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REAL AF C^*-ALGEBRAS WITH K_0 OF SMALL RANK

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A *real AF C^*-algebra* is the norm closure of a direct limit of finite dimensional real C^*-algebras (with real *-algebra maps). When we use the unadorned "AF C^*-algebra", we mean the usual complex version.

Let R be a simple AF C^*-algebra such that $K_0(R)$ is free of rank 2 or 3. The problem is to find (up to Morita equivalence) all real AF C^*-algebras A such that $A \otimes \mathbf{C} \cong R$. This is closely related to the problem of finding all involutions on R [**3**], [**10**].

For example, when the rank is 2, generically there are 8 such classes. The exceptional cases arise when the ratio of the two generators in $K_0(R)$ is a quadratic (algebraic) number, and here there are 4, 5, or 8 Morita equivalence classes, the number depending largely on the behaviour of the prime 2 in the relevant algebraic number field.

If the rank is 3, the situation is more complicated. Let S be a subset of $\{r, c, h\}$. Following [**3**] and [**4**], we say a real AF C^*-algebra A is of type S if it can be written as a limit of a direct sum of matrix algebras over the fields indicated in S (with r representing $\mathbf{R}$, c representing $\mathbf{C}$, and h standing for $\mathbf{H}$). Thus, if A is of type r, then A can be written as a limit of direct sums of matrix algebras over $\mathbf{R}$. If A is of type rc, then it can be obtained using finite dimensional matrix algebras over $\mathbf{R}$ and $\mathbf{C}$.

When $K_0(R)$ has rank 3, the corresponding A must be of one of the types, r, h, rh, rc, or ch (but it cannot be of type c). Note that if A is of type r, then it is also of type rh. If R has comparability, then A must be of type rh [**4**, 7.10].

The classification of the type rh algebras is then analogous to that of the rank 2 situation above. Generically, there are 58 Morita equivalence classes (arising essentially from the action of $GL(3, 2)$ on the pairs of complementary subspaces of a dimension 3 vector space over the 2-element field). The non-generic situation occurs only when there is a corresponding cubic field extension associated to $K_0(R)$.

If R does not have comparability, there are far fewer rh equivalence classes, and the situation is rather similar to that occurring in rank 2.

In the case of rc (and ch is similar, since we can always go from one to the other by tensoring with the quaternions), there is a very tight structure theorem available. In particular, there is a unique trace on R. There are generically just 3 Morita equivalence classes of such A, and in the exceptional cases (again

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corresponding to a quadratic image in the reals), there are 2 or 1. As in the rank 2 situation, we construct all the Morita equivalence classes of algebras.

The first results concern the algebras whose K_0 is of rank 2. Sections 3 and 4 discuss the rank 3 situation.

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1. Preliminaries. We first remind the reader of the various classification results for real AF C^*-algebras. An algebra A is a real AF C^*-algebra if it contains a dense subalgebra that can be written as a union of finite dimensional real C^*- algebras (that is, each such finite dimensional algebra is a direct sum of various size matrix algebras over **R**, **C**, or **H**) and so that the inclusion from this dense subalgebra to A is an isometry. In fact, we could work instead with the dense subalgebra with no changes to the classification.

In the classification achieved by Goodearl and Handelman [**4**, Theorem 5.1] (and independently by Stacey [**10**, 2.4]), a complete invariant (with respect to stable equivalence, which we use as a replacement for Morita equivalence— since we are working primarily with unital algebras (or we could so insist), we could take as our definition of Morita equivalence that for unital rings; this amounts to the same thing as isomorphism on tensoring with the algebra of compact operators on real separable Hilbert space; see also the comment in [**4**, p. v]—for real AF C^*-algebras is simply the triple (with maps) of dimension groups, $K_0(A) \longrightarrow K_0(A \otimes \mathbf{C}) \longrightarrow K_0(A \otimes \mathbf{H})$, with the maps being induced from the inclusions $\mathbf{R} \longrightarrow \mathbf{C} \longrightarrow \mathbf{H}$. The triples that arise from such an algebra have the following general form.

Given the original triple arising from A, there are dimension groups $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{K}$ with $\mathfrak{H}$ equipped with an order automorphism of period 2, denoted * (and called an involution of $\mathfrak{H}$), such that $\mathfrak{G}$ and $\mathfrak{K}$ are subgroups of $\mathfrak{H}^{sa} = \{h \in H \mid h^* = h\}$, equipped with the relative ordering from $\mathfrak{H}$ (thus $\mathfrak{G}^+ = \mathfrak{H}^+ \cap \mathfrak{G}$, etc), such that $\mathfrak{G}^+ + \mathfrak{K}^+ = (\mathfrak{H}^{sa})^+$, and $\mathfrak{G} \cap \mathfrak{K} = (1 + {}^*)\mathfrak{H}$. In addition, $(1 + {}^*)\mathfrak{H}^+ = \{(1 + {}^*)\mathfrak{H}\}^+$ and $(1 - {}^*)\mathfrak{H} = \mathfrak{H}^{ss}$, the latter being $\{h \in \mathfrak{H} \mid h^* = -h\}$ (all these properties are easily deduced from the triples arising from **R**, **C**, and **H** and can be found in [**4**, Section 7]). There is an (order) isomorphism of triples,

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A \otimes \mathbf{C}) & \longrightarrow & K_0(A \otimes \mathbf{H}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{G} & \xrightarrow{\subset} & \mathfrak{H} & \xrightarrow{\times(1+{}^*)} & \mathfrak{K} \end{array}.$$

In other words, our original triple can be assumed to be in this very specific form. It is at present unknown whether the properties ascribed to $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{K}$ and the maps in the triple are sufficient for it to arise as the triple coming from a real AF algebra as in the upper row, but many special cases are known. In the case that $\mathfrak{H}$ is simple, these are now known to be sufficient [**5**].

Another complete invariant (with respect to stable equivalence) for real AF algebras is defined in [**3**]. If A is a real AF algebra, the invariant involves the (real) K-groups of A (recall that A being a real Banach algebra, its K-theory is periodic, with period 8). More precisely, it consists of three K-groups $(K_0(A), K_2(A), K_4(A))$, an ordering on $K_0(A) \oplus (K_2(A)/K_1(A))$, and three natural homomorphisms between the K-groups. Details can be found in [op. cit.], and a lexicon for translating the two types of invariant occurs in [ibid., §9]. In this article, we shall primarily work with the triple invariant; however, in this section, we briefly indicate how some of the properties, given in terms of the triples, translate to the second invariant.

For any real Banach algebra A, let $r : K_*(A \otimes \mathbf{C}) \longrightarrow K_*(A)$ be the homomorphism of groups induced by the map $\mathbf{C} \longrightarrow M_2\mathbf{R}$ (the "realification" in [**3**]), and let

$$\beta : K_0(A \otimes \mathbf{C}) \longrightarrow K_2(A \otimes \mathbf{C})$$

be the Bott isomorphism. If A is a real AF algebra, there is a natural injective homomorphism,

$$\psi : K_0(A \otimes \mathbf{C}) \longrightarrow K_0(A) \oplus K_2(A),$$

defined via

$$\psi(x) = (r(x), r\beta(x)) \quad \text{for all } x \text{ in } K_0(A \otimes \mathbf{C}).$$

The homomorphism induced by ψ, with domain $K_0(A \otimes \mathbf{C})$ and range $K_0(A) \oplus (K_2(A)/K_1(A))$, is order-preserving.

If

$$t : K_0(A) \oplus K_2(A) \longrightarrow K_0(A) \oplus K_2(A)$$

is the involution defined by $t(x, y) = (x, -y)$, then the involution $*$ on $K_0(A \otimes \mathbf{C})$ satisfies $\psi(x^*) = t\psi(x)$ for all x in $K_0(A \otimes \mathbf{C})$.

Let $c : K_*(A) \longrightarrow K_*(A \otimes \mathbf{C})$ be the homomorphism induced by complexification. One checks easily that the subgroups $\mathfrak{G} = c(K_0(A))$, and $\mathfrak{K} = c\beta(K_4(A))$ of $\mathfrak{H} = K_0(A \otimes \mathbf{C})$ are precisely the groups obtained from the triple invariant, above.

The real AF C^*-algebra A is said to be of *type r* if it can be realized as a limit of finite dimensional algebras all of which are direct sums of matrix algebras over the reals; it is of *type rc* if it can be realized as a limit of direct sums of matrix algebras with coefficients which can be the reals or the complexes.

Similarly, types *rh*, *h*, *c*, *ch*, *rc* are defined. Note that a type *c* algebra need not be a complex algebra, because the maps between the finite dimensional algebras are merely real algebra maps.

Each of the 6 types, *r*, *c*, *h*, *rc*, *rh*, *ch* have been characterized by their triples, as well as by the K-group invariants. For example, if the triple $\mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{K}$ (we suppress labels on the maps) arises from a real AF algebra, then it is of type *rh* if and only if * is the identity; it is of type *c* if and only if $\mathfrak{G} = \mathfrak{K} = (1 + {}^*)\mathfrak{H}$; it is of type *rc* if and only if $\mathfrak{K} = (1 + {}^*)\mathfrak{H}$, and so on [**4**, §7].

