Theorems on Summation of Series.

By Alexander Holm, M.A.

(*Read* 12*th March*. *Received* 28*th April* 1909).

1. *Theorem I.* If $\Sigma_a^n f(n) = F(n)$, where a and n are positive integers, and if $f(n)$ is a finite single-valued function of n for values of $n \geqq a$, then $\Sigma_a^n \Delta f(n) = \Delta F(n) + \text{const.}$

By supposition $f(a) + f(a+1) + \ldots + f(n) = F(n)$, hence $F(n)$, being the sum of a finite number of finite single-valued functions, must be a finite single-valued function of n.

Hence also $\Delta f(n)$ and $\Delta F(n)$ must be finite single-valued functions of n.

Now
$$f(n) = F(n) - F(n-1)$$
$$\therefore \quad \Delta f(n) = \Delta F(n) - \Delta F(n-1)$$
$$\therefore \quad \Delta f(n-1) = \Delta F(n-1) - \Delta F(n-2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Delta f(a+1) = \Delta F(a+1) - \Delta F(a)$$
$$\Delta f(a) = \Delta F(a) - \Delta F(a-1)$$
$$\therefore \quad \Sigma_a^n \Delta f(n) = \Delta F(n) - \Delta F(a-1).$$
$$\therefore \quad \Sigma_a^n \Delta f(n) = \Delta F(n) + \text{const.}$$

Ex. From $\Sigma_1^n \dfrac{1}{n(n+1)} = -\dfrac{1}{n+1} + 1$,

we derive $\Sigma_1^n \left[\dfrac{1}{(n+1)(n+2)} - \dfrac{1}{n(n+1)}\right] = -\dfrac{1}{n+2} + \dfrac{1}{n+1} + \text{const.}$

$$\therefore \quad \Sigma_1^n \frac{1}{n(n+1)(n+2)} = -\frac{1}{2(n+1)(n+2)} + \text{const.}$$

Putting $n = 1$, const. $= \frac{1}{4}$ $\therefore \Sigma_1^n \dfrac{1}{n(n+1)(n+2)} = -\dfrac{1}{2(n+1)(n+2)} + \frac{1}{4}$.

2. *Theorem II.* If $\Sigma_a^n f(n) = F(n)$, where a and n are positive integers, and if $f(x)$, $F(x)$ and their first derivatives $f'(x)$, $F'(x)$ are finite single-valued *continuous* functions of x for finite values of $x \geqq a$,

then $$\Sigma_a^n f'(n) = F'(n) + \text{const.}$$

we have
$$f(x) = F(x) - F(x-1)$$
$$\therefore \quad f'(x) = F'(x) - F'(x-1)$$
$$\therefore \quad f'(n) = F'(n) - F'(n-1)$$
$$f'(n-1) = F'(n-1) - F'(n-2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f'(a+1) = F'(a+1) - F'(a)$$
$$f'(a) = F'(a) - F'(a-1)$$
$$\therefore \quad \Sigma_a^n f'(n) = F'(n) - F'(a-1)$$
$$\therefore \quad \Sigma_a^n f'(n) = F'(n) + \text{const.}$$

Ex. $$\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \ldots + \log\frac{n}{n+1} = \log\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n}{n+1}$$
$$\therefore \quad \Sigma_1^n \log\frac{n}{n+1} = \log\frac{1}{n+1}$$
$$\therefore \quad \Sigma_1^n[\log n - \log(n+1)] = -\log(n+1).$$

Here $$f(x) = \log x - \log(x+1), \quad F(x) = -\log(x+1)$$
$$f'(x) = \frac{1}{x} - \frac{1}{x+1}, \quad F'(x) = -\frac{1}{x+1}$$

$\therefore$ $f(x)$, $F(x)$, $f'(x)$, $F'(x)$ are all finite single-valued continuous functions of x for finite values of $x \geqq 1$.

$$\therefore \quad \Sigma_1^n\left[\frac{1}{n} - \frac{1}{n+1}\right] = -\frac{1}{n+1} + \text{const.}$$
$$\therefore \quad \Sigma_1^n \frac{1}{n(n+1)} = -\frac{1}{n+1} + \text{const.}$$

Putting $n = 1$, const. $= 1$

$$\therefore \quad \Sigma_1^n \frac{1}{n(n+1)} = -\frac{1}{n+1} + 1.$$

3. *Theorem III.* If $\Sigma_a^n f(n) = F(n)$, where a and n are positive integers, and if $f(x)$, $F(x)$ and their integrals with respect to x, $f^{-1}(x)$, $F^{-1}(x)$ are finite single-valued *continuous* functions of x for finite values of $x \geqq a$, then $\Sigma_a^n f^{-1}(n) = F^{-1}(n) + Cn + C'$, where C and C' are constants.

We have $f(x) = F(x) - F(x-1)$.

Hence on integrating with respect to x, we have for finite values of $x \geqq a$,

$$f^{-1}(x) = F^{-1}(x) - F^{-1}(x-1) + C$$

where C is a constant.

$$\therefore \quad f^{-1}(n) = F^{-1}(n) - F^{-1}(n-1) + C$$
$$f^{-1}(n-1) = F^{-1}(n-1) - F^{-1}(n-2) + C$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f^{-1}(a+1) = F^{-1}(a+1) - F^{-1}(a) + C$$
$$f^{-1}(a) = F^{-1}(a) - F^{-1}(a-1) + C$$
$$\therefore \quad \Sigma_a^n f^{-1}(n) = F^{-1}(n) - F^{-1}(a-1) + C(n-a+1)$$
$$\therefore \quad \Sigma_a^n f^{-1}(n) = F^{-1}(n) + Cn + C'$$

where C and C′ are constants.

Ex. $$\Sigma_1^n(n) = \frac{n^2}{2} + \frac{n}{2}.$$

Here $$f(x) = x, \quad F(x) = \frac{x^2}{2} + \frac{x}{2}.$$

$$f^{-1}(x) = \frac{x^2}{2} + \text{const.}, \quad F^{-1}(x) = \frac{x^3}{6} + \frac{x^2}{4} + \text{const.}$$

$\therefore$ $f(x)$, $F(x)$, $f^{-1}(x)$, $F^{-1}(x)$ are all finite single-valued continuous functions of x for finite values of $x \geqq 1$.

$$\therefore \quad \Sigma_1^n\left(\frac{n^2}{2}\right) = \frac{n^3}{6} + \frac{n^2}{4} + Cn + C'.$$

Putting $n = 0$, $C' = 0$, and $n = 1$, $C = \frac{1}{12}$,

$$\therefore \quad \Sigma_1^n(n^2) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

From which we derive in a similar manner

$$\Sigma_1^n\left(\frac{n^3}{3}\right) = \frac{n^4}{12} + \frac{n^3}{6} + \frac{n^2}{12} + Cn + C',$$

and putting $n = 0$, $C' = 0$, and $n = 1$, $C = 0$

$$\therefore \quad \Sigma_1^n(n^3) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

and so on.

https://doi.org/10.1017/S0013091500002170 Published online by Cambridge University Press