Canad. Math. Bull. Vol. **23** (2), 1980

FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS

BY
THAKYIN HU

ABSTRACT. Two fixed point theorems for multi-valued mappings in a complete, ε-chainable metric space are proved. The theorems, thus established, extend result of M. Edelstein, Peter K. F. Kuhfittig, Hwei-mei Ko and Yueh-hsia Tsai, S. B. Nadler, Jr. and S. Reich.

1. Introduction. Following Edelstein [3], Kelly [4], H. Covitz and S. B. Nadler, Jr. [2], we shall define some basic concepts as follows:

If (X, d) is a metric space, then

(a) $CB(X)=\{A \mid A$ is a nonempty closed and bounded subset of $X\}$,

(b) $N(A, \varepsilon)=\{x\in X \mid d(x, a)<\varepsilon$ for some $a\in A\}$ if $\varepsilon>0$ and $A\in CB(X)$,

(c) $H(A, B)=\inf\{\varepsilon>0 \mid A\subseteq N(B, \varepsilon)$ and $B\subseteq N(A, \varepsilon)\}$ if $A, B\in CB(X)$.

The pair (X, H) is a metric space and H is called the Hausdorff metric induced by d. A metric space is said to be ε-chainable if and only if given x, y in X, there is an ε-chain from x to y (i.e., a finite set of points $z_0=x, z_1, z_2, z_3, \ldots, z_n=y$ such that $d(z_{i-1}, z_i)<\varepsilon$ for all $i=1, 2, \ldots, n$). A function $F: X\to CB(X)$ is called a multi-valued contraction mapping if and only if there exists a fixed real number $\lambda<1$ such that $H(F(x), F(y))\le\lambda d(x, y)$ for all x, y in X. A function $F: X\to CB(X)$ is called an (ε, λ)-uniformly local contraction mapping (where $\varepsilon>0$ and $0<\lambda<1$) if and only if $H(F(x), F(y))\le \lambda\, d(x, y)$ for all x, y in X with $d(x, y)<\varepsilon$. Let $F: X\to CB(X)$ be a function and let $x\in X$. A sequence $\{x_i\}$ of points of X is said to be an iterative sequence of F at x if and only if $x_i\in F(x_{i-1})$ for each $i=1, 2, 3, \ldots$: a point $p\in X$ is a fixed point of F if and only if $p\in F(p)$.

S. Reich proved the following theorem in 1972.

THEOREM 1. *Let (X, d) be a complete ε-chainable metric space. Suppose $k:(0, \varepsilon)\to[0, 1)$ is a function with the following properties:*

(P1) *For each t in the domain of k, there exists $\delta(t)>0$, $s(t)<1$ such that $0\le r-t<\delta(t)$ implies $k(r)\le s(t)<1$.*

(P2) *There exists $b\in(0, \frac{1}{2}\varepsilon)$ such that $\frac{1}{2}\varepsilon-b<t<\varepsilon/2$ implies $\delta(t)\ge t$.*

Received by the editors June 27, 1978 and, in revised form, November 8, 1978.

https://doi.org/10.4153/CMB-1980-026-2 Published online by Cambridge University Press

Assume also $T: X \to X$ is a mapping that satisfies

$$0 < d(x, y) < \varepsilon \quad \text{implies} \quad d(Tx, Ty) \leq k(d(x, y))\, d(x, y).$$

Then T has a unique fixed point in X.

Reich posed the question whether property (P2) is indispensable. Ko and Tsai [5] showed that (P2) is redundant. We prove that Ko and Tsai's result can be extended.

We shall make use of the following lemmas, which are noted in Nadler [7] and Assad and Kirk [1].

LEMMA 1. *If $A, B \in CB(X)$ and $a \in A$, then for each positive number α, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \alpha$.*

LEMMA 2. *Let $\{X_n\}$ be a sequence of sets in $CB(X)$, and assume that $\lim_{n\to\infty} H(X_n, X_0) = 0$ where $X_0 \in CB(X)$. Then if $x_n \in X_n$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} x_n = x_0$, it follows that $x_0 \in X_0$.*

We shall state the following lemma without proof. It is an easy consequence of the definition of the Hausdorff metric and also follows immediately from Lemma 1.

LEMMA 3. *If $A, B \in CB(X)$ with $H(A, B) < \varepsilon$, then for each $a \in A$, there exists an element $b \in B$ such that $d(a, b) < \varepsilon$.*

2. Fixed point theorems.

We state our main result as the following theorem.