Here is a table which gives all 6 characterizations [**4**, §7] and [**3**, §6]:

Characterization of Types by means of their Invariants

Type	Property of Triple	Property of Real K-Groups
rh	$* =$ identity	$K_2(A)$ is a torsion group
rc	$\mathfrak{K} = (1 + {}^*)\mathfrak{H}$	$K_5(A) = \mathbf{0}$
ch	$\mathfrak{G} = (1 + {}^*)\mathfrak{H}$	$K_1(A) = \mathbf{0}$
r	$\mathfrak{G} = \mathfrak{H}$ or $\mathfrak{K} = \mathfrak{H}$	$K_6(A) = \mathbf{0}$
c	$\mathfrak{G} = \mathfrak{K} = (1 + {}^*)\mathfrak{H}$	$K_1(A) = K_5(A) = \mathbf{0}$
h	$\mathfrak{G} = 2\mathfrak{H}$	$K_2(A) = \mathbf{0}$

We say a triple is of *type rh* if * is the identity, of *type c* if $\mathfrak{G} = \mathfrak{K} = (1 + {}^*)\mathfrak{H}$, and so on; in other words, it is of the type it would have to have from the table, if it did arise from a real AF algebra.

Any triple of one of the types *r*, *h*, *rh*, *c* can be realized as the triple arising from a real AF algebra [**4**, 9.2, 10.6]. If $\mathfrak{H}$ is simple, then *ch* and *rc* can also be so realized [**5**].

In the case that A has comparability of projections (equivalently, $K_0(A \otimes \mathbf{C}) = \mathfrak{H}$ is totally ordered), then A (and so the corresponding triple) is of type *rh*; this is obtained, either directly from [**4**, 7.10], or via the following remark (which is equivalent to the proof of the result just cited): If B is a finite dimensional real C^*-algebra, then the kernel of the "realification map" $r : K_0(B \otimes \mathbf{C}) \longrightarrow K_0(B)$, does not contain a nonzero projection, hence the same is true if B is any real AF C^*-algebra. When $K_0(A \otimes \mathbf{C})$ is totally ordered, r is one to one. By [**3**, Theorem 4.5], the sequence

$$\mathbf{0} \longrightarrow K_1(A) \longrightarrow K_2(A) \longrightarrow K_2(A \otimes \mathbf{C}) \xrightarrow{r} K_0(A) \longrightarrow K_1(A) \longrightarrow \mathbf{0}$$

is exact. Thus $K_2(A) \cong K_1(A)$, so both must be torsion groups.

2. Rank two. In this section, we determine (up to Morita equivalence) all real AF algebras A such that $K_0(A \otimes \mathbf{C}) = \mathfrak{H}$ is free of rank two. If A is other than an elementary C^*-algebra (it would have to be a direct sum of two copies of the algebra of compact operators on real Hilbert space) $\mathfrak{H}$ will be totally ordered

[**1**]. If we assume that A is simple, then $\mathfrak{H}$ may be regarded as a subgroup of $\mathbf{R}$, equipped with the relative total ordering. We thus may write $\mathfrak{H} = \mathbf{Z}+\theta\mathbf{Z}$ where θ is an irrational real number. In particular, A (and thus the triple $\mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{K}$) must be of type rh.

Now the characterization of the triples of type rh, namely that * is the identity entails that $2\mathfrak{H} \subset \mathfrak{G}$, $\quad \mathfrak{K} \subset \mathfrak{H}$, $\quad \mathfrak{G}+\mathfrak{K} = \mathfrak{H}$ and $\mathfrak{G}\cap\mathfrak{K} = 2\mathfrak{H}$; the map $\mathfrak{H} \longrightarrow \mathfrak{K}$ is simply multiplication by 2. Clearly, the $\mathbf{Z}_2$-vector subspaces of $\mathfrak{H}/2\mathfrak{H}$ satisfy $\mathfrak{G}/2\mathfrak{H} \oplus \mathfrak{K}/2\mathfrak{H} = \mathfrak{H}/2\mathfrak{H}$. If $\mathfrak{G}$ and $\mathfrak{K}$ satisfy this property, we say that the ordered pair $(\mathfrak{G}, \mathfrak{K})$ is $\mathbf{Z}_2$-complementary. Given a triple of dimension groups, $\mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{K}$, with $(\mathfrak{G}, \mathfrak{K})$ being a being a $\mathbf{Z}_2$-complementary pair, and with $\mathfrak{G}$ and $\mathfrak{K}$ inheriting the relative ordering from $\mathfrak{H}$, and with $\mathfrak{H}$ simple, this triple can be realized by a real AF algebra [**4**, 9.2], which necessarily is of type rh. We also give a direct construction of all the algebras, without reference to [**4**]; the latter does not give a method for constructing the algebras.

So the problem of classifying Morita equivalence classes arising in the rank 2 situation reduces to determining all the $\mathbf{Z}_2$-complementary pairs for a given $\mathfrak{H}$, and then deciding which among them are isomorphic.

The first problem is easy; there are precisely 6 non-trivial $\mathbf{Z}_2$-complementary pairs (that is without either $\mathfrak{G} = \mathfrak{H}$ or $\mathfrak{K} = \mathfrak{H}$)—simply note that the number of ordered complementary pairs $(\bar{\mathfrak{G}}, \bar{\mathfrak{K}})$ of subspaces of a k-dimensional $\mathbf{Z}_2$-vector space with dim $\bar{\mathfrak{G}} = 1$ is $2^k \cdot (2^k - 1)$; here $k = 2$ (c.f., [**10**, 3.5–3.7]). Now according to [**4**, 9.2], these triples can all be realized. However, the proof given there is not constructive, that is, there is in general no way to write down the algebra maps. Here we present a construction of the real AF algebras that yield the corresponding triples.

We show that each of the six ordered pairs $(\mathfrak{G}, \mathfrak{K})$ of distinct complementary $\mathbf{Z}_2$-subgroups of $\mathfrak{H} = \mathbf{Z}+\theta\mathbf{Z}$ arise on the algebra level. As preparation, we first note that $GL(2, \mathbf{Z}) = GL(\mathfrak{H})$ acts on the set of pairs $\{(\mathfrak{G}, \mathfrak{K})\}$ via the action obtained from $GL(\mathfrak{H}/2\mathfrak{H}) = GL(2, 2) = S_3$; that $GL(\mathfrak{H})$ acts transitively on these pairs follows from the map $GL(2, \mathbf{Z}) \longrightarrow GL(2, 2)$ being onto.

We construct a real AF C^*-algebra A_1 yielding one of the pairs, and then we use this to obtain the remaining five. Subsequently, we determine which of the pairs are equivalent to any others.

Let $\{a_j\}_{j\geqq 1}$ be a sequence of elements of a group; a *telescoping* of this sequence is a new sequence obtained by multiplying consecutive subsets of them in order, e.g., $\{a_{k(1)} \cdot a_{k(1)-1} \ldots \cdot a_1, a_{k(2)} \cdot a_{k(2)-1} \cdot \ldots \cdot a_{k(1)+1}, a_{k(3)} \cdot \ldots \cdot a_{k(2)+1}, \ldots\}$.

LEMMA 2.1. *Let $\{a_j\}_{j\geqq 1}$ be an (ordered) sequence of elements of a finite group J. Then after deleting a finite initial set of terms, there exists a telescoping in which all terms are the identity element.*

Proof. Define $s_m = a_m \cdot a_{m-1} \cdot \ldots \cdot a_1 \in J$. As J is finite, there exist infinitely many integers $m(1) < m(2) < \ldots$, such that $s_{m(1)} = s_{m(2)} = \ldots = s_{m(k)}$ for all k. Delete $\{a_j\}_{1\leqq j\leqq m(1)}$ and observe that

$$a_{m(k+1)} \cdot \ldots \cdot a_{m(k)+1} = s_{m(k+1)} \cdot s_{m(k)}^{-1} = e$$

and we are done.

Let $[c_0, c_1, \ldots, c_m, \ldots]$ be the continued fraction expansion of the positive irrational number θ. Recall (viz. [**2**]) that $\mathfrak{H} = \mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$ is order isomorphic to the direct limit,

$$\varinjlim \gamma_n : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2$$

where

$$\gamma_n = \begin{bmatrix} c_n & 1 \\ 1 & 0 \end{bmatrix}.$$

Indeed, for $m > n$, define

$$\gamma_{n,m} = \gamma_{m-1} \cdot \ldots \cdot \gamma_n = \gamma_{0,m} \cdot \gamma_{0,n}^{-1}.$$

Then we have the following diagram:

$$(1) \qquad \begin{array}{ccccccccc}
\mathbf{Z}^2 & \xrightarrow{\gamma_{N,N+2}} & \mathbf{Z}^2 & \xrightarrow{\gamma_{N+2,N+4}} & \mathbf{Z}^2 & \longrightarrow & \ldots & \varinjlim(\gamma_n : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2) \\
\nu \big\downarrow & & \big\downarrow \nu\gamma_{N,N+2}^{-1} & & \big\downarrow \nu\gamma_{N,N+4}^{-1} & & & \big\downarrow \tilde{\nu} \\
\mathbf{R} & \xrightarrow{=} & \mathbf{R} & \xrightarrow{=} & \mathbf{R} & \longrightarrow & \ldots & \mathbf{R}
\end{array}$$

where the row vector is given by

$$\nu = \begin{cases} (\theta, 1)\gamma_{0,N}^{-1} & \text{if } N \text{ is even} \\ (1, \theta)\gamma_{0,N}^{-1} & \text{if } N \text{ is odd} \end{cases}$$

and acts by matrix multiplication. We see that the image of the limit group under $\tilde{\nu}$ is $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$.

