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ON THE CENTRAL KERNEL OF A GROUP

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Abstract

The *central kernel* $K(G)$ of a group G is the (characteristic) subgroup consisting of all elements $x \in G$ such that $x^\gamma = x$ for every central automorphism γ of G. We prove that if G is a finite-by-nilpotent group whose central kernel has finite index, then the full automorphism group $Aut(G)$ of G is finite. Some applications of this result are given.

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1. Introduction

An automorphism of a group G is said to be a *central automorphism* if it acts trivially on the centre factor group $G/Z(G)$. It is easy to show that an automorphism of G is central if and only if it is *normal*, that is, it commutes with all inner automorphisms of G. It follows that the set $Aut_c(G)$ of all central automorphisms of G is a normal subgroup of the full automorphism group $Aut(G)$ of G. Clearly, if G has trivial centre, then the identity is the unique central automorphism of G, while all automorphisms of G are central if G is abelian. We also note that each inner automorphism of G is central if and only if G is nilpotent with class at most 2.

Recall that an automorphism α of a group G is said to be a *power automorphism* if $X^\alpha = X$ for each subgroup X of G. A relevant theorem by Cooper [5] ensures that the (normal) subgroup $PAut(G)$ of $Aut(G)$ consisting of all power automorphisms of G is contained in $Aut_c(G)$. This result was extended to *cyclic automorphisms* in [7]. (Recall here that an automorphism α of a group G is said to be a cyclic automorphism if the subgroup $\langle x, x^\alpha\rangle$ is cyclic for any element $x \in G$ (see also [3]).)

Let G be a group. It is easy to show that if γ is a central automorphism of a group G, then the map

$$f_\gamma : x \in G \mapsto x^{-1}x^\gamma \in Z(G)$$

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is a homomorphism from G into $Z(G)$. In particular, γ acts trivially over the commutator subgroup G' of G. Following [4], we define the *central kernel* of G as the subgroup

$$K(G) = \bigcap_{\gamma \in Aut_c(G)} \ker f_\gamma$$

of G consisting of all elements $x \in G$ such that $x^\gamma = x$ for every central automorphism γ of G. Obviously, $K(G)$ is a characteristic subgroup of G since $Aut_c(G)$ is a normal subgroup of $Aut(G)$. The subgroup $K(G)$ was first considered by Haimo [8] in 1955 and its importance was pointed out later by Pettet [11]. More recently, Catino *et al.* [4] have investigated finite-by-nilpotent groups in which the central kernel is *large* in some sense. (Recall here that a group G is said to be *finite-by-nilpotent* if there exists a positive integer n such that the term $\gamma_n(G)$ of the lower central series of G is finite.) They proved, among other results, that a finite-by-nilpotent group G is central-by-finite whenever its central kernel $K(G)$ has finite index (and hence the celebrated Schur's theorem [12, Theorem 10.1.4] yields that G is finite-by-abelian). Moreover, the subgroup consisting of all elements of finite order of G is finite (see [4, Theorem A]).

In this short article, we improve the latter result showing that if G is a finite-by-nilpotent group whose central kernel has finite index, then the full automorphism group $Aut(G)$ of G is finite. As a consequence, we obtain the theorem of Hegarty [9] stating that the *autocommutator subgroup* $[G, Aut(G)]$ of a group G is finite whenever its *absolute centre* $C_G(Aut(G))$ has finite index.

Most of our notation is standard and can be found in [12].

2. Results

Let G be a group, and consider a group Γ of automorphisms of G. The interaction between Γ and the subgroups

$$C_G(\Gamma) = \{g \in G \mid g^\gamma = g \text{ for all } \gamma \in \Gamma\}$$

and

$$[G, \Gamma] = \langle [g, \gamma] \mid g \in G, \gamma \in \Gamma \rangle$$

of G was first investigated in 1952 by Baer [2], who proved in particular the following result.

