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ON POINTED HOPF ALGEBRAS OF DIMENSION p^5

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Abstract. We describe all possible coradically graded pointed Hopf algebras of dimension p^5 (where p is an odd prime number) over an algebraically closed field of characteristic 0.

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1. Introduction. The *lifting procedure* described in [**2**] is a powerful tool for classifying pointed Hopf algebras. It has been applied successfully to the classification of pointed Hopf algebras of dimension p^3 in [**2**] and dimension p^4 in [**4**]. It has been used also in the classification of pointed Hopf algebras of dimension 32 in [**10**]. We describe here all pointed coradically graded Hopf algebras of dimension p^5 (we assume p is odd since the case $p = 2$ is treated in [**10**]). Some of these algebras are known and can be found in the referred articles as well as in [**3**], [**8**]. Classification problems of pointed Hopf algebras have been also treated in [**6**], [**9**] and [**7**].

Our main references for Hopf algebras are [**13**] and [**11**]. For Nichols algebras we refer to [**12**] and [**1**].

The article is organized as follows: in Section 2 we give the notation and definitions we use and the first results we need. In Section 3 we describe all possible Nichols algebras of dimension p^{5-j} over groups of order p^j ($j = 1, \dots, 4$). In Section 4 we prove necessary auxiliary results; some of them have interest on their own, e.g. Theorem 4.3. In Section 5 we prove that any pointed Hopf algebra of dimension p^5 over $\mathbf{k}$ is generated by group-like and skew-primitive elements. In other words, any coradically graded pointed Hopf algebra of dimension p^5 can be recovered by bosonization (or biproduct) from one of the Nichols algebras appearing in Theorem 3.2. Furthermore, this proves also that any pointed Hopf algebra of dimension p^5 can be recovered by lifting (in the sense of [**2**]) of one of these bosonizations. Thus the classification of the pointed Hopf algebras of dimension p^5 could be done in principle using the lifting procedure. This article contains the first steps in this direction. In Section 6 we address the remaining steps and consider some illustrating examples.

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2. Notation and preliminary results. The letter $\mathbf{k}$ will stand for an algebraically closed field of characteristic 0. All Hopf algebras are $\mathbf{k}$-algebras. For Γ a group and $g \in \Gamma$ we denote by Γ_g the isotropy subgroup $\Gamma_g = \{h \in \Gamma \mid hg = gh\}$. Let $q \in \mathbf{k}$. For $n \geq m \in \mathbb{N}$, we use the standard notation

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$$(n)_q = \sum_{i=0}^{n-1} q^i,\ (0)_q = 1; \quad (n)_q^! = \prod_{i=1}^{n} (i)_q; \quad \binom{n}{m}_q = \frac{(n)_q^!}{(m)_q^!(n-m)_q^!}.$$

For A a Hopf algebra, we say that A is *pointed* if and only if the simple subcoalgebras of A are 1-dimensional (if and only if the irreducible representations of A^* are 1-dimensional).

Let $A = \oplus_{i\geq 0} A(i)$ be a graded Hopf algebra. We say that A is *coradically graded* if the graduation corresponds to the coradical filtration of A; i.e. if $A_r = \oplus_{i=0}^{r} A(i)$ $\forall r \geq 0$, where $A_0 \subseteq A_1 \subseteq \dots$ stands for the coradical filtration of A. In particular, A being coradically graded and pointed implies that $A(0) \simeq \mathbf{k}\Gamma$, where Γ is the group of group-likes of A.

Let H be a Hopf algebra. We denote by ${}^H_H\mathcal{YD}$ the category of (left-left) Yetter–Drinfeld modules over H (see [**11**]) and by c its braiding. Let A be a coradically graded pointed Hopf algebra and $A(0) = \mathbf{k}\Gamma$; then

$$R = A^{\mathrm{co}A(0)} = \{x \in A \mid (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\} = \oplus_i R(i), \qquad (2.1)$$

(where $\pi : A \to A(0)$ is the canonical projection), is a braided Hopf algebra in the category ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$. The Hopf algebra A can be recovered by bosonization: $A = R\#\mathbf{k}\Gamma$. Furthermore, R is coradically graded and $R(0) = \mathbf{k}1$. If moreover R is generated as an algebra by $R(1)$, then we say that R is a *Nichols algebra*.

If R is a Nichols algebra, then R is uniquely determined (up to isomorphism) by $V = R(1)$, which coincides with the space of primitive elements $\mathcal{P}(R)$. We write $R = \mathfrak{B}(V)$.