For all non-negative integers n, let

$$\bar{\gamma}_n = \begin{bmatrix} \bar{c}_n & 1 \\ 1 & 0 \end{bmatrix} \in GL(2, \mathbf{Z}_4)$$

(the bar over c_n denotes its class modulo 4). Applying Lemma 2.1 to $GL(2, \mathbf{Z}_4)$, we deduce that there exists an integer N and an increasing sequence $\{n(k)\}_{k \geqq 0}$ with $n_0 = N + 1$ and

$$\overline{\gamma_{n(k)-1}} \cdot \overline{\gamma_{n(k)-2}} \cdot \ldots \cdot \overline{\gamma_{n(k-1}} = \mathbf{I} \quad \text{for all } k \geqq 1$$

Then define

$$\Psi_k = \gamma_{n(k)-1} \cdot \ldots \gamma_{n(k-1)} = \begin{bmatrix} a_k & 4b_k \\ 4e_k & d_k \end{bmatrix}.$$

By construction, the diagonal entries of the matrices are congruent to 1 modulo 4. Let

$$\phi_k = \begin{bmatrix} a_k & 2b_k \\ 2e_k & d_k \end{bmatrix} \quad \text{in } GL(2, \mathbf{Z})^+$$

and let A_1 be the real AF algebra given as the direct limit of the inductive system,

$$\mathbf{R} \oplus \mathbf{H} \xrightarrow{\Phi_1} M_{r(1)}\mathbf{R} \oplus M_{h(1)}\mathbf{H} \xrightarrow{\Phi_2} M_{r(2)}\mathbf{R} \oplus M_{h(2)}\mathbf{H} \longrightarrow \dots,$$

where Φ_k denotes the standard homomorphism determined by ϕ_k (for example, the (1, 2) entry, $2b_k$, represents the direct sum of $2b_k$ copies of the map $\mathbf{H} \longrightarrow M_4\mathbf{R}$); of course,

$$r(1) = a_1 + 8b_1, h(1) = 2e_1 + d_1, \dots, r(k) = a_k r(k-1) + 8b_k h(k-1), \dots.$$

Then $A_1 \otimes \mathbf{C}$ is given by tensoring the maps and algebras with $\mathbf{C}$. It is easy to check the following:

$$\begin{aligned} K_0(A_1) &= \lim \begin{bmatrix} a_k & 8b_k \\ 2e_k & d_k \end{bmatrix} : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2 \\ K_0(A_1 \otimes \mathbf{H}) &= \lim \begin{bmatrix} a_k & 2b_k \\ 8e_k & d_k \end{bmatrix} : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2; \\ K_0(A_1 \otimes \mathbf{C}) &= \lim \psi_k : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2 \\ c(K_0(A_1)) &= \lim \psi_k : \mathbf{Z} \oplus 2\mathbf{Z} \longrightarrow \mathbf{Z} \oplus 2\mathbf{Z}; \\ c(K_0(A_1 \otimes \mathbf{H})) &= \lim \psi_k : 2\mathbf{Z} \oplus \mathbf{Z} \longrightarrow 2\mathbf{Z} \oplus \mathbf{Z}. \end{aligned}$$

As in Section 1, for A a real C^*-algebra, c will denote the natural inclusion,

$$K_0(A) \longrightarrow K_0(A \otimes \mathbf{C}),$$

induced by complexifying. Now $\mathfrak{G} = c(K_0(A_1))$ and $\mathfrak{K} = c(K_0(A_1 \otimes \mathbf{H}))$ are index two subgroups of $\mathfrak{H} = K_0(A_1 \otimes \mathbf{C})$. The outcome of this is that we have constructed one of the possible 6 Morita equivalence classes of rh algebras with

$$K_0(A \otimes \mathbf{C}) = \mathbf{Z} + \theta\mathbf{Z} \quad \text{and} \quad K_0(A \otimes \mathbf{C})/K_0(A) = \mathbf{Z}_2.$$

We can obtain the remaining ones as follows.

Write the eigenvector $\nu = (r, s)$ with entries in $\mathbf{Z} + \theta\mathbf{Z}$; this yields a map (also called ν) $\nu : \mathbf{Z}^2 \longrightarrow \mathbf{R}$; necessarily $r\mathbf{Z} + s\mathbf{Z} = \mathbf{Z} + \theta\mathbf{Z}$. Next, notice that for the algebra above,

$$\begin{aligned} &\tilde{\nu}(c(K_0(A_1))) = \tilde{\nu}(\mathfrak{G}) = r\mathbf{Z} + 2s\mathbf{Z}, \quad \text{and} \\ &\tilde{\nu}(c(K_0(A_1 \otimes \mathbf{H}))) = 2r\mathbf{Z} + s\mathbf{Z}. \end{aligned}$$

Now define

$$P_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{in } GL(2, \mathbf{Z}).$$

Then

$$P_1 \gamma_n P_1^{-1} = \begin{bmatrix} c_n - 1 & 1 \\ c_n & 1 \end{bmatrix};$$

this has no negative coefficients, so we can construct another algebra A_2 by using these matrices in place of the original choice,

$$\begin{bmatrix} c_n & 1 \\ 1 & 0 \end{bmatrix}.$$

Applying P_1 to the diagram (1), we see that the image of

$$\lim P_1 \gamma_n P_1^{-1} : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2$$

under $\tilde{\nu}$ is simply that of $\tilde{\nu} P_1$ on $\lim \gamma_n : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2$. We deduce that

$$\tilde{\nu}(c(K_0(A_2))) = \tilde{\nu} P_1(c(K_0(A_1))) = (r+s, s)(\mathbf{Z} \oplus 2\mathbf{Z}) = (r+s)\mathbf{Z} + 2s\mathbf{Z}.$$

Similarly,

$$\tilde{\nu}(c(K_0(A_2 \otimes \mathbf{H}))) = 2r\mathbf{Z} + s\mathbf{Z}.$$

As $(r+s)\mathbf{Z} + 2s\mathbf{Z} \neq r\mathbf{Z} + 2s\mathbf{Z}$ (neither r nor s lies in the former), we have a distinct $\mathbf{Z}_2$-complementary pair, $(\mathfrak{G}^2, \mathfrak{K}^2)$, of subgroups of $\mathfrak{H}$.

Next, we apply the matrix

$$P_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

to the first diagram. Then we obtain

$$\tilde{\nu}(c(K_0(A_3))) = \tilde{\nu} P_2(c(K_0(A_1))) = r\mathbf{Z} + 2s\mathbf{Z}, \quad \text{and}$$
$$\tilde{\nu}(c(K_0(A_3 \otimes \mathbf{H}))) = \tilde{\nu} P_2(c(K_0(A_1 \otimes \mathbf{H}))) = 2r\mathbf{Z} + (s-r)\mathbf{Z}.$$

This defines $(\mathfrak{G}^3, \mathfrak{K}^3)$.

Now for each of the three algebras, we can obtain a new one by replacing the algebra by its quaternionification. The effect of this is to replace the ordered pair $(\mathfrak{G}^i, \mathfrak{K}^i)$ by $(\mathfrak{K}^i, \mathfrak{G}^i)$. Hence we obtain 6 ordered pairs of index two subgroups of $\mathbf{Z} + \theta\mathbf{Z}$. It is quite easy to check that these 6 are distinct, and so we have constructed all of the possible complementary pairs.

Next we proceed to investigate Morita equivalences among the 6 *rh* algebras A with

$$K_0(A \otimes \mathbf{C}) = \mathbf{Z} + \theta\mathbf{Z} \quad \text{and} \quad K_0(A \otimes \mathbf{C})/K_0(A) = \mathbf{Z}_2.$$

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In order that the algebras be Morita equivalent, it is necessary and sufficient by [**4**, Theorem 5.1], [**3**, 8.3], or [**10**, 2.4], that there exist an order isomorphism

$$\phi : \mathbf{Z} + \theta\mathbf{Z} \longrightarrow \mathbf{Z} + \theta\mathbf{Z}$$

so that $\phi(\mathfrak{G}) = \mathfrak{G}'$ and $\phi(\mathfrak{K}) = \mathfrak{K}'$, where $(\mathfrak{G}, \mathfrak{K})$ and $(\mathfrak{G}', \mathfrak{K}')$ are the respective $\mathbf{Z}_2$-complementary pairs.