LEMMA 2.1. *Let G be a group and let Γ be a group of automorphisms of G. Then the finiteness of any two between Γ, $|G : C_G(\Gamma)|$ and $[G, \Gamma]$ implies the finiteness of the third.*

Our next result gives information of this type concerning the group of central automorphisms of a group.

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LEMMA 2.2. *Let G be any group. If the subgroup $A = K(G) \cap Z(G)$ has finite index in G, then the subgroup $[G, Aut_c(G)]$ of G has finite exponent.*

PROOF. Put $|G : A| = n$. As the subgroup A lies in the centre of G, the transfer homomorphism of G into A is the map

$$\tau : g \in G \mapsto g^n \in A.$$

Let g and γ be elements of G and $Aut_c(G)$, respectively. Then $(g^\gamma)^n = (g^n)^\gamma = g^n$, so

$$[g, \gamma]^\tau = g^{-\tau} g^{\gamma\tau} = g^{-n} g^n = 1.$$

It follows that $[G, Aut_c(G)]$ is contained in the kernel of τ and $[G, Aut_c(G)]^n = \{1\}$. □

As in many investigations concerning automorphisms, we will use in our arguments some cohomological methods. Let

$$\Sigma : A \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$$

be a *central extension* of a group A by a group Q. Without loss of generality, it can be assumed that A is a central subgroup of G, μ is the embedding of A into G, Q is the factor group G/A and ε is the natural projection. Recall that a *transversal map*

$$\tau : Q \longrightarrow G$$

of Σ is a function such that $\tau\varepsilon = \iota_Q$, so that the set

$$\{x^\tau \mid x \in Q\}$$

is a transversal to A in G. If $x, y \in Q$, we have

$$(x^\tau y^\tau)^\varepsilon = xy = ((xy)^\tau)^\varepsilon.$$

It follows that there exists a *unique* element $\varphi(x, y) \in A (= \ker \varepsilon)$ such that

$$x^\tau y^\tau = (xy)^\tau \varphi(x, y).$$

If x, y, z are elements of Q, from the equality $x^\tau(y^\tau z^\tau) = (x^\tau y^\tau) z^\tau$, it follows that

$$\varphi(x, yz) + \varphi(y, z) = \varphi(xy, z) + \varphi(x, y).$$

Therefore, the map

$$\varphi : Q \times Q \longrightarrow A$$

is a *2-cocycle* of Q in A. The coset

$$\Delta = \varphi + B^2(Q, A)$$

is an element of the *second cohomology group* $H^2(Q, A)$ of Q with coefficients in A, which depends only on the extension Σ and not on the choice of the transversal function. The element Δ is called the *cohomology class* of Σ.

Let α be an automorphism of A. An easy application of [13, Proposition II 4.3] shows that α can be extended to an automorphism β of G inducing the

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identity on G/A if

$$\varphi(x,y)^{\alpha} = \varphi(x,y)$$

for all $x, y \in Q$. Clearly, β is a central automorphism G. As an immediate consequence of the previous considerations, we have the following result which is useful for constructing central automorphisms.

LEMMA 2.3. *Let G be a group and consider the central extension*

$$\Sigma : A \rightarrowtail G \twoheadrightarrow G/A,$$

where $A = K(G) \cap Z(G)$. If $\Delta = \varphi + B^2(Q, A)$ is the cohomology class of Σ and α is an automorphism of A such that

$$\varphi(xA, yA)^{\alpha} = \varphi(xA, yA)$$

for all $xA, yA \in G/A$, then α is the identity of A.

Now we are in a position to prove our main result.

THEOREM 2.4. *Let G be a finite-by-nilpotent group in which the central kernel $K(G)$ has finite index. Then the full automorphism group $Aut(G)$ of G is finite. In particular, the subgroup of all elements of finite order of G is finite.*

PROOF. First we note that the factor group $G/Z(G)$ is finite by [4, Theorem A]. As $A = K(G) \cap Z(G)$ is a characteristic subgroup of G, then every automorphism γ of G induces an automorphism $\bar{\gamma}$ on the finite group G/A. Therefore, we may consider the homomorphism

$$f : \gamma \in Aut(G) \mapsto \bar{\gamma} \in Aut(G/A).$$

Clearly, the kernel Γ of f is a subgroup of $Aut_c(G)$, and hence the subgroup $[G, \Gamma]$ has finite exponent by Lemma 2.2.