We refer to the survey [**1**] for details on these constructions (Nichols algebras are called TOBAs in that article).

Proposition 2.2. *Let* $\mathbf{f}$ *be any field, and let* H *be a Hopf algebra over* $\mathbf{f}$. *Let* V *be an object in* ${}^H_H\mathcal{YD}$. *Suppose* V *has a basis* $\{x_1, \dots, x_\theta\}$ *such that* $c(x_i \otimes x_j) = b_{ij} x_j \otimes x_i$ *for certain* $b_{ij} \in \mathbf{f}$ *(since* c *is an automorphism,* $b_{ij} \in \mathbf{f}^\times$*). We take for each* $i = 1, \dots, \theta$

$$N_i = \begin{cases} \text{order of } b_{ii} & \text{if } b_{ii} \neq 1 \text{ and is a root of unity,} \\ \infty & \text{if } b_{ii} \text{ is not a root of unity,} \\ \infty & \text{if } b_{ii} = 1 \text{ and } \operatorname{char} \mathbf{f} = 0, \\ \operatorname{char} \mathbf{f} & \text{if } b_{ii} = 1 \text{ and } \operatorname{char} \mathbf{f} > 0. \end{cases}$$

Then $\dim \mathfrak{B}(V) \geq \prod_i N_i$. *Moreover, if* $\mathfrak{B}(V)$ *is finite dimensional, then the equality holds if and only if* $b_{ij}b_{ji} = 1$, $\forall i \neq j$.

Proof. See [**2**, §3]. □

We recall (see for instance [**1**]) that if Γ is a finite group, the category ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ is semisimple. The simple objects are the modules $M(g, \rho)$ defined as follows: let $g \in \Gamma$, ρ an irreducible representation of the isotropy group Γ_g. Let W be the space affording ρ, and take

$$M(g, \rho) = \mathrm{Ind}_{\Gamma_g}^{\Gamma} W = \mathbf{k}\Gamma \otimes_{\mathbf{k}\Gamma_g} W,$$

with the usual module structure and the comodule structure given by

$$\delta(h \otimes w) = hgh^{-1} \otimes (h \otimes w) \in \mathbf{k}\Gamma \otimes M(g, \rho).$$

REMARK 2.3. Since g is central in Γ_g, if ρ is an irreducible representation of Γ_g then the Schur lemma says that $\rho(g) = q\mathrm{id}$, for some $q \in \mathbf{k}^{\times}$.

DEFINITION 2.4. We say that $V \in {}^H_H\mathcal{YD}$ *has a matrix* (b_{ij}) if it has a basis $\{x_1, \dots, x_\theta\}$ such that $c(x_i \otimes x_j) = b_{ij}x_j \otimes x_i$.

This happens for instance if Γ is abelian. This happens also under a weaker condition: let $V = \oplus_i M(g_i, \rho_i)$ and suppose that the subgroup Γ' of Γ generated by the conjugacy classes of all the g_i is abelian. Then V *comes from the abelian case* in the sense of [**1**, Definition 3.1.8] and consequently has a matrix. In this case V can be considered as a Yetter–Drinfeld module over Γ' and $\mathfrak{B}(V)\#\mathbf{k}\Gamma$ can be reconstructed as an extension of Γ/Γ' by $\mathfrak{B}(V)\#\mathbf{k}\Gamma'$. A sufficient condition for Γ' to be abelian in the case $V = M(g, \rho)$ is that the isotropy subgroup Γ_g be invariant in Γ (see [**1**, Lemma 3.1.9]). Since we are working in characteristic 0, if V has a matrix (b_{ij}) and $\mathfrak{B}(V)$ is finite dimensional then, by Proposition 2.2, $b_{ii} \neq 1, \forall i$.

