s, a, r) is fixed, then $G(s, a, r)_i$ restricted to S_u is a polynomial in i of degree at most $r-1$ (Theorem 2.2, Corollary 2.3).

The intervals $K_u(j)$ we are about to define are the values of i for which the u-th syzygies of L^j may enter into the strings of the Koszul part of the resolution (see Thin Algebra Resuolution Conjecture in Section 0). The following definitions and Lemma translate these intervals into the corresponding intervals S_u for $G(s, a, r)_i$. We denote by $\mathbb{N}$ the positive integers.

DEFINITION 2.1A. Koszul intervals for HGEN (s, j, r). Given the positive integer j, if $0 \leqslant u$ we denote by $K_u(j)$ the interval $uj \leqslant i < (u+1)j \subset \mathbb{N}$.

DEFINITION 2.1B. Koszul intervals for $G(s, a, r)$. Given the positive integer a, we define sets $S_u(a) \subset \mathbb{N}$, by

$$i \in S_u(a) \quad \text{iff } i \in K_u(j), \quad j = i + 1 - a.$$

We let $S_\infty(a) = [1, a-1]$. We define excess functions

$$e_{a,u}(i) = i - u(i + 1 - a). \tag{2.1}$$

LEMMA 2.1C. *The positive integers* $\mathbb{N}$ *are decomposed into no more than* $a+1$ *disjoint intervals* $S_u(a)$: $S_1(a) > S_2(a) > \cdots > S_a(a) > S_\infty(a)$, *some of which may be empty. The interval* $S_u(a)$ *satisfies*

$$\begin{aligned} &S_1 : 2(a-1) < i, \\ &S_u : \left(\tfrac{u+1}{u}\right)(a-1) < i \leqslant \left(\tfrac{u}{u-1}\right)(a-1), \\ &S_\infty : i \leqslant a-1. \end{aligned} \tag{2.2}$$

REMARK. As we shall see, if P is a set of points in $\mathbb{P}^{r-1}$ the region S_u correspond to where the conjectural upper boundy $G(s,a,r)_i$ for HPOINTS $(s,a,r)_i$ involves u steps in the Koszul resolution of L_P^{i+1-a}.

Definition 2.1D. Koszul dimensions. We suppose a is fixed, that $i \in S_u$ and define

$$c(s,a,r)_i = \sum_{1 \leqslant t \leqslant \min(u,r,s)} c_t(s,a,r)_i, \quad \text{with}$$

$$c_t(s,a,r)_i = \begin{cases} (-1)^{t+1}(\dim_k R_{e_{a,t(i)}}) \begin{pmatrix} s \\ t \end{pmatrix}, & \text{if } e_{a,t}(i) \geqslant 0, \\ 0, \text{ if } e_{a,t}(i) < 0. \end{cases} \tag{2.3}$$

DEFINITION 2.1E. We let Ord $(G(s,a,r)) = \min\{i \mid G(s,a,r)_i < r_i\}$, and set SOCDEG $(s,j,r) = \max\{i \mid F(s,j,r)_i \neq 0\}$.

Recall that $F'(s,j,r)_i$ denotes the coefficient of $(1-\mathrm{Z})^{-r}(1-\mathrm{Z}^j)^s$ on Z^i (Definition 0.1). We set

$$\tau(s,j,r) = \begin{cases} \min\,(i \mid F'(s,j,r)_i < 0), & \text{or} \\ +\infty \quad if\ F'(s,j,r)_i \geqslant 0 & \text{for all } i. \end{cases}$$

Then we have SOCDEG $(s,j,r) < \tau(s,j,r)$ and

$$\mathrm{Ord}(G(s,a,r) = \min\{i \,|\, c(s,a,r)_i < r_i \text{ and } i < \mathrm{SOCDEG}\,(s, i+1-a, r)\}. \tag{2.4}$$

REMARK. When $(s,j,r) = (5,2,3)$, the $F'(5,2,3)$ series is $(1,3,1,-5,-5,1,3,1)$, to be replaced by $F(5,2,3) = (1,3,1,0,0\ldots)$. The second condition in (2.4) requires, paradoxically, that i be *large enough* so that $F(s,i+1-a,r)_i = F'(s,i+1-a,r)_i$. Thus, Ord $(G(5,4,3)) = 9$, and $G(5,4,3) =$ HGP $(50,3)$ (50-generic position). Here, $c(5,4,3)_5 = 50 - 3(10) = 20$, but $G(5,4,3)_5 = r_5 - F(5,2,3)_5 = 21$. In practice $G(s,a,r) =$ HGP (μ, r) unless (1.3) is satisfied. For large a the order of $G(s,a,r)$ may be accurately estimated as $b_{s,r} \cdot a$, where $b_{s,r}$ is a known constant (see Theorem 2.11 and Example 0.9).

THEOREM 2.2. *Koszul intervals for the function* $G(s,a,r)$. *Suppose* $a \geqslant 2, 2 \leqslant s \leqslant \dim_k R_{i-(a-1)}$, *and* $i \geqslant$ Ord $(G(s,a,r))$. *Then*

$$G(s,a,r)_i = c(s,a,r)_i. \tag{2.5}$$

Proof. Let $j = i+1-a$. If $i < \tau(s,j,r)$ we have $F(s,j,r) = F'(s,j,r)$, hence by Definition 0.1,

$$F(s,j,r)_i = \dim_k R_i + \sum_{1 \leqslant t \leqslant \min(\lfloor i/j \rfloor, r, s)} (-1)^t (\dim_k R_{i-tj}) \cdot \begin{pmatrix} s \\ t \end{pmatrix}.$$

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If $i \in S_u$ then $[i/j] = u$, and $e_{a,t}(i) = i - tj$. Theorem 1.15 implies that $G(s,a,r)_i = \dim_k R_i - F(s,j,r)_i$. This and (2.4) show (2.5).