THEOREM 2. *Suppose (X, d) is a complete ε-chainable metric space and $T: X \to CB(X)$ is a mapping that satisfies the following condition:*

$$(C)\ 0 < d(x, y) < \varepsilon \quad \text{implies} \quad H(Tx, Ty) < k(d(x, y))\, d(x, y),$$

where $k: (0, \varepsilon) \to [0, 1)$ is a function satisfying property (P1). Then for each $x_0 \in X$, there exists an iterative sequence $\{x_n\}$ of T at x_0 such that x_n converges to a fixed point of T.

Proof. Our method is constructive. Given $x_0 \in X$, we shall define an iterative sequence $\{x_n\}$ of T at x_0 as follows. Let $x_1 \in Tx_0$ be arbitrary and let

$$x_0 = z_{(1,0)}, z_{(1,1)}, z_{(1,2)}, \ldots, z_{(1,m)} = x_1 \in Tx_0$$

be an arbitrary ε-chain from x_0 to x_1. We shall construct the remaining terms in the diagram shown below as follows:

We rename x_1 as $z_{(2,0)}$ and place it right below $x_0=z_{(1,0)}$ as shown.

$$
\begin{aligned}
&x_0 = z_{(1,0)}, z_{(1,1)}, z_{(1,2)}, \ldots, z_{(1,m)} = x_1 \in Tx_0\\
&x_1 = z_{(2,0)}, z_{(2,1)}, z_{(2,2)}, \ldots, z_{(2,m)} = x_2 \in Tx_1\\
&x_2 = z_{(3,0)}, z_{(3,1)}, z_{(3,2)}, \ldots, z_{(3,m)} = x_3 \in Tx_2\\
&\ldots\\
&\ldots\\
&x_{n-1} = z_{(n,0)}, z_{(n,1)}, \ldots, z_{(n,m)} = x_n \in Tx_{n-1}\\
&x_n = z_{(n+1,0)}, z_{(n+1,1)}, \ldots, z_{(n+1,m)} = x_{n+1} \in Tx_n\\
&\ldots\\
&\ldots
\end{aligned}
$$

Since $d(z_{(1,0)}, z_{(1,1)})<\varepsilon$ and $T:X\to CB(X)$ satisfies property (P1), we get

$$
\begin{aligned}
H(Tz_{(1,0)}, Tz_{(1,1)}) &< k[d(z_{(1,0)}, z_{(1,1)})]\, d(z_{(1,0)}, z_{(1,1)})\\
&< d(z_{(1,0)}, z_{(1,1)}) < \varepsilon.
\end{aligned}
$$

Since $z_{(2,0)}\in Tz_{(1,0)}$, we may use Lemma 3 to get an element $z_{(2,1)}$ in $Tz_{(1,1)}$ such that

$$
\begin{aligned}
d(z_{(2,0)}, z_{(2,1)}) &< k[d(z_{(1,0)}, z_{(1,1)})]\, d(z_{(1,0)}, z_{(1,1)})\\
&< d(z_{(1,0)}, z_{(1,1)}) < \varepsilon.
\end{aligned}
$$

By the same procedure, we get $z_{(2,j)}\in Tz_{(1,j)}$ with

$$
\begin{aligned}
d(z_{(2,j)}, z_{(2,j+1)}) &< k[d(z_{(1,j)}, z_{(1,j+1)})]\, d(z_{(1,j)}, z_{(1,j+1)})\\
&< d(z_{(1,j)}, z_{(1,j+1)}) < \varepsilon, \quad \text{for} \quad j=0,1,\ldots,(m-1).
\end{aligned}
$$

In particular, $z_{(2,m)}\in Tz_{(1,m)}=Tx_1$ and we let $x_2=z_{(2,m)}$. Inductively, assume that the nth row has been obtained, we may then use the same argument as above to construct the $(n+1)$th row. From construction, we get

$$
\text{(A)} \qquad
\begin{aligned}
d(z_{(n+1,i)}, z_{(n+1,i+1)}) &< k[d(z_{(n,i)}, z_{(n,i+1)})]\, d(z_{(n,i)}, z_{(n,i+1)})\\
&< d(z_{(n,i)}, z_{(n,i+1)}) < \varepsilon,
\end{aligned}
$$

for $i=0,1,2,\ldots,(m-1)$, and for all n. Also $z_{(n+1,i)}\in Tz_{(n,i)}$ for $i=0,1,2,\ldots,m$ and for all n.