If θ is not quadratic, then none of $2\mathbf{Z} + \theta\mathbf{Z}$, $\mathbf{Z} + 2\theta\mathbf{Z}$, $(\theta + 1)\mathbf{Z} + (\theta - 1)\mathbf{Z}$ are order isomorphic; to see this, simply observe that any order isomorphism among them will extend (after regarding them, as we have, as subgroups of $\mathbf{R}$) to a multiplication by a real number, call it ρ. Then ρ will induce an order automorphism of the rational vector space $\mathbf{Q}+\theta\mathbf{Q}$. Hence each of $1, \rho, \rho^2$ belongs to $\mathbf{Q}+\theta\mathbf{Q}$. As the latter is two dimensional, ρ satisfies a quadratic equation, which is impossible unless θ is quadratic. One may also apply Lemma 2.2 below.

Hence in the non-quadratic case, it follows that not only are there no isomorphisms of the triples (hence no Morita equivalences of the algebras) but the pair $(K_0(A), K_0(A \otimes \mathbf{H}))$ without reference to the maps, determines the Morita equivalence class. In contrast, we shall give an example of two *rh* algebras, A_1, A_2, such that all six groups, $K_0(A_1)$, $K_0(A_1 \otimes \mathbf{H})$, $K_0(A_1 \otimes \mathbf{C})$, $K_0(A_2)$, $K_0(A_2 \otimes \mathbf{H})$, $K_0(A_2 \otimes \mathbf{C})$ are order isomorphic, yet with A_1 and A_2 not being Morita equivalent.

The following two lemmas are well-known; details may be found in [**6**].

LEMMA 2.2. *Let $\mathfrak{H}$ be a subgroup of $\mathbf{R}$, equipped with the relative total ordering. Then any order-preserving group homomorphism $\phi : \mathfrak{H} \longrightarrow \mathfrak{H}$ is given by a multiplication by a positive real number.*

Let $\mathfrak{H} = \mathbf{Z} + \theta\mathbf{Z}$, and following [**6**, §5], define $E = E(\mathfrak{H}) = \mathrm{End}_c(\mathfrak{H})$ (the group of endomorphisms of $\mathfrak{H}$ that are continuous with respect to the topology of $\mathfrak{H}$ inherited from $\mathbf{R}$; in this case,

$$E = \{\rho \in \mathbf{R} \mid \rho, \rho\theta \in \mathbf{Z} + \theta\mathbf{Z}\}.$$

Then E is a subring of $\mathbf{R}$ that is free of rank 2 (additively), $\mathfrak{H}$ viewed as an E-module is isomorphic to an ideal of E, and $\mathrm{End}_E(\mathfrak{H})$ (the E-module endomorphisms of $\mathfrak{H}$) is naturally isomorphic to E. Moreover, if $\mathfrak{H}$ and $\mathfrak{H}'$ are free subgroups of $\mathbf{R}$ of rank 2, with the same E, then they are order isomorphic if and only if they are E-module isomorphic. See [**6**, §5] for more details.

LEMMA 2.3. *Let $\mathfrak{H}$ be a subgroup of $\mathbf{R}$, equipped with the relative total ordering. Suppose additionally that the rank of $\mathfrak{H}$ (as an abelian group) is a prime number. Then if $\mathfrak{H}$ admits a non-trivial (that is, not multiplication by an integer) order-preserving endomorphism, the ring of continuous endomorphisms of $\mathfrak{H}$, E, has the same additive rank as $\mathfrak{H}$, and $\mathfrak{H}$ is a rank one E-module. If $\mathfrak{H}$ is free, then E is integral over $\mathbf{Z}$, and E^* and* $\mathrm{Aut}_0(\mathfrak{H})$ *are finitely generated abelian groups.*

In particular, if θ is a quadratic integer, then $E = \mathbf{Z} + \theta\mathbf{Z} = \mathbf{Z}[\theta] = \mathfrak{H}$ already. If θ is in the $PGL(2, \mathbf{Z})$-orbit of an algebraic integer, we may assume θ is already such, so that $\mathfrak{H}$ corresponds to the principal ideal class of $E = \mathbf{Z}[\theta]$ (non-principal ideals give rise to choices for θ that are not $PGL(2, \mathbf{Z})$- equivalent to an algebraic integer).

If $2\mathfrak{H} \subset \mathfrak{G} \subset \mathfrak{H} = \mathbf{Z} + \theta\mathbf{Z}$, there is in general no inclusion $E(\mathfrak{G}) \subset E(\mathfrak{H})$ or vice versa. Hence $E(\mathfrak{G})$ can be used to distinguish some order-isomorphism classes of $\mathbf{Z}_2$-complementary pairs $(\mathfrak{G}, \mathfrak{K})$. The units (invertible elements) of E induce automorphisms of $\mathfrak{H}$; since $\mathbf{R}$ is totally ordered, for any unit Ψ in E, one of $\pm\Psi$ will induce an order automorphism of $\mathfrak{H}$. Moreover, since E has additive rank 2 and $E \subset \mathbf{R}$, there exists Ψ (exceeding 1 as an element of $\mathbf{R}$) in E (called the fundamental unit) such that

$$E^* = \{\pm 1\} \times \{\Psi^n \mid n \in \mathbf{Z}\};$$

that is, $E^*/\{\pm 1\}$ is cyclic (see e.g., [**9**, ¶4.6]).

Let $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ be the three index two subgroups of $\mathfrak{H}$. There are three possible situations:

(i) $\Psi\mathfrak{G}_i = \mathfrak{G}_i$ for all i. In this case, Ψ and therefore all of its powers do not implement any equivalences among the six pairs, so that all six algebras are mutually inequivalent.

(ii) $\Psi\mathfrak{G}_1 = \mathfrak{G}_1$, $\Psi\mathfrak{G}_2 = \mathfrak{G}_3$, and $\Psi\mathfrak{G}_3 = \mathfrak{G}_2$. In this case Ψ fixes one of the subgroups and interchanges the other two. Then Ψ induces isomorphisms among the pairs as follows:

$$(\mathfrak{G}_1, \mathfrak{G}_2) \cong (\mathfrak{G}_1, \mathfrak{G}_3); (\mathfrak{G}_2, \mathfrak{G}_3) \cong (\mathfrak{G}_3, \mathfrak{G}_2); (\mathfrak{G}_2, \mathfrak{G}_1) \cong (\mathfrak{G}_3, \mathfrak{G}_1).$$

Even powers of Ψ induce no isomorphisms, while odd powers induce the same ones as Ψ. Hence there are 3 Morita equivalence classes of algebras.

(iii) $\Psi\mathfrak{G}_1 = \mathfrak{G}_2$, $\Psi\mathfrak{G}_2 = \mathfrak{G}_3$, and $\Psi\mathfrak{G}_3 = \mathfrak{G}_1$ (up to relabelling). Then Ψ and Ψ^2 induce

$$(\mathfrak{G}_1, \mathfrak{G}_2) \cong (\mathfrak{G}_2, \mathfrak{G}_3) \cong (\mathfrak{G}_3, \mathfrak{G}_1); (\mathfrak{G}_2, \mathfrak{G}_1) \cong (\mathfrak{G}_3, \mathfrak{G}_2) \cong (\mathfrak{G}_1, \mathfrak{G}_3).$$

Here there are exactly two Morita equivalence classes.

This exhausts the possibilities, as $(E^*)^+$ must act as a group of permutations on $\{\mathfrak{G}_i\}$. Hence we can obtain 2, 3, or 6 equivalence classes of algebras (with neither $\mathfrak{K} = \mathfrak{H}$, corresponding to type h, nor with $\mathfrak{G} = \mathfrak{H}$, corresponding to type r).

Now case (i) occurs if and only if $\Psi \in \cap E(\mathfrak{G}_i)$, that is, $E^* \subset E(\mathfrak{G}_i)$ for $i = 1, 2, 3$. Case (ii) occurs if and only if Ψ belongs to exactly one of the $E(\mathfrak{G}_i)$, and (iii) arises precisely when E^* is not contained in any of the $E(\mathfrak{G}_i)$.

We now present some examples to illustrate each of these situations described here, together with some analysis.

Let $\mathfrak{H}$ and $E = E(\mathfrak{H})$ be fixed, with $\mathfrak{H} = \mathbf{Z} + \theta\mathbf{Z}$ where θ is some quadratic (not necessarily integral) real number. If E happens to be integrally closed, then the ideal of E generated by 2, $2E$, behaves in one of the three following ways:

(a) $2E = p$, where p is a prime ideal of E (2 is *inert*);

(b) $2E = p^2$, where p is prime (2 *ramifies*);

(c) $2E = pq$, where p and q are distinct prime ideals (2 *splits*).

For example, if

$$E = \mathbf{Z}\left[\frac{1+\sqrt{5}}{2}\right],$$

then $2E$ is prime, as there are no solutions to $x^2 + x + 1 = 0$ modulo 2, so that in this case, 2 is inert. If $E = \mathbf{Z}[\sqrt{2}]$, then $2E = (\sqrt{2}E)^2$, so that 2 ramifies in this case. If $E = \mathbf{Z}[\sqrt{7}]$, then

$$2E = \{(3+\sqrt{7})E\} \cdot \{(3-\sqrt{7})E\},$$

and here 2 splits.