COROLLARY 2.3. *$G(s,a,r)$ is piecewise polynomial. $G(s,a,r)$ satisfies,*

A If $i \geqslant$ Ord $(G(s,a,r)), u \in [1,a]$ then for $i \in S_u, G(s,a,r)_i$ is a polynomial in i of degree at most $r-1$.
B When i is in S_u and $t \leqslant u$, then $c_t(s,a,r)_i$ has degree t as a function of s; if i is in S_u and $t > u$ then $c_t(s,a,r)_i$ is zero.
C If $i \leqslant a-1$, or if $i = a$ and $s \geqslant r$, then $G(s,a,r)_i = r_i$.
D If $i \geqslant 2a-1$, then $G(s,a,r)_i =$ HGP $(\mu,r)_i = \min(\mu, r_i)$.
E If $s \geqslant 2^{r-1}$, then $G(s,a,r) =$ HGP (μ,r).

Proof. The excess function $e_{a,u}(i) = i - u(i+1-a)$ is linear in i, and the function $r_i = \dim_k R_i$ is a polynomial of degree $r-1$ in i. Corollary 2.3A thus follows from (2.3) and (2.4). B is immediate from Definition 2.1D, C and D follow from Theorem 2.15 and the definition of $F(s,j,r)$. The elementary inequality $2^{r-1}r_{j-1} \geqslant r_{2j-1}$, implies that if $s \geqslant 2^{r-1}$, then the socle degree of $F(s,j,r)$ is at most $2j-2$, in the K_1 interval of $\mathbb{N}$; this implies that $G(s,a,r) =$ HGP (μ,r).

SUMMARY. Assume $s < 2^{r-1}$. For i in $S_u, i \geqslant$ Ord $(G(s,a,r))$, the function $G(s,a,r)_i$ is a sum of min (s,u,r) terms whose t-th term $c_t(s,a,r)_i$ is polynomial of degree t in s, and degree $r-1$ in i. The value of Ord $(G(s,a,r))$ is determined by (2.11), but is not simply expressed in terms of (s,a,r). When $i =$ Ord $(G(s,a,r))$ the most number of terms $c_t(s,a,r)$ are required; the number of terms decreases as i increases. For $i \geqslant 2a-1$, $G(s,a,r)_i =$ HGP (μ,r).

EXAMPLE 2.4A. Koszul intervals. When $a = 3$, the intervals are

$$S_1: 4 < i, \quad G(s,3,r)_i = \text{HGP } (\mu,r)_i = \min\,(s \cdot r_2, r_i),$$

$$S_2: i = 4, \quad G(s,3,r)_4 = \min\left(\mu - \binom{s}{2}, r_4\right),$$

$$S_3: i = 3, \quad G(s,a,r)_3 = \begin{cases} r_3 - \binom{r-s+2}{3} & \text{if } s \leqslant r, \\ r_3 & \text{otherwise.} \end{cases}$$

EXAMPLE 2.4B. Koszul intervals and μ-generic position for $G(s,3,4)$, s small. In Table II we give $G(s,3,4)$ for $2 \leqslant s \leqslant 5$.

For $s \leqslant 3$, the scheme Spec $(R/\mathfrak{I}_P^{(a)})$ becomes regular only in degree 6. For $s = 4$, the scheme is not in 40-generic position, because there is at least one quartic vanishing on it. For $s = 5$, the ideal $\mathfrak{I}_P^{(3)}$ has degree $\mu = 50$, and is in 50-generic position. Note that as s increases, with (a,r) fixed, the scheme approaches μ-generic position.

Table II. Values for HPOINTS $(s,3,4) = G(s,3,4)$, *when* $r = 4, 2 \leqslant s \leqslant 5$. (See Example 2.4 A,B)

s	The sequence $G(s,3,4)$	Comment
2	1 4 10 16 19 20 20	Regularity $i = 6$.
3	1 4 10 19 27 30 30	Regularity $i = 6$.
4	1 4 10 20 34 40 40	" "
5	1 4 10 20 35 50 50	50-generic position

EXAMPLE 2.4C. Koszul intervals and μ-generic position for $G(s,3,r), s$ large. When $a = 3, r \leqslant 7$ and $s \geqslant r + 2$, then HPOINTS $(s,3,r) \leqslant G(s,3,r) =$ HGP(μ, r), but this imposes no nontrivial restriction. However, if we fix b, set $s = r + b$, and increase r, we soon find a contradiction to μ-generic behavior for $i = 4$ in the S_2 region. The multiplicity $\mu(\mathfrak{I}_P^{(3)}) = s \cdot r_2 \cong sr^2/2 \cong r^3/2$, but $\dim_k R_4 \cong r^4/4$. Thus, when r is large enough the scheme Spec $(R/\mathfrak{I}_P^{(a)})$ cannot be in μ-generic position.