Assume for a contradiction that Γ is infinite, so that $[G, \Gamma]$ is likewise infinite by Lemma 2.1. Since $[G, \Gamma]$ is a subgroup of A with finite exponent, it follows that the p-component P of A has infinite rank for some prime p. Let D be the largest divisible subgroup of P. Then A splits over D, so that $A = D \times E$ and hence $P = D \times R$, where $R = E \cap P$ is a reduced subgroup. First suppose that the subgroup D has finite rank. Then there exists a sequence $(X_n)_{n \in \mathbb{N}}$ of cyclic nontrivial subgroups of R such that

$$R = X_1 \times \cdots \times X_n \times R_n$$

for *all* positive integers n and suitable subgroups R_n (see [12, Propositions 4.3.3 and 4.3.8]). Moreover, since $X_1 \times \cdots \times X_n$ is a finite direct factor of P and P is pure in A for each n, there exists a subgroup A_n of A such that

$$A = X_1 \times \cdots \times X_n \times A_n.$$

Clearly, if D has infinite rank, we have a similar decomposition of A for each n, where all the subgroups $X_1, \ldots, X_n$ are of type p^{∞}.

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Let $\Delta = \varphi + B^2(G/A, A)$ be the cohomology class of the central extension

$$A \rightarrowtail G \twoheadrightarrow G/A.$$

By hypothesis, the set $\{\varphi(xA, yA) \mid xA, yA \in G/A\}$ is finite. It follows that for a sufficiently large n, there exist two direct factors U and V of the decomposition $A = X_1 \times \cdots \times X_n \times A_n$ of A such that

$$\langle \varphi(xA, yA) \rangle \cap (U \times V) = \{1\}$$

for all $xA, yA \in G/A$. Let α be a nonidentity automorphism of $U \times V$. Clearly, α can be extended to an automorphism β of A acting trivially over all direct factors other than U and V of the decomposition $A = X_1 \times \cdots \times X_n \times A_n$ of A. This is a contradiction by Lemma 2.3. Therefore, the full automorphism group $Aut(G)$ of G is finite as required.

Finally, as the commutator subgroup G' of G is finite by Schur's theorem, it follows that the set T of all elements of finite order of G is a subgroup, and hence T is finite by a result of Nagrebeckiĭ [10]. □

COROLLARY 2.5. *Let G be a finitely generated infinite nilpotent group such that the index $|G : K(G)|$ is finite. Then G contains a central infinite cyclic subgroup of finite index.*

PROOF. By Theorem 2.4, the full automorphism group $Aut(G)$ of G is finite, and hence the statement follows from a celebrated result by Alperin (see [1, Theorem 1]). □

We note that the above result has been proved with different arguments also in [4, Proposition 2.1].

Let G be a group. Following [9], the *absolute centre* (or *autocentre*) of G is the characteristic subgroup $C_G(Aut(G))$ of G consisting of all elements of G fixed by every automorphism of G. Clearly, $C_G(Aut(G))$ is contained in the central kernel $K(G)$ of G. Therefore, if the index $|G : C_G(Aut(G))|$ is finite, then Theorem 2.4 yields that $Aut(G)$ is finite, and hence the *autocommutator subgroup* $[G, Aut(G)]$ of G is likewise finite by Lemma 2.1. Thus, we have obtained the following result that was first proved by Hegarty [9].

COROLLARY 2.6. *If the absolute centre $C_G(Aut(G))$ of a group G has finite index, then the autocommutator subgroup $[G, Aut(G)]$ is finite.*

We point out that another generalisation of Hegarty's theorem was obtained by de Giovanni, Newell and the author [6] in 2014.