If 2 splits, then $p\mathfrak{H}$ and $q\mathfrak{H}$ are distinct subgroups of $\mathfrak{H}$ of index 2 (as $p\mathfrak{H}/2\mathfrak{H} \cong \mathbf{Z}_2$). Any order-automorphism Ψ of $\mathfrak{H}$ belongs to E^*, so that we must have $\Psi(p\mathfrak{H}) = p\mathfrak{H}$ and $\Psi(q\mathfrak{H}) = q\mathfrak{H}$. Obviously this forces Ψ to fix the third intermediate subgroup of $\mathfrak{H}$, so that the permutation action of Ψ is trivial. Hence, 6 distinct Morita equivalence classes of algebras occur here.

If 2 ramifies, any Ψ in E^* must fix $p\mathfrak{H}$, hence its permutation action is either trivial, or of order 2. Either situation can occur; for example, if $\theta = \sqrt{2}$, then $\mathfrak{H} = E = \mathbf{Z}[\sqrt{2}]$, and an easy computation reveals that the fundamental unit, $1+\sqrt{2}$, interchanges the subgroups of $\mathfrak{H}$ other than $p = p\mathfrak{H}$. Thus for $\theta = \sqrt{2}$, there are exactly three distinct Morita equivalence classes. On the other hand, if $\theta = \sqrt{6}$, then $\mathfrak{H} = E = \mathbf{Z}[\sqrt{6}]$; its fundamental unit is $\Psi = 5 + 2\sqrt{6}$, which belongs to $\mathbf{Z}[2\sqrt{6}]$. As $\mathbf{Z}[2\sqrt{6}] \subset \cap E_i$, $\Psi\mathfrak{G}_i = \mathfrak{G}_i$ for $i = 1, 2, 3$. Thus 6 distinct Morita equivalence classes result.

Finally, suppose 2 is inert. In all these cases (and even the non-integrally closed cases), we may write $E = \mathbf{Z}[\tau]$, where τ satisfies the quadratic equation, $x^2 + (\operatorname{tr}\tau)x + N(\tau) = 0$. Then 2 is inert precisely if the factor ring, $E/2E = \mathbf{Z}_2[\bar{\tau}]$ is a field. This happens precisely when both $\operatorname{tr}\tau$ and $N(\tau)$ are odd (since x^2+x+1 is the only irreducible quadratic over $\mathbf{Z}_2$). If under these circumstances, Ψ in E^* has order 2 as a permutation of $\{\mathfrak{G}_i\}$, then $\Psi^2 - 1$ belongs to $2E$, but $\Psi - 1$ does not. However, modulo 2, $\Psi^2 - 1$ is congruent to $(\Psi + 1)^2$; as the factor ring is a field, this implies $\Psi \pm 1$ belongs to $2E$, a contradiction. Hence the order of the permutation induced by Ψ can only be 1 or 3, corresponding to 6 or 2 respectively Morita equivalence classes.

If $\theta = (1+\sqrt{5})/2$, then $E = \mathfrak{H} = \mathbf{Z}[\theta]$ and $\Psi = \theta$. As

$$\Psi^3 - 1 = (\Psi - 1)(\Psi^2 + \Psi + 1) \in \mathbf{Z}[2\theta] \quad \text{and}$$
$$\Psi - 1 \notin \mathbf{Z}[2\theta],$$

we deduce that Ψ has order 3 as a permutation of $\{\mathfrak{G}_i\}$. Thus there are 2 Morita equivalence classes in this example.

On the other hand, if $\theta = (1+\sqrt{37})/2$, then $E = \mathfrak{H} = \mathbf{Z}[\theta]$ and the fundamental unit is $\Psi = 6+\sqrt{37} = 5+2\theta$. Hence $\Psi - 1$ belongs to $2E$, so that Ψ can only act as a trivial permutation and so there are 6 distinct Morita equivalence classes. As $\operatorname{tr}\theta = 1$ and $N(\theta) = -9$, 2 is inert.

In all of the examples thus far, $E = \mathfrak{H} = \mathbf{Z}[\theta]$ and E was integrally closed. If θ' is $PGL(2,\mathbf{Z})$-equivalent to θ and the latter is an algebraic integer, we may replace θ' by θ. If however, θ is quadratic, but is not $PGL(2,\mathbf{Z})$-equivalent to an algebraic integer, then $\mathfrak{H} = \mathbf{Z}+\theta\mathbf{Z}$ will correspond to a non-trivial ideal class of E. This really makes no difference to the analysis of the possibilities, since the relevant properties are preserved by localization at 2. We give an example anyway.

Let $\theta = \sqrt{15}/3$. Then $E = \mathbf{Z}[\sqrt{15}]$ which is integrally closed. Now $\mathfrak{H}$ is E-module isomorphic to the ideal of E, $\mathfrak{b} = (3,\sqrt{15})$. This is not principal, as the equation $a^2 - 15b^2 = \pm 3$ has no integer solutions (neither of ± 3 is a square modulo 5). Next

$$2E = (2, 1+\sqrt{15})^2 = p^2,$$

so that 2 ramifies. In particular, $p\mathfrak{b}$ is a subgroup of $\mathfrak{b}$ of index 2, and obviously is invariant under any Ψ in E^*. The other two index 2 subgroups are

$$\mathfrak{G}_2 = 6\mathbf{Z}+\sqrt{15}\mathbf{Z} \quad \text{and} \quad \mathfrak{G}_3 = 3\mathbf{Z}+2\sqrt{15}\mathbf{Z}.$$

The fundamental unit of E is $\Psi = 4+\sqrt{15}$. We observe that 15 belongs to $\sqrt{15}\mathfrak{G}_2$, so that $\Psi\mathfrak{G}_2 \neq \mathfrak{G}_2$. This forces Ψ to interchange $\mathfrak{G}_2$ and $\mathfrak{G}_3$, so that its permutation order is 2, and thus there are three Morita equivalence classes.

If we drop the requirement that E be integrally closed, all of the previous analysis (concerning the behaviour of 2 and sizes of orbits) applies, except that there is an fourth phenomenon, which is rather similar to ramification.

(d) There exists a prime ideal p of E such that $p^2 \subset 2E \subset p$ with the inclusions being strict.

In this case (as in the case with ramification), $p\mathfrak{H}$ is the only E-module of index 2 in $\mathfrak{H}$, and of course it is invariant under E^*. Thus the orbits of $\{\mathfrak{G}_i\}$ under the action of E^* can only be of sizes one and two. In particular, either there are 6 distinct Morita equivalence classes or there are 3.

If $\theta = 2\sqrt{2}$, then $\mathfrak{H} = E = \mathbf{Z}[\theta]$, which is not integrally closed. Now

$$\mathfrak{G}_1 = p = 2\mathbf{Z}+2\sqrt{2}\mathbf{Z}, \quad \mathfrak{G}_2 = \mathbf{Z}+4\sqrt{2}\mathbf{Z}, \quad \text{and}$$
$$\mathfrak{G}_3 = (2\sqrt{2}+1)\mathbf{Z}+(2\sqrt{2}-1)\mathbf{Z}.$$

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It is easy to check that $p^2 \subset 2E \subset p$, the inclusions being strict. The fundamental unit is

$$\Psi = (1+\sqrt{2})^2 = 5 + 2\sqrt{2}.$$

As $2\sqrt{2}$ belongs to $2\sqrt{2}\mathfrak{G}_2$, but not to $\mathfrak{G}_2$, it follows that Ψ must permute $\mathfrak{G}_2$ and $\mathfrak{G}_3$, whence 3 Morita equivalence classes result.

If instead $\theta = 2\sqrt{14}$, we observe that the fundamental unit of $\mathbf{Z}[\sqrt{14}]$ is

$$\Psi = 15 + 4\sqrt{14} = 15 + 2\theta;$$

it is thus the fundamental unit of $E = \mathbf{Z}[\theta]$ as well. Since Ψ belongs to $\mathbf{Z}[2\theta]$, it follows immediately that $\Psi\mathfrak{G}_i = \mathfrak{G}_i$ and thus all 6 choices are mutually inequivalent.

3. Rank three, not of type ***rh*****.** Let $\mathfrak{H}$ be free of rank 3, and such that * is not the identity. In this case, we show successively that there is a unique state, and the corresponding algebras giving rise to $\mathfrak{H}$ (in the sense that $(\mathfrak{H},^*) \cong (K_0(A \otimes \mathbf{C}),^*)$) must be of type *rc* or *ch* (but cannot be of type *c*).

We first observe that as * is not trivial, the rank of $\mathfrak{G}$ is less than or equal to 2; however, $A \otimes \mathbf{C}$ being simple implies that A is simple, and thus $\mathfrak{G}$ is a simple dimension group, and is clearly not $\mathbf{Z}$ (as this would entail that A be finite dimensional). Hence the rank of $\mathfrak{G}$ is 2, and being free, $\mathfrak{G}$ must be a subgroup of $\mathbf{R}$, with the relative ordering; in particular, it has just one state.