When $(s,a,r) = (10,3,8)$, ten points on $\mathbb{P}^7$, we have by (1.4) of Theorem I, HPOINTS $(10,3,8)_4 \leqslant (36)(10) - 45 = 315$, which is less than $\dim_k R_4 = 330$, so $\mathfrak{I}_P^{(3)}$ is *not* in μ-generic position in $\mathbb{P}^7$.

Likewise, when $(s,a,r) = (11,3,9)$, for eleven points on $\mathbb{P}^8$ we have HPOINTS $(11,3,9)_4 \leqslant (45)(11) - 55 = 445$, which is less than the degree $\mu = 495 = 11(\dim_k R_2)$, again preventing μ-genericity.

For $(s,a,r) = (12,3,9)$, twelve points on $\mathbb{P}^8$, we have HPOINTS $(12,3,9)_4 \leqslant (45)(12) - 66 = 485$, so $\mathfrak{I}_P^{(3)}$ is not in μ-generic position. For $a = 3$ and $s = 13$ points, we must take $r > 9$ to obtain non μ-generic position for the upper bound $G(13,3,r)$.

EXAMPLE 2.5. Koszul intervals and μ-genericity for $a = 4$. When $a = 4$, the Koszul intervals are

$$S_1,\ i \geqslant 7; \quad S_2,\ i = 5,6; \quad S_3,\ i = 4; \quad \text{and } S_\infty, i < 4.$$

Again, by Theorem 1.6, we have HPOINTS $(s,4,r) \leqslant G(s,4,r)$. If $s \geqslant r + 1$ and we take $i = 6$ then $(s,a,r) = (8,4,6)$ is the example with lowest embedding dimension r where Theorem I requires non μ-generic behavior for $\mathfrak{I}_P^{(4)}$. When $i = 5$ the first such example is $(r,a,s) = (10,4,12)$.

Notation for Table III
The $i=6$ column of Table III below lists under 'dim' the upper bound $G(s,4,r)_i$ for $H(s,4,r)_i = H(R/\mathfrak{I}_P^{(4)})$. Since $i=6$ is in the S_2 region, we have

$$G(s,4,r)_i = \min\left(\mu(\mathfrak{I}_P^{(4)}) - \binom{s}{2}, \dim_k \mathfrak{R}_i\right).$$

We list the bound in boldface, when it is smaller than $\dim_k \mathfrak{R}_i$ and so satisfies (1.4), preventing μ-genericity of $\mathfrak{I}_P^{(4)}$. We then list the codimension cod $= \dim_k \mathfrak{R}_6 - G(s,4,r)_6$ which is a lower bound for $\dim_k(\mathfrak{I}_P^{(4)})$. We next list the *difference* of $G(s,4,r)_6$ from μ-generic position,

$$\text{diff} = \min(\mu, r_6) - G(s,4,r)_6. \tag{2.6}$$

Finally we list in boldface the points Fröberg defect

$$\delta = \text{PFER}(s,4,r)_6 = G(s,4,r)_6 - \text{HPOINTS}(s,4,r)_6 = \text{LD}(s,3,r)_6, \tag{2.7}$$

between the actual value of HPOINTS $(s,4,r)_6$ as calculated in 'Macaulay', and $G(s,4,r)_6$ (when available).

The $i=7$ *column* of Table III lists $G(s,4,r)_7 = \mu(\mathfrak{I}_P^{(4)})$, the degree of the fat point.

Rows in Table III
For each r, $6 \leqslant r \leqslant 10$, we begin with $s = r+2$, and end with the highest value of s, for which the difference of (2.7) is nonzero in degree 6. For $r=9$, $(\mathfrak{I}_P^{(4)})_6$ has diff $\neq 0$ for $11 \leqslant s \leqslant 19$, but diff $=0$ for $s \geqslant 20$. A striking aspect of Table III is that δ is nonzero only in the exceptional case $(s,r) = (8,6)$! (See Conjecture 0.6, Table I, and [15] for further discussion).

Remark on Table III
The value $\delta_6 = 0$ (or 1 when $(s,r) = (8,6)$), was checked by calculation in 'Macaulay' for the highest s value for each $r \geqslant 7$, and implies $\delta = 0$ for lower s. See Example 1.5.3 for $(r,s) = (10,24)$. The value of δ_7 in Table III was not available; we believe $\delta_7 = 0$ because the codimensions are large.

EXAMPLE 2.6A. Koszul intervals. When $a=7$, we have

$$S_1,\ 12 < i; \quad S_2,\ 10 \leqslant i \leqslant 12; \quad S_3,\ i=9; \quad S_4,\ i=8;$$
$$S_7,\ i=7; \quad S_\infty,\ i<7.$$

EXAMPLE 2.6B. Koszul intervals and μ genericity, $a=7$, s small. We suppose that $r=4$, and $a=7$. Table IV lists under 'dim' the value $G(s,7,4)_i =$ for $s =$

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Table III. Upper bound $G(s,4,r)$ for HPOINTS $(s,4,r)$ in the S_2 region $i = 5,6$. The bound for $i = 6$ prevents μ-generic position in each case (See Example 2.5).