Let G be a nilpotent group with class at most 2. Then $Inn(G) \leq Aut_c(G)$ and hence $K(G) \leq Z(G)$. Conversely, as a central automorphism acts trivially over the commutator subgroup G' of G, we see that G is nilpotent with class at most 2 if the central kernel of G is contained in $Z(G)$. Now we construct an infinite nonabelian group G such that $Z(G) = K(G) \neq C_G(Aut(G))$.

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Let $A = \langle x \rangle$ and $B = \langle y \rangle$ be a cyclic group of order 9 and an infinite cyclic group, respectively. Consider the semidirect product

$$G = B \ltimes A,$$

where $x^y = x^4$ and $[x, B^3] = \{1\}$. Clearly, $Z(G) = B^3 \times \langle x^3 \rangle$ and $G' = \langle x^3 \rangle$. Let γ be a central automorphism of G. Then $y^\gamma = yz$ for some central element z. Moreover, γ induces an automorphism over the characteristic subgroup B^9. First suppose that $(y^9)^\gamma = y^{-9}$. If $z = y^{3i}x^{3j}$ for some integers i and j, then

$$y^{-9} = (y^9)^\gamma = y^9 z^9 = y^{27i+9}x^{27j} = y^{27i+9},$$

and so $y^{27i+18} = 1$, which is a contradiction. Therefore, $y^9 = (y^9)^\gamma = y^9 z^9$ and hence $z = x^{3t}$ for some nonnegative integer t. It follows that $(y^3)^\gamma = y^3$ so that γ acts trivially on $Z(G)$. Thus, $K(G) = Z(G)$. Clearly, every inner automorphism of G is central. However, it is easy to show that $Aut_c(G) \simeq \mathrm{Hom}(G/Z(G), Z(G)) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, so that $Inn(G) = Aut_c(G)$.

Finally, consider the automorphism $\alpha : x \mapsto x^2$ of A and $\beta = \iota_B$ of B. Since

$$(x^y)^\alpha = (x^4)^\alpha = x^8 = (x^2)^y = (x^\alpha)^{y^\beta},$$

there exists an automorphism γ of G inducing α and β. We note that γ cannot be central since $(x^3)^\alpha = x^{-3}$. It follows that $C_G(Aut(G)) \neq K(G)$.

Let G be a group, and denote by $\overline{K}(G)$ the set of all elements x of G such that $x^\alpha = x$ for every power automorphism α of G. Clearly, $\overline{K}(G)$ is a characteristic subgroup of G containing the central kernel $K(G)$ of G and the subgroup $G[2] = \langle g \in G \mid g^2 = 1 \rangle$. Note that the consideration of the direct product $G = A \times B$, where A is a cyclic group of order 3 and B is a countably infinite abelian group of exponent 2, shows that we cannot replace in Theorem 2.4 the central kernel by the subgroup $\overline{K}(G)$. Nevertheless, it is easy to prove the following result.

PROPOSITION 2.7. *Let G be a finite-by-nilpotent group such that the index $|G : \overline{K}(G)|$ is finite. Then the group $PAut(G)$ of all power automorphisms of G is finite.*

PROOF. First suppose that G is a nonperiodic group. By hypothesis, the term $\gamma_n(G)$ of the lower central series of G is finite for some positive integer n. It follows that the set of periodic elements of G is a subgroup, so that G is a *weak* group. Thus, $PAut(G)$ is finite of order at most 2 (see [5, Corollary 4.2.3]).

If G is periodic, then there exists a finite subgroup F of G such that $G = F\overline{K}(G)$. Let α be a power automorphism of G. Then $F^\alpha = F$ and $x^\alpha = x$ for every $x \in \overline{K}(G)$. It follows that the map

$$f : \alpha \in PAut(G) \mapsto \alpha_F \in Aut(F)$$

is injective and hence $PAut(G)$ is again finite. □

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