Claim 1. For fixed $i=0,1,\ldots,(m-1)$, it must be the case that $\lim_{n\to\infty} d(z_{(n,i)}, z_{(n,i+1)})=0$.

Proof of Claim 1. From (A), we see that $\lim_{n\to\infty} d(z_{(n,i)}, z_{(n,i+1)})$ exists and must be a number in $[0,\varepsilon)$. Let $\lim_{n\to\infty} d(z_{(n,i)}, z_{(n,i+1)})=t$. If $t>0$, by (P1), there exists $\delta(t)>0$, $s(t)<1$ such that $0\le r-t<\delta(t)$ implies $k(r)\le s(t)<1$. For this $\delta(t)>0$, there exists an integer N such that $0 \le d(z_{(n,i)}, z_{(n,i+1)}) - t < \delta(t)$ and hence

$$k[d(z_{(n,i)}, z_{(n,i+1)})] \le s(t) < 1 \quad \text{whenever} \quad n \ge N.$$

Let $K = \max\{k_0, k_1, \ldots, k_N, s(t)\} < 1$ where $k_j = k[d(z_{(j,i)}, z_{(j,i+1)})]$ for $j = 0, 1, 2, \ldots, N$. Then

$$\begin{aligned} d(z_{(n,i)}, z_{(n,i+)}) &< k[d(z_{(n-1,i)}, z_{(n-1,i+1)})]\, d(z_{(n-1,i)}, z_{(n-1,i+1)}) \\ &\le K\, d(z_{(n-1,i)}, z_{(n-1,i+1)}) \quad \text{for} \quad n = 1, 2, 3, \ldots . \end{aligned}$$

Thus $d(z_{(n,i)}, z_{(n,i+1)}) \le K^n\, d(z_{(0,i)}, z_{(0,i+1)}) \to 0$ as $n \to \infty$. That is a contraction to $t > 0$. Consequently, $t = \lim_{n\to\infty} d(z_{(n,i)}, z_{(n,i+1)}) = 0$.

CLAIM 2. $d(x_{n-1}, x_n) \to 0$ as $n \to \infty$.

Proof of Claim 2. From our construction, $x_{n-1} = z_{(n,0)}$ and $x_n = z_{(n,m)}$. Thus

$$d(x_{n-1}, x_n) = d(z_{(n,0)}, z_{(n,m)}) \le \sum_{i=0}^{m-1} d(z_{(n,i)}, z_{(n,i+1)})$$

where the right hand side converges to zero because of Claim 1. Consequently, $d(x_{n-1}, x_n) \to 0$ as $n \to \infty$ as claimed.

CLAIM 3. $\{x_n\}$ is a Cauchy sequence.

Proof of Claim 3. We prove by contradiction. Suppose $\{x_n\}$ is not a Cauchy sequence. Then there exists a number $t > 0$ (we may assume $t < \varepsilon$ without loss of generality) and two subsequences $\{n_i\}$, $\{m_i\}$ of the natural numbers with $n_i < m_i$ and such that

$$d(x_{n_i}, x_{m_i}) \ge t, \qquad d(x_{n_i}, x_{m_i-1}) < t, \qquad \text{for } i = 1, 2, 3, \ldots .$$

Then $t \le d(x_{n_i}, x_{m_i}) \le d(x_{n_i}, x_{m_i-1}) + d(x_{m_i-1}, x_{m_i})$. Letting $i \to \infty$, we get

$$t \le \lim_{i\to\infty} d(x_{n_i}, x_{m_i}) \le \lim_{i\to\infty} d(x_{n_i}, x_{m_i-1}) + \lim_{i\to\infty} d(x_{m_i-1}, x_{m_i}) \le t + 0 = t.$$