$r; s\backslash i$	5 dim/cod/δ		6 dim/cod/diff/δ		7 dim $= \mu$
6; 8	252/0/	**0**	**420**/42/28 /	**1**	448
7; 9	462/0/	**0**	**720**/204/36/	**0**	756
7;10	462/0		**795**/129/45/	**0**	840
7;11	462/0		**869**/55/55/	**0**	924
8;10	792/0/	**0**	**1155**/561/45/	**0**	1200
8;15	792/0		**1695**/21/21/	**0**	1800
9;11	1287/0/	**0**	**1760**/1243/55/	**0**	1815
9;19	1287/0		**2964**/39/39/	**0**	3135
10;12	**1980**/22/	**0**	**2574**/2431/66/	**0**	2640
10;13	2002/0/	**0**	**2782**/2123/78/	**0**	2860
10;24	2002/0		**5004**/1/1/	**0**	5280

Table IV. Upper bounds for HPOINTS $(s,7,4)$ *when* $r = 4, 2 \leqslant s \leqslant 7$. (See Example 2.6B.). The three nonzero values of δ are in bold.

$s\backslash i$	6	7	8	9	10	11	12/δ	13/δ	14:μ
2	84	**112**	**133**	**148**	**158**	**164**	**167**	168	168
3	84	**119**	**157**	**193**	**222**	**240**	**249**	252	252
4	84	120	165	220	**276**	**312**	**330**	336	336
5	84	120	165	220	286	364	**410**	420	420
6	84	120	165	220	286	364	455/**1**	504/**4**	504
7	84	120	165	220	286	364	455	560/**1**	588

$2,\dots,6$. In each case, $G(s,7,4)$ is regular by degree $i = 15$. A value is listed in boldface when it prevents $G(s,7,4)$ from being in μ-generic position.

When $s \leqslant 5, H(s,7,4)_i = G(s,7,4)_i$ by Theorem 1.6. For $s \geqslant 6, G(s,7,4) =$ HGP $(\mu,4)$, the Hilbert function of an ideal in μ-generic position, $\mu = 84s$. We list the defect $\delta =$ PFER $(s,7,4)_i = G(s,7,4)_i -$ HPOINTS $(s,7,4)_i$ in boldface, when it is nonzero – when $s = 6, i = 12, 13$, or $s = 7, i = 13$. δ *is otherwise zero* in Table IV.

EXAMPLE 2.6C. Koszul intervals and μ genericity, $a = 7, r = 9, 10, S_3$ region. If $a = 7, s \geqslant r+2$, the case $(s,r) = (11,9)$ is the smallest r for which $G(s,7,r)$ impacts the S_3 region, $i = 9$. See Table V.

Notation for Table V

We follow the notation of Table III. In degrees $i = 10 - 12$ of the S_2 region we give the predicted difference, usually diff $= r_{12-i} \cdot s(s-1)/2$, from the μ-generic-position value HGP $(\mu,r)_i$. The entry $G(s,7,r)$ is in boldface when diff $\neq 0$,

Table V. Comparison of $G(s,7,r)$, HGP (μ,r) and HPOINTS $(s,7,r)$, in the S_3 and S_2 regions, $r=9$, 10. See Example 2.6C.

$r;s\backslash i$	9 dim/cod/δ	10 dim/cod/diff	11 diff	12 diff	degree
9;10	**22725**/1585	**28005**/15753/45·45	9·45	45	30030
9;11	**24123/187/154**	**31558**/12200/45·55	9· 55	55	33033
9;12	24310/0/**0**	**34066**/9692/45·66	9·66	66	36036
9;16	" /0/**0**	**42648**/1110/1110	9· 120	120	48048
9;17	" /0/**0**	43758/0/0	9· 136	136	51051
10;12	**45760**/2860	**56430**/35948/55·66	10·66	66	60060
10;13	**48191**/429	**60775**/31603/55·78	10·78	78	65065
10;20	48620/0	**89650**/2728/2728	10·190	190	100100
10;21	" /0	92378/0/0	10·210	210	105105

preventing μ-generic position. The four entries $\delta=$ PFER in bold when $i=9$ are the defects we found using 'Macaulay'.

REMARK. The values of δ for $i>9$ and those not listed when $i=9$ have not been checked by 'Macaulay', as they are out of the effective range of the available computer. We calculated LFER $(11,3,9)=154$ in the second row of Table V in char $(k)=997$. Inaccuracy can arise in several ways in the computer calculations: through a too-special set of linear forms, errors in 'Macaulay' (we found some), or special behavior of HPOWLIN in the characteristics we used. However, we believe the values we list are accurate when char $(k)=0$, or even if char $(k)=p>i$.

2.2. ORDER OF HPOINTS (s,a,r), ASSUMING SFC

Recall that the *socle degree* of an Artinian R-module M is the largest degree i such that $M_i\neq 0$; the *order* of a graded ideal $\mathfrak{I}$ of R is the smallest i such that $\mathfrak{I}_i\neq 0$. When $s\geqslant r$, and the set $L=\{L_1,\ldots,L_s\}$ of linear forms is general enough, the algebra $\mathfrak{R}/L^j$ is Artin, and the socle degree of $\mathfrak{R}/L^j$ is the largest i such that HPOWLIN $(s,j,r)_i\neq 0$. Because of the relation

$$\text{HPOINTS }(s,a,r)_i=\dim_k R_i-\text{HPOWLIN }(s,i+1-a,r)_i,$$

the socle degree of HPOWLIN (s,j,r) is connected with the *order* $\nu(\mathfrak{I}_P^{(a)})$ of fat point ideals $\mathfrak{I}_P^{(a)}$ defining algebras $R/\mathfrak{I}_P^{(a)}$ having Hilbert function HPOINTS (s,a,r). The Main Conjecture 0.6 would imply that HPOWLIN (s,j,r) is the same as $F(s,j,r)$ in most cases when $s\geqslant r+4$. If so, the socle degrees SOCDEG (s,j,r) of the functions $\{F(s,j,r)\mid j\in\mathbb{N}\}$ would determine the order $\nu(\mathfrak{I}_P^{(a)})$.