If V has a matrix (b_{ij}) with $b_{ij}b_{ji} = 1,\ \forall i \neq j$, then it can be shown that $\mathfrak{B}(V)$ has a PBW basis of the form

$$\{x_1^{n_1} \cdots x_\theta^{n_\theta} \mid 0 \leq n_i < N_i\},$$

where N_i is defined as in Proposition 2.2. The relations are given by

$$x_i^{N_i} = 0, \quad x_ix_j = b_{ij}x_jx_i,\ \forall i > j.$$

Thus $\mathfrak{B}(V)$ is a quantum linear space as an algebra. We notice that the lines $\mathbf{k}x_i$ $(i = 1, \dots, \theta)$ are not Yetter–Drinfeld submodules in general. In order to agree with the terminology of [**2**], we shall denote such an algebra by QLS only when the lines $\mathbf{k}x_i$ are Yetter–Drinfeld modules $\forall i$. Thus, a QLS in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ is given by a module $V = \oplus_{i=1}^{\theta} M(g_i, \chi_i)$, where

$$\begin{cases} g_1, \dots, g_\theta \in \Gamma \text{ are central elements, and} \\ \chi_1, \dots, \chi_\theta \in \hat{\Gamma} \text{ are characters such that} \\ \chi_i(g_j)\chi_j(g_i) = 1,\ \forall i \neq j. \end{cases} \tag{2.5}$$

For $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$, we shall say that $\mathfrak{B}(V)$ is a QLS over $\Gamma' \subset \Gamma$ if V is a Yetter–Drinfeld module in ${}^{\mathbf{k}\Gamma'}_{\mathbf{k}\Gamma'}\mathcal{YD}$ and the conditions 2.5 hold for Γ'. A 1-dimensional QLS will be called also *Quantum Line* (or QL), and a 2-dimensional QLS will be called also *Quantum Plane* (or QP).

According to [**3**], if V has a matrix (b_{ij}) we say that V *is of Cartan type* if there exists a (generalized) Cartan matrix (a_{ij}) such that

$$b_{ij}b_{ji} = b_{ii}^{a_{ij}}, \quad \forall i, j = 1, \dots, \theta.$$

We transfer to V the terminology over the Cartan matrix (a_{ij}).

LEMMA 2.6. *Let g be central in Γ and ρ an irreducible representation of Γ. Let $V = M(g, \rho)$. By 2.3, g acts by a scalar on V, say q. Let N be the order of q. Then* $\dim \mathfrak{B}(V) \geq N^{\deg \rho}$.

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Proof. Since g is central, V comes from the abelian case, and consequently c has a matrix (b_{ij}). It is straightforward to see that $b_{ij} = q$, $\forall i, j$. Then Proposition 2.2 applies and the result follows. $\square$

LEMMA 2.7. *Let $g \in \Gamma$ and ρ an irreducible representation of Γ_g. Let $V = M(g, \rho)$. Suppose that* $\dim V < p$, *where p is the smallest prime dividing $|\Gamma|$. Then g is central,* $\deg \rho = 1$ *and thus $\mathfrak{B}(V)$ is a QL over Γ with* $\dim \mathfrak{B}(V) = N$, *where N is the order of $\rho(g)$.*

Proof. We have $\dim V = [\Gamma : \Gamma_g] \deg(\rho) < p$. Since $[\Gamma : \Gamma_g]$ and $\deg(\rho)$ both divide $|\Gamma|$, necessarily $[\Gamma : \Gamma_g] = 1$, whence g is central, and $\deg(\rho) = 1$. The result follows from Proposition 2.2. $\square$

REMARK 2.8. Since $\dim \mathfrak{B}(V) \geq 1 + \dim V$, the hypothesis of the preceding lemma is satisfied if $\dim \mathfrak{B}(V) \leq p$.

REMARK 2.9. By Lemma 2.7, we have that if $V = \oplus_i M(g_i, \rho_i)$ is such that $\dim V < p$, then g_i is central and ρ_i is a character $\forall i$, and furthermore $\dim \mathfrak{B}(V) \geq \prod_i N_i$, where N_i is the order of $\rho_i(g_i)$.

3. Main results.

LEMMA 3.1. *Let A be a coradically graded pointed Hopf algebra of dimension p^5. Let $\Gamma = G(A)$ be the group of group-likes of A. Let $R = A^{\mathrm{co}\,\mathbf{k}\Gamma} \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ be the coinvariants (thus $A = R\#\mathbf{k}\Gamma$) and let $V = R(1)$ be the primitive elements of R. Assume that V generates R as an algebra (i.e. $R = \mathfrak{B}(V)$). Then the following possibilities arise.*