Consequently, $\lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = t \in (0, \varepsilon)$. For this $t > 0$, by property (P1), there exists $\delta(t) > 0$, $s(t) < 1$ such that $0 \le r - t < \delta(t) \Rightarrow k(r) \le s(t) < 1$. For this $\delta(t) > 0$, there exists an integer N such that $i \ge N$ implies $0 \le d(x_{n_i}, x_{m_i}) - t < \delta(t)$ and hence $k[d(x_{n_i}, x_{m_i})] < s(t)$ if $i \ge N$. Thus

$$\begin{aligned} d(x_{n_i}, x_{m_i}) &\le d(x_{n_i}, x_{n_i+1}) + d(x_{n_i+1}, x_{m_i+1}) + d(x_{m_i+1}, x_{m_i}) \\ &\le d(x_{n_i}, x_{n_i+1}) + k[d(x_{n_i}, x_{m_i})]\, d(x_{n_i}, x_{m_i}) + d(x_{m_i+1}, x_{m_i}) \\ &\le d(x_{n_i}, x_{n_i+1}) + s(t)\, d(x_{n_i}, x_{m_i}) + d(x_{m_i+1}, x_{m_i}). \end{aligned}$$

Letting $i \to \infty$, we get $t \le s(t)t < t$. That is a contradiction. Consequently, $\{x_n\}$ is Cauchy and Claim 3 is proved.

https://doi.org/10.4153/CMB-1980-026-2 Published online by Cambridge University Press

By completeness of the space, there exists an element $p \in X$ such that $d(x_n, p) \to 0$ as $n \to \infty$. Hence there exists an integer $N_1 > 0$ such that $n \geq N_1$ implies $d(x_n, p) < \varepsilon$. Thus for $n \geq N_1$, we have

$$H(Tx_n, Tp) \leq k[d(x_n, p)]\, d(x_n, p) < d(x_n, p).$$

Consequently, $H(Tx_n, Tp) \to 0$ as $n \to \infty$. Since $x_{n+1} \in Tx_n$ for all n and Tp is closed, it follows from Lemma 2 that $p \in Tp$ and the proof is complete.

The following Theorem is an immediate consequence of Theorem 2.

THEOREM 3. *Let (X, d) be a complete metric space. Suppose $T : X \to CB(X)$ is a mapping that satisfies $H(Tx, Ty) < k[d(x, y)]\, d(x, y)$ for all x, y (where $k : (0, \infty) \to [0, 1)$ is a function satisfying property $(P1)$). Then T has a fixed point in X.*

Obviously, Theorem 2 is a better result than Theorem 1 (see Reich [8]). Also, our fixed point theorems extend Theorems of Ko and Tsai [5], Theorems 5 and 6 of Nadler, Jr. [7], Theorem 5.2 of Edelstein [3] and Theorem 1 of Kuhfittig [6].

REMARK. Suppose $k : (0, b) \to [0, 1)$ is a function satisfying property (P1), then the function $g : (0, b) \to [0, 1)$ defined by $g(t) = \sqrt{k(t)}$ also satisfies (P1). Consequently, the condition that $H(Tx, Ty) < k[d(x, y)]\, d(x, y)$ as stated in the hypothesis of Theorems 2 and 3 can be replaced by $H(Tx, Ty) \leq k[d(x, y)]\, d(x, y)$ without affecting the validity of the Theorems. We intentionally use strict inequality so that proofs are substantially simplified with the help of Lemma 3.

ACKNOWLEDGEMENTS. The author is grateful to Professor W. A. Kirk whose comments on Kuhfittig's paper has initiated this research. Also, we call attention to the obvious fact that we have relied on ideas of Ko and Tsai, P. Kuhfittig and S. B. Nadler, Jr.

REFERENCES

1. N. A. Assad and W. A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math., **43** (1972), 553–562.

2. H. Covitz and S. B. Nadler, Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math., **8** (1970), 5–11.

3. M. Edelstein, *An extension of Banach's Contraction Principle*, Proc. Amer. Math. Soc., **12** (1961), 7–10.

4. J. L. Kelly, *General Topology*, D. Van Nostrand Co., Inc., Princeton, New Jersey, 1959.

5. Hwei-mei Ko and Yueh-Hsia Tsai, *Fixed point theorems with localized property*, Tamkang J. Math., Vol. **8,** No. 1 (1977), 81–85.

6. Peter K. Kuhfittig, *Fixed points of locally contractive and non-expansive set-valued mappings*, Pacific J. Math., Vol. **65,** No. 2 (1976), 399–403.

7. S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488.

8. S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital. (4), **5** (1972), 26–42.

DEPARTMENT OF MATHEMATICS
TAMKANG COLLEGE OF ARTS AND SCIENCES
TAMSUI, TAIPEI HSIEN, TAIWAN 251
REPUBLIC OF CHINA