We first show that the socle degree of $F(s,j,r)$ is asymptotic to $b_{s,r}\cdot j$, where $b_{s,r}$ is a constant depending on s (Propositions 2.8, Theorem 2.9). We then show that if

PFER $(s,a,r)=0$, (if HPOINTS $(s,a,r)=G(s,a,r)$), then the ath order vanishing ideal at s general points of $\mathbb{P}^{r-1}$ has order $\nu(\mathfrak{I}_P^{(a)})=c_{s,r}a+O(1)$, asymptotic to a constant multiple of a (Theorem 2.11). If $s<2^{r-1}$, we find $c_{s,r}<s^{1/(r-1)}$ (Remark 2.12, Example 2.13). This result suggests how Nagata's conjecture concerning the order $\nu(\mathfrak{I}_P^{(a)})$ (Conjecture N in Sect. 0), should be modified for the case $r+4\leqslant s<2^{r-1}$ (Conjecure N′).

The first Lemma 2.7 and Proposition 2.8 concern the less remarkable case $s\geqslant 2^{r-1}$, but prepare for Theorem 2.9.

LEMMA 2.7. *Assume the Strong Fröberg Conjecture* HGEN $(s,j,r)=F(s,j,r)$ *for the triple* (s,j,r). *Suppose* $r\geqslant 2$, *and that a constant* b *satisfying* $1<b\leqslant 2$ *is given. If* j *is large enough and* s *satisfies*

$$s\geqslant\left(\frac{b}{b-1}\right)^{r-1}\geqslant 2^{r-1}, \tag{2.8}$$

then the socle degree SOCDEG (s,j,r), *of a thin algebra* $A=R/(F)$ *determined by a set* F *of* s *degree-*j *forms in* r *variables, satisfies*

$$\text{SOCDEG}\,(s,j,r)\leqslant bj. \tag{2.9}$$

Equality in (2.9) *for a value* $b\leqslant 2$, *implies the asymptotic equality* $s(b-1)^{r-1}=b^{r-1}+O(j^{-1})$. *Conversely, if* $b\leqslant 2$ *is defined by* $s=(b/(b-1))^{r-1}$, *then under Strong Fröberg, we have asympotically*

$$\text{SOCDEG}\,(s,j,r)=bj+O(1). \tag{2.10}$$

Proof. We want SOCDEG $(A)\leqslant bj-1$. Since $bj-1\leqslant 2j-1$ and we have assumed the strong Fröberg conjecture, we have

$$\dim_k(R/F)_{[bj]}=\dim_k R_{[bj]}-s\cdot\dim_k R_{[bj]-j},$$

unless $b=2$ (which we handle as a special case). To show (2.9) for $b<2$, it suffices to show that $s(\dim_k R_{[(bj]-j})\geqslant\dim_k R_{[bj]}$, or equivalently, that

$$s\binom{\lfloor bj\rfloor-j+r-1}{r-1}\geqslant\binom{\lfloor bj\rfloor+r-1}{r-1}. \tag{2.11}$$

Let

$$f(x)=(xj+r-1)_{r-1}\cdot j^{-r+1}=\prod_{i=1}^{r-1}\left(x+\frac{i}{j}\right).$$

It is easy to see that if s satisfies (2.8) and if b' satisfies $b'j = [bj]$, the greatest integer in bj, then

$$s \cdot f(b' - 1) \geqslant f(b'), \tag{2.12}$$

which is equivalent to (2.11). When $b = 2$ one can similarly show that $j > r$ and Strong Fröberg imply that the socle degree of R/F is no greater than 2j.

Now, for $b < 2$, equality in

$$s \cdot f(b - 1) = f(b) \tag{2.13}$$

implies that the socle degree of R/F (under Strong Fröberg) is in the interval $[bj] - 1, [bj] + 1$. Since $f(x) = x^{r-1} + \binom{r}{2} j^{-1} + O(j^{-2})$ the equality (2.13) implies $s(b - 1)^{r-1} = b^{r-1} + O(j^{-1})$, where we may take $O(j^{-1})$ to mean $\mid s(b - 1)^{r-1} - b^{r-1} \mid \leqslant r^2/2j$. From this one sees readily that if b satisfies $s(b - 1)^{r-1} = b^{r-1}$, with $b < 2$, then the socle degree of R/F under Strong Fröberg satisfies SOCDEG $(s, j, r) = b'j$, where

$$\mid b' - b \mid \leqslant W(s, r) \cdot j^{-1}, \quad W(s, r) \approx \frac{r^2}{2(r - 1) \mid (s(b - 1)^{r-2} - b^{r-2}) \mid'} \tag{2.14}$$

when j is large. This implies (2.10).

PROPOSITION 2.8. Socle degree of thin algebras. *If $s \geqslant 2^{r-1}$ and the Strong Fröberg Conjecture is true for (s, j, r), then the socle degree $j' =$ SOCDEG (s, j, r) of a thin algebra satisfies*

$$\text{SOCDEG } (s, j, r) \approx bj + O(1), \quad \text{where } b = 1 + \frac{1}{s^{1/(r-1)} - 1}. \tag{2.15}$$

The Weak Fröberg Conjecture implies that the socle degree of a thin power algebra is at least the right side of (2.15). *The limit constant* $O(1)$ *in* (2.15) *may be taken to be* $W(s, r)$ *of* (2.14).

