MEAN VALUE PROPERTIES OF GENERALISED EIGENFUNCTIONS †

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Some Hilbert spaces of continuous functions satisfying a mean value property are studied in which the generalised eigenfunctions of any selfadjoint operator again satisfy the same mean value property. Applications are made to nullspaces of some differential operators. The classes of functions involved in these applications are less general than those studied by K. Maurin (6); however, the Hilbert space norms may be arbitrary, while Maurin only considered L^2-norms.

Let $\{\mathscr{F}, D\}$ be a proper functional Hilbert space whose elements are continuous on the domain D of $\mathbf{R}^n$, and whose reproducing kernel K has the property that $K(x, x) = \| K(., x)\|^2_{\mathscr{F}}$ is bounded on compact subsets of D. In other words, $\{\mathscr{F}, D\}$ is a Hilbert-subspace of the space $\mathscr{C}(D)$ of all continuous functions on D with uniform convergence on compacts. Suppose that there is a Hilbert-subspace Φ of $\mathscr{F}$, with Hilbert-Schmidt imbedding into $\mathscr{F}$, which contains all $K(., x)$, $x \in D$. (Then we know that the space $\mathscr{F}$ is Hilbert-Schmidt expansible, cf. (2), Definitions 14.I and 4. II.)

Proposition 1. *Let $\{\mathscr{F}, D\}$ and Φ be as described above, and suppose that also $\| K(., x)\|_\Phi$ is bounded on compact subsets of D. Then, if the functions f in $\mathscr{F}$ satisfy a mean value property in the sense of Pietsch* **(9, 10)** *or Wloka* **(11)**, *the generalised eigenfunctions of any selfadjoint operator in $\mathscr{F}$ (as introduced in* **(2)**, § II.1) *satisfy the same mean value property.*

Proof. We deal with the mean value property in the form given by Pietsch **(9)** (cf. also **(10)**). There is a fixed Radon measure ρ on D, and to each compact $C \subset D$ there exist a compact $H \subset D$, $C \subset H$, and a ρ-measurable function $M(x, t)$ on $H \times C$ so that

$$f(t) = \int_H f(s)M(s, t)d\rho \quad \textit{for all } t \in C, f \in \mathscr{F}.$$

Since Φ contains all the $K(., x)$, it is dense in $\mathscr{F}$, and consequently we have the chain of Hilbert-Schmidt imbeddings $\Phi \subset \mathscr{F} \subset \Phi^*$, where Φ^* is the anti-dual of Φ. The pairing of Φ and Φ^* is represented by the scalar product of $\mathscr{F}$; the elements of Φ^* are identified with functions on D by

$$F(x) = (F, K(., x))_{\Phi^*, \Phi}.$$

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By construction, $\mathscr{F}$ is dense in Φ^*. For an arbitrary $F \in \Phi^*$, let

$$F = \Phi^* - \lim_{N\to\infty} f_N, \quad f_N \in \mathscr{F}.$$

Let C, H, and $M(s, t)$ be as above. On the compact set $H, f_N(s) \to F(s)$ uniformly since

$$|f_N(s) - F(s)| = |(f_N - F, K(., s))_{\Phi^*, \Phi}| \leqq \| f_N - F \|_{\Phi^*} \| K(., s) \|_{\Phi}$$

where the first factor on the right tends to zero as $N\to\infty$ and the second is bounded on H by hypothesis. On account of the uniform convergence we may interchange limit and integral and obtain

$$F(t) = \lim_{N\to\infty} f_N(t) = \lim_{N\to\infty} \int_H f_N(s)M(s, t)d\rho$$

$$= \int_H [\lim_{N\to\infty} f_N(s)]M(s, t)d\rho = \int_H F(s)M(s, t)d\rho$$

for every $t \in C$. Thus the functions in Φ^* have the same mean value property as those in $\mathscr{F}$. It is known that the generalised eigenfunctions of any selfadjoint operator in $\mathscr{F}$ lie in the space Φ^* (cf. for instance K. Maurin (7) § 2, Satz 2, or G. I. Kac (4), Theorems 1.1 and 2.1); consequently the generalised eigenfunctions have the desired mean value property.

Remarks. (i) According to Pietsch (9), the space $\{\mathscr{F}, D\}$ with the topology of uniform convergence on compacts in D is nuclear, and on account of Proposition 1 now also Φ, Φ^* and (almost) all generalised eigenspaces $\mathscr{F}^{(\lambda)}$ of an arbitrary selfadjoint operator in $\mathscr{F}$ are nuclear in the topology of uniform convergence on compacts in D.

(ii) Proposition 1 holds also in the case where D is any locally compact Hausdorff space which is σ-compact.

As a corollary we have the following result.

Theorem 2. *Let $\{\mathscr{F}, D\}$ be a proper functional Hilbert space of analytic functions in the domain D ($\subset \mathbf{R}^n$), and suppose that all the elements f of $\mathscr{F}$ satisfy the equation $Lf = 0$ in D where L is any second order linear elliptic differential operator (with analytic coefficients). Then for every selfadjoint operator A in $\mathscr{F}$ (with spectral measure μ), for μ-almost all λ, the generalised eigenfunctions f_λ in $\mathscr{F}^{(\lambda)}$ are real-analytic and satisfy the equation $Lf_\lambda = 0$ in D.*

Proof. Analyticity of the generalised eigenfunctions is known from (3). Let us cover D by a sequence of closed bounded " nice " domains—say balls—contained in D. On every ball, the solutions of $Lu = 0$ are characterised by a suitable mean value property (see for instance S. G. Mikhlin (8), § 22). By the results of (2), pages 570-572 and (3) there is a Hilbert-Schmidt subspace Φ of $\mathscr{F}$, with $K(., x) \in \Phi$ for all x in the ball B under consideration (K the

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reproducing kernel of $\mathscr{F}$), and the function $x \mapsto K(.,x)$ of B into Φ is strongly real-analytic (analyticity in the norm of Φ is implicitly contained in the constructions of (2), Ch. III), hence $\| K(.,x)\|_{\Phi}$ is bounded on compact subsets of B. Now apply Proposition 1, which concludes the proof.

Remark. Theorem 2 remains valid for any other linear differential operator L whose nullspace (say in the space $\mathscr{C}(D)$ of continuous functions on D) is characterised by a mean value property.

Example. Let the functions f in $\mathscr{F}$ of Proposition 1 satisfy the following special mean value property. There is a non-negative Borel measure μ on the unit ball of $\mathbf{R}^n$, with compact support M such that M is contained in no subspace of $\mathbf{R}^n$ of lower dimension, and $\mu(M) = 1$; for every $f \in \mathscr{F}$, every $x \in D$, and every $\delta > 0$, $x + \delta M \subset D$ implies $f(x) = \int_M f(x+\delta t)d\mu(t)$. Then, according to a theorem of A. Friedman and W. Littman ((1), p. 168) (cf. also K. O. Leland (5)), the elements of $\mathscr{F}$ satisfy a (common) elliptic partial differential equation (determined by the mean value property, with the particular μ above) and are real-analytic. Consequently Theorem 2 applies to $\mathscr{F}$. A special case is that of *harmonic functions.*

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