Now $\mathfrak{H}$ being simple and free of rank 3, cannot have more than 2 pure states [**1**, 4.1], and as they must agree on $\mathfrak{G}$, they are related via the *. That is, if the two pure states of $\mathfrak{H}$ are τ_1 and τ_2, we have that $(\tau_1)^* = \tau_2$. Since τ_1 is faithful on $\mathfrak{G}$, it must be that the map, $(\tau_1, \tau_2) : \mathfrak{H} \longrightarrow \mathbf{R}^2$ is one to one. Since $\mathfrak{H}$ separates the points, and is a simple dimension group, its image is dense.

Next we observe that

$$2\mathfrak{H} \subset \mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss} \subset \mathfrak{H}.$$

As the image of $2\mathfrak{H}$ is clearly dense (from that of $\mathfrak{H}$ being dense), the middle term has dense image. Also $\mathfrak{G} \subset \mathfrak{H}^{sa}$, and $\mathfrak{H}^{ss} \neq (0)$, so that the rank of $\mathfrak{H}^{sa}$ is 2, and that of $\mathfrak{H}^{ss}$ is 1. We may normalize the states so that the image of $\mathfrak{H}^{sa}$ under the map is $(1, 1)\mathbf{Z} + (r, r)\mathbf{Z}$ for some irrational number r. Also the generator of $\mathfrak{H}^{ss}$ will be sent to something of the form (a, b). We have that $\{(1, 1), (r, r), (a, b)\}$ generates a dense subgroup of $\mathbf{R}^2$. This is impossible, as

$$\det \begin{bmatrix} 1 & 1 \\ r & r \end{bmatrix} = 0$$

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([**1**, 4.2]). This contradiction yields that $\mathfrak{H}$ may have only one state.

Now consider,

$$(1 + {}^*)\mathfrak{H} \oplus \mathfrak{H}^{ss} \subset \mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss} \subset \mathfrak{H}.$$

We observe that $(1+{}^*)\mathfrak{H} \subset \mathfrak{G} \subset \mathfrak{H}^{sa}$, and the rank of each of these three groups is 2. Hence the torsion part of $\mathfrak{H}/(1+{}^*)\mathfrak{H}$ is either $\mathbf{Z}_2$ or $(\mathbf{Z}_2)^2$.

Next we observe that for any triple $\mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{K}$ (arising from a real AF algebra), $\mathfrak{H} = \mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss}$ implies that H^{ss} is 2-divisible. Simply note that $\mathfrak{H}^{ss} = (1 - {}^*)\mathfrak{H}$ for all such triples, so that on applying $(1 - {}^*)$, we deduce that $\mathfrak{H}^{ss} = 2\mathfrak{H}^{ss}$.

Up to this point, we know that $\mathfrak{H} \neq \mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss}$. We note however, that there are strict inclusions,

$$\mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss} \subset \mathfrak{H} \subset \tfrac{1}{2}\mathfrak{H}^{sa} \oplus \tfrac{1}{2}\mathfrak{H}^{ss}.$$

Select (x, y) in $\mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss}$ such that $\frac{1}{2}(x, y) \in \mathfrak{H}\backslash(\mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss})$, and let $\mathfrak{H}'$ be the subgroup of $\mathfrak{H}$ generated by $\frac{1}{2}(x, y)$ and $\mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss}$. Then

$$(1 - {}^*)\mathfrak{H}' = 2\mathfrak{H}^{ss} + y\mathbf{Z};$$

if we choose y not to belong to $2\mathfrak{H}^{ss}$, then we would have that $(1 - {}^*)\mathfrak{H}' = \mathfrak{H}^{ss}$ (because $\mathfrak{H}^{ss}$ is cyclic). Similarly, select x not to belong to $2\mathfrak{H}^{sa}$. Then we claim that $\mathfrak{H} = \mathfrak{H}'$.

If $\mathfrak{H} \neq \mathfrak{H}'$, then we may find $\frac{1}{2}(x', y') \in \mathfrak{H}\backslash\mathfrak{H}'$, with (x', y') in $\mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss}$. If $\frac{1}{2}(y'+y)$ belonged to H^{ss}, we would obtain that $\frac{1}{2}x'$ belongs to $\mathfrak{H}\backslash\mathfrak{H}'$, whence it belongs to $\mathfrak{H}^{sa}$, a contradiction. Thus $\frac{1}{2}(y'+y)$ does not belong to $\mathfrak{H}^{ss}$. However, $\frac{1}{2}\mathfrak{H}^{ss}/\mathfrak{H}^{ss} = \mathbf{Z}_2$, so that at least one of $\frac{1}{2}y$, $\frac{1}{2}y'$, $\frac{1}{2}(y + y')$ belongs to $\mathfrak{H}^{ss}$. This forces $\frac{1}{2}x'$ to belong to $\mathfrak{H}$, but not to $\mathfrak{H}'$. As x' and $\frac{1}{2}x'$ are self-adjoint, it follows that they belong to $\mathfrak{H}^{sa}$, which is contained in $\mathfrak{H}'$, again a contradiction. Thus $\mathfrak{H} = \mathfrak{H}'$. The conclusion of this argument is that

$$\mathfrak{H} = \mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss} + \tfrac{1}{2}(x, y)\mathbf{Z},$$

where $x \in \mathfrak{H}^{sa}\backslash 2\mathfrak{H}^{sa}$, and $y \in \mathfrak{H}^{sa}\backslash 2\mathfrak{H}^{sa}$.

Thus (with this choice of x), $(1+{}^*)\mathfrak{H} = x\mathbf{Z} + 2\mathfrak{H}^{sa}$, so that $\mathfrak{H}^{sa}/(1+{}^*)\mathfrak{H}$ has order 2 (as $\mathfrak{H}^{sa}$ is free of rank 2 and x does not belong to $2\mathfrak{H}^{sa}$). We deduce from $\mathfrak{G} + \mathfrak{K} = \mathfrak{H}^{sa}$ and $\mathfrak{G} \cap \mathfrak{K} = (1+{}^*)\mathfrak{H}$, that either $\mathfrak{G}$ or $\mathfrak{K}$ is $(1+{}^*)\mathfrak{H}$, which corresponds to *ch* or *rc*. Notice that $\mathfrak{H}$ cannot be of type c, for this would be equivalent to $\mathfrak{H}^{sa} = (1 + {}^*)\mathfrak{H}$.

Next we consider the isomorphism classes.

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PROPOSITION 3.1. *Let $\mathfrak{H}$ be a dimension group free of rank* 3, *with a unique trace τ whose kernel is of rank* 1, *say with generator z. Let $\{z, \alpha, \beta\}$ be a basis for $\mathfrak{H}$. Then with respect to this basis, every order-automorphism of $\mathfrak{H}$ is of one of the forms:*

$$\begin{bmatrix} \pm 1 & p & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad or \quad \begin{bmatrix} \pm 1 & p & s \\ 0 & q & t \\ 0 & r & u \end{bmatrix} \quad where \quad \Lambda = \begin{bmatrix} q & t \\ r & u \end{bmatrix}$$

is an automorphism of $\tau(\mathfrak{H})$ (induced by multiplication by some quadratic unit λ). These are all order automorphisms.

Proof. Let ϕ denote the order automorphism. Then $\phi(\ker \tau) = \ker \tau$, so that $\phi(z) = \pm z$, and clearly ϕ induces an automorphism on $\mathfrak{H}/\ker \tau$, and the desired matrices result. To check that these are order automorphisms is straightforward.

PROPOSITION 3.2. *Let $\mathfrak{H}$ be a dimension group free of rank* 3, *and having a unique pure trace τ such that*

(i) $\ker \tau$ *is of rank* 1

(ii) $\mathfrak{H}^+ \backslash \{0\} = \{h \in \mathfrak{H} \mid \tau(h) > 0\}$.

*Let * be an order-preserving involution of $\mathfrak{H}$, and suppose that $\{z, \alpha, \beta\}$ is a basis for $\mathfrak{H}$ such that* $\ker \tau = z\mathbf{Z}$.