Proof. Immediate from the last statement of Lemma 2.7, as $s(b - 1)^{r-1} = b^{r-1}$ implies b satisfies the equation of (2.15).

REMARK. When the limit ratio b of (2.15) is irrational it follows that under Strong Fröberg, the integer SOCDEG (s, j, r) cannot be simply expressed in terms of j or of j mod n, for some fixed integer n.

When $s < 2^{r-1}$, the expression for the limit

$$b_{s,r} = \lim_{j \to \infty} \text{SOCDEG } (s, j, r)/j \tag{2.16}$$

is more complicated. If $2 \leqslant b < 3$, corresponding to roughly, $(3/2)^{r-1} < s \leqslant 2^{r-1}$ then if the error FER $(s, j, r) = 0$, a refinement of the proof of Lemma 2.7 shows

THEOREM 2.9. *If* $(3/2)^{r-1} < s \leqslant 2^{r-1}$, *and* FER $(s,j,r) = 0$, *then* $b = b_{s,r}$ *defined in* (2.16) *satisfies* $2 \leqslant b < 3$, *and*

$$b^{r-1} - s(b-1)^{r-1} + \binom{s}{2}(b-2)^{r-1} = 0. \tag{2.17}$$

Furthermore, SOCDEG $(s,j,r) = bj + 0(1)$.

When N *is a positive integer such that* $N \leqslant b < N+1$, *the corresponding equation relating* s *and* $b = b_{s,r}$ *is*

$$\sum_{0 \leqslant k \leqslant \min(N,r)} \binom{s}{k}(-1)^k(b-k)^{r-1} = 0. \tag{2.18}$$

Note on proof. These equations arise from assuming that min (N,r) steps in the Koszul resolution are involved in determining the socle degree.

EXAMPLE 2.10. When $(s,r) = (10,5)$ we obtain from (2.17) a limit ratio $b = 2.293765553$. When $(s,r) = (8,5)$ we obtain a limit ratio $b = 2.509833693$. When $(s,r) = (7,4)$ we obtain $b = 2.096961266$. Here, we calculated solutions to (2.17) using the Maple software. Note that when $s = r+2$ or $r+3$ we conjecture that HPOWLIN $(s,j,r) \neq$ HGEN (s,j,r); if so, the socle degree of thin power algebras will be even greater than the value $b_{s,r}$ for thin algebras.

We now obtain information about the order of HPOINTS (s,a,r), under the assumption PFER $(s,a,r) = 0$. It is easy to show that a general set P of points in $\mathbb{P}^{r-1}$ satisfies $H(R/\mathfrak{I}_P^{(a)}) =$ HPOINTS (s,a,r); such a set of points P is in 'a-general position'.

THEOREM 2.11. *Fix* (s,r) *and assume that* PFER $(s,a,r) = 0$ *for all sufficiently large* a. *Suppose that* $b = b_{s,r}$ *satisfies* SOCDEG $(s,j,r) = b_{s,r}j + O(1)$ *and that the subset* P *of* $\mathbb{P}^{r-1}$ *is in 'a-general position'. Then the order* $\nu(\mathfrak{I}_P^{(a)})$ *satisfies*

$$\nu(\mathfrak{I}_P^{(a)}) = c_{s,r}a + O(1), \qquad c_{s,r} = \frac{b_{s,r}}{b_{s,r}-1}. \tag{2.19}$$

Proof. Given a, we must determine the smallest integer i such that $G(s,a,r)_i < \dim_k R_i$. Since $G(s,a,r)_i = \dim_k R_i - F(s,i+1-a,r)_i$, we must find

$$i \mid F(s,i+1-a,r)_i > 0, \quad \text{but } F(s,i-a,r)_{i-1} = 0.$$

Since SOCDEG $(s,j,r) = b_{s,r}j + O(1)$, we have for $j = i+1-a$ large, there is a constant d such that $i \leqslant b_{s,r}(i+1-a)+d$ but $i-1 \geqslant b_{s,r}(i+1-a)-d$, whence $i \cong c_{s,r}a + O(1)$.

REMARK 2.12. When $s \geqslant 2^{r-1}$, Theorem 2.11 gives the usual estimate $c_{s,r} \cong s^{1/(r-1)}$, obtained from assuming HPOINTS $(s,a,r) =$ HGP $(\mu,r), \mu = \mu(\mathfrak{I}_P^{(a)})$.

This estimate is consistent with Conjecture N of Section 0, generalizing Nagata's conjecture. But if $r+4 \leqslant s < 2^{r-1}$, combining (2.19) with (2.17) or (2.18) gives a new estimate, smaller than $s^{1/(r-1)}$. Recall that (s, r) is *exceptional* if $s = r+2$, or $r+3$, or $(s, r) = (7, 3), (8, 3), (9, 4)$ or $(14, 5)$.

CONJECTURE N′. Suppose $2 \leqslant n$, and let $P_1, \ldots, P_s$ be independent generic points of $\mathbb{P}^n$. Suppose that $n+5 \leqslant s$, and that $c_{s,r}, r = n+1$ is defined by (2.19) and (2.18) from s, and that (s, r) is not exceptional. If a hypersurface of degree d passes through each of the points with multiplicity $a(> 0)$, then d/a is greater than $c_{s,r}$. The minimum such degree, ORDER $(\mathfrak{I}_P^{(a)})$, is asymptotic to $c_{s,r}a$

When (s, r) is exceptional, $c_{s,r}$ in Conjecture N′ must be replaced by an even smaller number.

EXAMPLE 2.13. When $(s, r) = (10, 5)$, if PFER $(10, a, 5) = 0$ for large a, we have $c_{10,5} = 2.293765553/1.293765553 = 1.7729376$. This is strictly smaller than the value $10^{1/4} = 1.77827$, which is the limit $\lim_{a\to\infty}(\nu(J)/a)$ for an ideal J in μ-generic position, $H(R/J) =$ HGP (μ, r), if $\mu = \mu(\mathfrak{I}_P^{(a)}) = 10(r_{a-1})$.

COROLLARY 2.14. *Assuming WFC, if P is any set of s points in $\mathbb{P}^n$, the degrees i for which $H(R/\mathfrak{I}_P^{(a)})_i <$ HGP $(\mu, r)_i = \min(\mu, r_i)$, includes an interval asymptotic, for large a, to*

$$c_{s,r}a \leqslant i \leqslant 2a - 2.$$

Proof. Immediate from Theorems 2.6 and 2.11.

3. Application to splines

In this Section only we denote by $\Delta = \Delta(L)$ the polyhedron containing the origin in Euclidean space $\mathbb{R}^r$, formed by the set L of hyperplanes $L_1 = 0, \ldots, L_s = 0$, where the $L_1, \ldots, L_s$ are real linear polynomials in $R_{\mathbb{R}} = \mathbb{R}[x_1, \ldots, x_r]$. Consider the module $(C^d\Delta)_i$ of degree-i, d-differentiable piecewise polynomials on Δ: these are functions $f : \mathbb{R}^r \to \mathbb{R}$ that are polynomial in each of the regions defined by the hyperplanes. Such a complex is termed 'central' when the L_i are homogeneous (have zero constant term). The module $C^d\Delta = 7_i(C^d\Delta)_i$ is the $R_{\mathbb{R}}$-module of d-differentiable splines on Δ, the sum of its degree-i pieces. Recently, L. Rose has related the dimension of $(C^d\Delta)_i$ to the Hilbert function $H(R_{\mathbb{R}}/(L^j))$, for certain δ. She defines a dual graph $G(\Delta)$ whose vertices correspond $1-1$ to the r-polytopes of δ; two vertices of $G(\Delta)$ share an edge 'e' when the corresponding r-polytopes meet in an $r-1$ dimensional face $L_e = 0$. $G(\Delta)$ is *hereditary* when for each face σ of Δ, the dual graph of the star $G(st(\sigma))$ is connected.

Let $\mathfrak{C}$ denote the set of cycles of $G(\Delta)$, and define

$$B^d(\Delta) = \left\{(b_1, \ldots, b_s) \in \mathbb{R}^s : \text{for all } C \in \mathfrak{C}, \sum_{e\in C} b_e L_e^{d+1} = 0\right\}.$$

The following theorem of L. Rose does not require Δ to be central: the defining linear equations L_i of Δ may have constant terms. However, in the subsequent results, Δ will be central.

LEMMA (L. Rose, Theorem 1.12 of [R]). *If Δ is hereditary, the space $(C^d\Delta)$ of C^d splines on Δ satisfies*

$$(C^d\Delta) \cong (R_{\mathbb{R}}) \oplus B^d(\Delta). \tag{3.1}$$

We let $r_i = \dim_k R_i$. Recall from (1.12a) *that $\mathfrak{F}_i(L^j)$ denotes the vector space of 'degree-i' syzygies among the powers $L_1^j, \dots, L_s^j$*

$$\mathfrak{F}_i(L^j) = \left\langle \left\{ (b_1, \dots, b_s) \mid b_1, \dots, b_s \in \mathfrak{R}_{i-j} \text{ and } \sum_{1 \leqslant e \leqslant s} b_e L_e^j = 0 \right\} \right\rangle.$$

Given a linear form $\sum p_u X_u \in \mathfrak{R}_1$, we let $p = (p_1, \dots, p_r)$ be the corresponding point of $\mathbb{P}^{r-1}$; likewise, given a set $L = (L_1, \dots, L_s)$ of linear forms, we let $P = P_L$ denote the corresponding set of points in $\mathbb{P}^{r-1}$. In the notation of Section 1.3, $L = L(P_L)$. Recall that $\mathfrak{I}_{P_L}^{(a)}$ is the graded ideal in R, of functions vanishing to order at least a at each point of P_L.