(a) *Then the matrix for * is of the form*

$$A_{a,b} = \begin{bmatrix} -1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some choice of a and b in $\mathbf{Z}$, *and every such choice of a and b yields an automorphism.*

(b) *If * and # are two automorphisms with corresponding matrices $A_{a,b}$ and $A_{a',b'}$, and if $\tau(\mathfrak{H})$ admits no order automorphisms (which is generic), then $(\mathfrak{H}, ^*) \cong (\mathfrak{H}, \#)$ if and only if both $a \equiv a'$ and $b \equiv b'$ modulo* 2. *Generically, there are three equivalence classes of $(\mathfrak{H}, ^*)$ which can arise from a real AF algebra.*

(b′) *If in* (ii), *we assume that $\tau(\mathfrak{H})$ admits a non-trivial order automorphism, then its order automorphism group is cyclic with generator say Φ in $E = \mathrm{End}_{\mathbf{Q}}(\tau(\mathfrak{H}))$. We have that:*

There exist 3 *isomorphism classes for $(\mathfrak{H}, ^*)$ to arise from a real AF algebra if and only if $\Phi - 1$ belongs to $2E$.*

There exist 2 *isomorphism classes for $(\mathfrak{H}, ^*)$ to so arise if and only if $(\Psi - 1)^2 \in 2E$ and $\Phi - 1$ does not belong to $2E$.*

Otherwise there exists precisely 1 *isomorphism class for $(\mathfrak{H}, ^*)$ so obtained.*

Proof. As $z^* = -z$, by the previous result, $*$ must be given by the matrix

$$X = \begin{bmatrix} -1 & p & s \\ 0 & e & f \\ 0 & g & h \end{bmatrix}$$

for some p and s in $\mathbf{Z}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ in $GL(2, \mathbf{Z})$.

From $\alpha \longmapsto \alpha^* = pz + e\alpha + g\beta$ and $\tau(\alpha) = \tau(\alpha^*)$ (a priori, it could happen that $\tau(a) = q \cdot \tau(a^*)$ for all a in H, for some quadratic unit q; however, applying $*$ again yields $\tau(a^*) = q \cdot \tau(a^{**}) = q \cdot t(a)$, so that $q^2 = 1$) we deduce that

$$(e-1)\tau(\alpha) + g\tau(\beta) = 0.$$

However, $\{\tau(\alpha), \tau(\beta)\}$ is linearly independent over $\mathbf{Z}$, whence $e = 1$ and $g = 0$. Similarly, replacing α by β yields $h = 1$ and $f = 0$. Hence $X = A_{p,s}$ for some integers p, s. From $A_{p,s}^2 = \mathbf{I}$ and that τ is invariant under the involution, we deduce that every choice for p and s yields an order-automorphism.

Now let $\rho : (\mathfrak{H}, ^*) \longrightarrow (\mathfrak{H}, \#)$ be an order-automorphism that preserves the involutions. Again by the previous result, the matrix of ρ with respect to the basis $\{z, \alpha, \beta\}$ is

$$\begin{bmatrix} -1 & v & w \\ 0 & & \\ & \Lambda & \\ 0 & & \end{bmatrix}$$

where Λ induces multiplication by some quadratic unit (possibly 1) on $\tau(\mathfrak{H})$. In case (b), $\Lambda = \mathbf{I}$, so that the matrix of ρ is especially simple. We observe that ρ induces an order-automorphism preserving the involution if and only if the diagram

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{A_{a,b}} & \mathfrak{H} \\ \rho\downarrow & & \downarrow\rho \\ \mathfrak{H} & \xrightarrow[A_{a',b'}]{} & \mathfrak{H} \end{array}$$

commutes. That is $A_{a',b'}\rho = \rho A_{a,b}$. We deduce immediately that this occurs if and only if $2v = a' + a$ and $2w = b' + b$. Thus such a ρ exists if and only if $a \equiv a'$ and $b \equiv b'$ modulo 2. Hence there exist at most 4 equivalence classes for involutions (the congruence classes of a and b modulo 2). In fact one of these cannot arise in our context.

We observe that if a and b are both even, then $\mathfrak{H} = \mathfrak{H}^{sa} \oplus \mathfrak{H}^{ss}$, which as we have seen previously, cannot occur if $(\mathfrak{H}, ^*)$ is to arise from a real AF algebra for which * is not trivial. On the other hand, it is routine to verify that each of the other three choices for a and b do yield suitable involutions (for which the corresponding type can be rc or ch, depending on the choice of $\mathfrak{G} = \mathfrak{H}^{sa}$ or $\mathfrak{G} = (1 + {}^*)\mathfrak{H}$).

In the case that $\tau(\mathfrak{H})$ is quadratic, we have that ρ is now represented by a matrix with Λ in it, and some of the 3 equivalence classes previously obtained may now become equivalent. The equation resulting from commutativity of the diagram is now

$$2(v, w) = (a', b')\Lambda \pm (a, b).$$

In order words, such a Λ will exist if and only if $(a', b')\Lambda \equiv (a, b)$ modulo 2. Now mod 2, Λ can only be equivalent to one of the 6 matrices,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The first (the identity) yields no new equivalence relations. The second, third and sixth are of order 2, and yield one new equivalence relation (for example, the sixth yields $(1, 0) \sim (0, 1)$), and the fourth and fifth generate each other and are of order 3, and act transitively on the three equivalence classes. It is routine to translate the orders of the matrices to the corresponding statement, as indicated in (b').

As in Section 2, we present a direct construction for the real AF algebras of type rc, A, such that $K_0(A \otimes \mathbf{C})$ is free of rank 3. This covers the type ch algebras as well: Merely tensor with the quaternions. (See also Section 4 for a discussion in the rh situation.) The construction is analogous to the rank 2 case, so we only present a detailed outline of it, leaving some details to the reader.

Let θ be a positive irrational number. Following [**11**] and [**2**], we denote by $(\mathbf{Z}^3, P_\theta)$ the dimension group that is free of rank three, with positive cone

$$P_\theta = \{(a, b, c)^T \in \mathbf{Z}^3 \mid \theta a + b + c \geqq 0\}.$$

LEMMA 3.3. *Let $\mathfrak{H}$ be a simple dimension group free of rank 3, having a unique state, whose kernel is of rank 1. Then there is a positive irrational number θ such that $\mathfrak{H} \cong (\mathbf{Z}^3, P_\theta)$ (as ordered groups).*

Proof. Let $\{z, \alpha, \beta\}$ be a basis for $\mathfrak{H}$ as in 3.1 and 3.2, with state τ, wherein $\tau(\alpha)$ and $\tau(\beta)$ are positive. Now

$$\mathfrak{H}^+ = \{\mathbf{0}\} \cup \{h \in \mathfrak{H} \mid \tau(h) > 0\}.$$

Since $\tau(\mathfrak{H})$ is a free rank 2 subgroup of $\mathbf{R}$ with generators $\{\tau(\alpha), \tau(\beta)\}$, it follows that the ratio, $\theta = \tau(\alpha)/\tau(\beta)$ is a positive irrational real number. Define a map $S : \mathfrak{H} \longrightarrow \mathbf{Z}^3$, given by extending the assignment,

$$S(z) = (0, 1, -1)^T, \quad S(\alpha) = (1, 0, 0)^T, \quad S(\beta) = (0, 1, 0)^T.$$

It is routine to verify that this extends to an order isomorphism between $\mathfrak{H}$ and $(\mathbf{Z}^3, P_\theta)$.

LEMMA 3.4. *Let α and α' be two positive irrational numbers. Then*

$$(\mathbf{Z}^3, P_\alpha) \cong (\mathbf{Z}^3, P_{\alpha'})$$

(as ordered groups) if and only if there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *in* $GL(2, \mathbf{Z})$ *such that*

(i) $\alpha c + d > 0$

$$\alpha' = \frac{\alpha a + b}{\alpha c + d}.$$

Proof. This just follows (with the appropriate translation of notation) from 3.1 above.

Given a positive irrational number ζ, let $[c_0, c_1, \dots c_n, \dots]$ be its continued fraction expansion. Recall ([**8**], for example) that $c_i \geqq 1$ for $i \geqq 1$, and $c_0 \geqq 0$. Define as usual,

$$\begin{aligned} p_0 &= c_0, \quad p_1 = c_1 c_0 + 1, \ \dots, \quad p_n = c_1 p_{n-1} + p_{n-2}, \dots \\ q_0 &= 1, \quad q_1 = c_1, \quad \dots, \quad q_n = c_1 q_{n-1} + q_{n-2}, \dots. \end{aligned}$$

Then:

(i) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$

(ii) $\dfrac{p_0}{q_0} < \dfrac{p_2}{q_2} < \dots < \dfrac{p_{2n}}{q_{2n}} < \dots < \zeta < \dots < \dfrac{p_{2n+1}}{q_{2n+1}} < \dots < \dfrac{p_3}{q_3} < \dfrac{p_1}{q_1}.$

Remark 3.5. Let $\zeta = [c_0, c_1, \dots c_n, \dots]$ be an irrational number between 0 and 1. From the continued fraction algorithm, we have that

$$\zeta^{-1} = [c_1, c_2, \dots, c_n, \dots].$$

We also deduce:

$$\begin{aligned} &p_0(\zeta^{-1}) = c_1 = q_1(\zeta); q_0(\zeta^{-1}) = 1 = p_1(\zeta); \\ &p_1(\zeta^{-1}) = c_1 c_2 + 1 = q_2(\zeta); q_1(\zeta^{-1}) = c_2 = p_2(\zeta). \end{aligned}$$

Thus

$$p_n(\zeta^{-1}) = q_{n+1}(\zeta) \quad \text{and} \quad q_n(\zeta^{-1}) = p_{n+1}(\zeta) \quad \text{for all } n \geqq 1.$$

Suppose that θ is a positive irrational number; we apply the preceding to $\theta/2$. Let $[c_0, c_1, \dots c_n, \dots]$ be the continued fraction expansion of $\theta/2$, and define the simple dimension group, $(\mathbf{Z}^2, P_{\theta/2})$ in the obvious way. As in Section 2, $(\mathbf{Z}^2, P_{\theta/2})$ is order isomorphic to

$$\lim \gamma_k : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2, \quad \text{where } \gamma_k = \begin{bmatrix} c_k & 1 \\ 1 & 0 \end{bmatrix}$$

([**2**, Theorem 3.2]).