LEMMA 3.1. *If $\Delta = \Delta(L)$ is central and hereditary and $G(\Delta)$ consists of a single cycle determined by L: $L_1 = 0, \dots, L_s = 0$, then $(C^d\Delta)_i \cong \mathfrak{R}_i \oplus \mathfrak{F}_i(L^j)$. We have*

$$\dim_{\mathbb{R}}(C^d\Delta)_i = s \cdot r_{i-d-1} + r_i - H(R/\mathfrak{I}_{P_L}^{(i-d)})_i, \tag{3.2}$$

$$\dim_{\mathbb{R}}(C^d\Delta)_i \geqslant s \cdot r_{i-d-1} + \mathrm{HPOWLIN}_{\mathbb{R}}(s, d+1, r)_i, \tag{3.3}$$

$$\dim_{\mathbb{R}}(C^d\Delta)_i \geqslant s \cdot r_{i-d-1} + r_i - \mathrm{HPOINTS}_{\mathbb{R}}(s, i-d, r)_i, \tag{3.4}$$

with equality in both (3.3) *and* (3.4) *if L satisfies $H(R/L^j)_i = \mathrm{HPOWLIN}_{\mathbb{R}}(s, d+1, r)_i$.*

Proof. The equality $(C^d\Delta)_i \cong \mathfrak{R}_i \oplus \mathfrak{F}_i(L^j)$ follows from L. Rose's Theorem, and the hypotheses on Δ and $G(\Delta)$. The formula (3.2) now follows from (1.12a–d); (3.3) follows from the definition of HPOWLIN, and Lemma A implies that (3.3) is equivalent to (3.4).

As a consequence of Lemma 3.1 and the Alexander–Hirschowitz Theorem (see Example 1.8), we have

PROPOSITION 3.2. *If $i = d+2, d \geqslant 1$, and if Δ satisfies the hypotheses of Proposition* 3.1, *then $\dim_{\mathbb{R}}(C^d(\Delta)_{d+2}$ satisfies,*

$$\dim_{\mathbb{R}}(C^d\Delta)_{d+2} \geqslant \max(sr, r_{d+2}). \tag{3.5}$$

With four exceptions there is equality in (3.5) *if* $L_1, \ldots, L_s$ *are 'general' (parametrized by a suitable open set in the sense of Lemma* 1.4.2). *In the four exceptional cases* $(s, r; d) = (5, 3; 2), (9, 4; 2), (5, 14; 2), (7, 5; 1)$, *the right side of* (3.5) *should be replaced by* $(1 + r_{d+2})$*; then there is equality in the modified equation for* L *general.*

Proof. The Alexander–Hirschowitz result is independent of the infinite field chosen, as is Lemma E. If we take $k = \mathbb{R}$, we obtain from Lemma 3.1 and the Alexander–Hirschowitz theorem

$$\dim_{\mathbb{R}}(C^d\Delta)_{d+2} \geqslant sr + r_{d+2} - \min(sr, r_{d+2}),$$

with four exceptions in which we must replace the minimum by $(sr - 1)$. This proves the Proposition.

QUESTION. Is L 'general' in the above sense, consistent with the hypotheses $G(\Delta)$ hereditary and consists of a single cycle?

PROPOSITION 3.3. *Under the same hypotheses as Proposition 3.1, the dimensions of the splines satisfies,*

$$\dim_{\mathbb{R}}(C^d\Delta)_{i+1} - \dim_{\mathbb{R}}(C^d\Delta)_i \leqslant r_{i+1} - r_i. \tag{3.6}$$

There is equality in (3.6) *for a given* Δ *and sufficiently high degrees* i. *Equality in* (3.6) *for a given degree* i, *implies equality for all higher degrees.*

Proof. From Lemma 3.1 and Lemma 1.4.3 we have, taking $k = \mathbb{R}$,

$$\begin{aligned}
&\dim_{\mathbb{R}}(C^{d+1}\Delta)_{i+1} - \dim_{\mathbb{R}}(C^d\Delta)_i \\
&\quad = \dim_{\mathbb{R}}(\mathfrak{R}_{i+1}/\mathfrak{R}_{i-d-1}L^{d+2}) - \dim_{\mathbb{R}}(\mathfrak{R}_i/\mathfrak{R}_{i-d-1}L^{d+1}) \\
&\quad = r_{i+1} - r_i + \dim_{\mathbb{R}}(\mathfrak{R}_{i-d-1}L^{d+1}) - \dim_{\mathbb{R}}(\mathfrak{R}_{i-d-1}L^{d+2}) \\
&\quad \leqslant r_{i+1} - r_i.
\end{aligned}$$

PROPOSITION 3.4 *Under the same hypotheses as Lemma* 3.1, *if also* $s < 2^{r-1}$ *and* $(d + 2)$ *is 'sufficiently large' for* (s, r), *in the sense that* $sr_d < r_{2d+1}$ *and*

$$s \cdot r_{d+1} - \binom{s}{2} < r_{2d+2},$$

then

$$\dim_{\mathbb{R}}(C^d\Delta)_{2d+2} \geqslant r_{2d+2} + \binom{s}{2}. \tag{3.7}$$

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If $s < \min(2^{r-1}, r_{d+1}), r \geqslant 4, d \geqslant 1$ *and* d *satisfies*

$$s \cdot r_{d+2} - r \cdot \binom{s}{2} < r_{2d+3},$$

then

$$\dim_{\mathbb{R}}(C^d\Delta)_{2d+3} \geqslant r_{2d+3} + r\binom{s}{2}. \tag{3.8}$$

Proof. The formula (3.7) follows from Theorem I, by setting $a = d + 2$ and applying Lemma 3.1. The formula (3.8) follows from Lemma 1.6.1, Corollary 1.6.2 and Lemma 3.1.

REMARK 3.5. Evidently, WFC for HPOWLIN $(s, d+1, r)_i$ implies

$$\dim_{\mathbb{R}}(C^d\Delta)_i \geqslant s \cdot r_{i-d-1} + F(s, d+1, r)_i,$$

and upper bounds for HPOINTS $(s, i-d, r)_i$ convert to lower bounds for the dimension of splines.

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