LEMMA 3.6. *Let θ, c_k, γ_k, be as in the preceding paragraphs. Suppose there is an increasing sequence of integers $\{n(k)\}_{k \geqq 0}$ with $n(0) = 0$, and such that*

$$\psi_k = \gamma_{n(k)-1} \cdot \gamma_{n(k)_2} \cdot \ldots \cdot \gamma_{n(k-1)} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{mod}(2)$$

Then there exists a simple real AF algebra A of type rc such that $K_0(A) \cong (\mathbf{Z}^2, P_{\theta/2})$ and $K_0(A \otimes \mathbf{C})$ is free of rank 3, with unique state.

Proof. This is similar to the construction in Section 2. Write

$$\psi_k = \begin{bmatrix} 2a_k + 1 & 2b_k \\ e_k & 2d_k + 1 \end{bmatrix},$$

and define algebra maps

$$\Phi_i M_{r(i-1)}\mathbf{R} \oplus M_{c(i-1)}\mathbf{C} \longrightarrow M_{r(i)}\mathbf{R} \oplus M_{c(i)}\mathbf{C}$$

as follows. The matrix

$$\begin{bmatrix} 2a_k + 1 & b_k \\ e_k & (d_k + 1, d_k) \end{bmatrix}$$

obtained from ψ_k yields a $\mathbf{R}$-algebra map in the following manner. The (1, 1) entry $2a_k + 1$ corresponds to the map $M_*\mathbf{R} \longrightarrow M_*\mathbf{R}$ which has multiplicity $2a_k + 1$; the (1, 2) entry, b_k corresponds to the direct sum of b_k copies of the map $\mathbf{C} \longrightarrow M_2\mathbf{R}$; and the (2, 1) entry refers to the multiplicity of the map $\mathbf{R} \longrightarrow \mathbf{C}$. Finally the pair, $(d_k + 1, d_k)$ represents the real algebra map $\mathbf{C} \longrightarrow M_{2d_k+1}\mathbf{C}$ given via:

$$z \longmapsto \mathrm{diag}\,(z, z, \ldots (d_k + 1 \text{ times}); \bar{z}, \bar{z}, \ldots (d_k \text{ times}))$$

where the bar denotes complex conjugation. We thus may construct the real AF algebra A as the limit of the sequence of maps,

$$\mathbf{R} \oplus \mathbf{C} \xrightarrow{\Phi_1} M_{r(1)}\mathbf{R} \oplus M_{c(1)}\mathbf{C} \xrightarrow{\Phi_2} M_{r(2)}\mathbf{R} \oplus M_{c(2)}\mathbf{C} \xrightarrow{\Phi_3} \ldots.$$

Straightforward computations reveal that

$$K_0(A) = \lim \psi_k : \mathbf{Z}^2 \longrightarrow \mathbf{Z}^2 \cong (\mathbf{Z}^2, P_{\theta/2});$$

in particular, A is simple, and it is easy to check that its centre is $\mathbf{R}$. Thus $A \otimes \mathbf{C}$ is simple. Now

$$K_0(A \otimes \mathbf{C}) = \lim \psi_k^c : \mathbf{Z}^3 \longrightarrow \mathbf{Z}^3,$$

and a routine computation (as in [**3**] or [**4**]) reveals that

$$\psi_k^c = \begin{bmatrix} 2a_k + 1 & b_k & b_k \\ e_k & d_k + 1 & d_k \\ e_k & d_k & d_k + 1 \end{bmatrix},$$

whose eigenvalues are those of ψ_k together with 1. In particular, $K_0(A \otimes \mathbf{C})$ is free of rank three.

Thus far, we have constructed an *rc* AF algebra with the "$\mathfrak{G}$" term being $\mathbf{Z} + \frac{1}{2}\theta\mathbf{Z}$. There are at most three possibilities for the "$\mathfrak{H}$" term so arising. If the triple we obtain is not the originally desired one, we proceed similarly to what was done in Section 2, with a minor modification. There, we applied certain matrices from $SL(2, \mathbf{Z})$ to what amounted to the matrices in the middle terms of the K_0-sequence. This time, we apply them to the left terms; the outcome is that the "$\mathfrak{G}$" term is unaffected, but the "$\mathfrak{H}$" term may be altered, and in fact, these $SL(2, \mathbf{Z})$ matrices do act transitively on the three classes, as is readily verified.

4. Rank three, of type *rh*. Suppose $\mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{K}$ is of type *rh*, with $\mathfrak{H}$ simple, and free of rank 3. Then there are three possibilities, and we examine each one separately.

To begin with, $\mathfrak{H}$ could have a unique state τ with no kernel. Here $\mathfrak{H}/2\mathfrak{H} \cong (\mathbf{Z}_2)^3$. If $\mathfrak{G}$ or $\mathfrak{K}$ equals $2\mathfrak{H}$ (correspondingly, $\mathfrak{K}$ or $\mathfrak{G}$ equals $\mathfrak{H}$), then $\mathfrak{H}$ is of type h or r, and this is essentially trivial in our context. If $\mathfrak{G}/2\mathfrak{H}$ is of order 2, then there are 7 choices for $\mathfrak{G}$ (as a subgroup of $\mathfrak{H}$ containing $2\mathfrak{H}$ such that $\mathfrak{G}/2\mathfrak{H}$ is one-dimensional over the field of two elements), and it is straightforward to check that to each choice for $\mathfrak{G}$ there are 4 complementary subspaces (observe that $GL(3, 2)$ acts transitively on the nonzero elements of $(\mathbf{Z}_2)^3$, and put the matrices in corresponding block form). This yields a total of 28 possibilities. Replacing $\mathfrak{G}$ by $\mathfrak{K}$ gives us another 28, for a grand total of 58 $(1 + 28 + 28 + 1)$. Generically, $\mathfrak{H}$ admits no order automorphisms; in this case each of the 58 possibilities are mutually inequivalent (2.2).

If $\mathfrak{H}$ admits an order automorphism (which essentially says that $\mathfrak{H}$ amounts to an ideal in an order in a cubic number field), then some of the 28 choices (once $\dim \mathfrak{G}/2\mathfrak{H}$ has been fixed) may be equivalent to each other. It is possible but rather tedious to analyze the possibilities in a fashion analogous to the rank 2 case. For an automorphism to exist, there would have to be a real number μ such that $\mu(\mathfrak{H} \otimes \mathbf{Q})$ is a cubic field, call it F.

If F has just one real embedding (e.g., $F = \mathbf{Q}(2^{1/3})$), then as in the quadratic case, there is a fundamental unit, and its behaviour determines the orbits (and thus the equivalence classes). On the other hand, F could have three real embeddings (e.g., $F = \mathbf{Q}(r)$, where $r^3 - r + 1 = 0$), and in this case the order automorphism group requires two generators, and the situation becomes more complicated.

Next, $\mathfrak{H}$ could admit one state which has a non-trivial kernel, necessarily of rank 1. As ordered abelian groups, these have the same structure as those discussed in the *rc* and *ch* cases, but now we do not have to consider involutions; on the other hand, there are choices to be made for $\mathfrak{G}$ and $\mathfrak{K}$ (in the *rc* case, once $\mathfrak{H}$ and * have been chosen, $\mathfrak{G}$ and $\mathfrak{K}$ are determined). If $\mathfrak{G}/2\mathfrak{H}$ has just 2 elements, then as in the previous paragraph, there are 28 choices for triples, $\mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{K}$. This time however, there are automatically order automorphisms, regardless of whether $\tau(\mathfrak{H})$ is quadratic or not.

If it is not quadratic, then the automorphisms of $\mathfrak{H}$ are given, as in Proposition 3.1, with the 2×2 lower right matirx being the identity. There are four automorphisms (mod 2) given according to the congruence classes of p, s modulo 2, and so there is a four element group acting on the 28 choices. A brief computation yields that there are precisely 7 orbits (of 4 elements each), so there are 7 Morita equivalence classes in this case (with $\mathfrak{G}/2\mathfrak{H}$ of order 2; if $\mathfrak{G}/2\mathfrak{H}$ is of order 4, there are an additional 7 orbits, obtained by interchanging $\mathfrak{G}$ with $\mathfrak{K}$).

Finally, $\mathfrak{H}$ could admit 2 pure states, and thus the results of [**7**] apply. The order automorphisms are again given by multiplications by positive real numbers, this time inside a cubic extension of the rational with three real embeddings, although of course generically, there are no non-trivial order automorphisms.

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