1. *If $|\Gamma| = p^5$, then $V = 0$ and $A = \mathbf{k}\Gamma$.*
2. *If $|\Gamma| = p^4$, then V is 1-dimensional and R is a QLS.*
3. *If $|\Gamma| = p^3$, then V may be 1 or 2-dimensional and R is a QLS over some subgroup Γ' of Γ.*
4. *If $\Gamma = C_p \times C_p$, then V is 2-dimensional (and then R is a twisting of a Nichols algebra of type A_2) or V is 3-dimensional (and R is a QLS).*
5. *If $\Gamma = C_{p^2}$, then V is 2-dimensional (and in this case R is a QLS or a twisting of a Nichols algebra of type A_2) or V is 3-dimensional (and R is a QLS).*
6. *If $\Gamma = C_p$, then either V is 2-dimensional, R is of type B_2 and necessarily* $p \equiv 1 \bmod 4$, *or V is 3-dimensional, R is of type $A_2 \times A_1$ and $p = 3$.*

Proof. We prove that $\mathfrak{B}(V)$ is of the form claimed.

1. This is immediate.

2. By Remark 2.9 we have $\dim V = 1$, $V = (x) = M(g, \chi)$, $g \in Z(\Gamma)$. Furthermore, $\chi(g) = q$ is such that $q^p = 1$ (and $q \neq 1$ since A is finite dimensional), whence the structure of R is given by

$$x^p = 0, \quad \varepsilon(x) = 0,$$

$$\Delta(x^r) = \sum_{i=0}^{r} \binom{r}{i}_q x^i \otimes x^{r-i},$$

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$$\delta(x) = g \otimes x, \quad h \rightharpoonup x = \chi(h)x.$$

Let $a = x\#1 \in A$. Then A is generated by Γ and a, with the structure given by

$$a^p = 0, \quad \varepsilon(a) = 0, \quad hah^{-1} = \chi(h)a \quad \forall h \in \Gamma,$$

$$\Delta(a^r) = \sum_{i=0}^{r} \binom{r}{i}_q (a^i g^{r-i}) \otimes a^{r-i}.$$

3. The bound $\dim V \leq 2$ is a consequence of 4.3 below. If V is 1-dimensional, then $V = M(g, \chi)$ and A is given exactly as in the case $|\Gamma| = p^4$ with the only exception being that q has order p^2 and the relation on a is $a^{p^2} = 0$. If V is 2-dimensional, [**1**, Proposition 3.1.11] applies and V comes from the abelian case; i.e. V has a basis $\{x_1, x_2\}$ with $(x_i) = M(g_i, \chi_i)$ (g_i and χ_i are respectively central elements and characters of a certain subgroup Γ' of Γ). Let N_i be the order of $\chi_i(g_i)$. Then, by 2.2, we have $p^2 \geq N_1 N_2$, whence $N_1 = N_2 = p$; ($\chi_i(g_i) \neq 1$ since A is finite dimensional). Again by Proposition 2.2 we have that $\mathfrak{B}(V)$ is a QLS over Γ'. Let $b_{12} = \chi_2(g_1)$, and for each $h \in \Gamma$ let the matrix $\rho(h)_{ij}$ be defined by $h \rightharpoonup x_j = \sum_{i=1}^{2} \rho(h)_{ij} x_i$. Then A is generated by Γ, a_1, a_2 with structure and relations given by

$$a_i^p = 0, \quad \varepsilon(a_i) = 0, \quad ha_jh^{-1} = \sum_{i=1}^{2} \rho(h)_{ij} a_i, \quad \forall h \in \Gamma,$$

$$\Delta(a_i) = g_i \otimes a_i + 1 \otimes a_i,$$

$$a_1 a_2 = b_{12} a_2 a_1.$$

4. The bounds $2 \leq \dim V \leq 3$ are immediate consequences of Proposition 2.2. If $\dim V = 2$, then by Lemma 4.10 below it is a twisting of an algebra of type A_2. If $\dim V = 3$, then by Proposition 2.2 it is a QLS.

5. As in the case $\Gamma = C_p \times C_p$, the bounds $2 \leq \dim V \leq 3$ are consequences of Proposition 2.2. Suppose that $\dim V = 2$, V has basis $\{x_1, x_2\}$ and c is given in this basis by the matrix (b_{ij}). If b_{11} (resp. b_{22}) has order p^2, then by Proposition 2.2 b_{22} (resp. b_{11}) has order p and $\mathfrak{B}(V)$ is a QLS. If both b_{11} and b_{22} have order p then, by Lemmas 4.9 and 4.10 below, $\mathfrak{B}(V)$ is a twisting of an algebra of type A_2. If $\dim V = 3$, then by Proposition 2.2 it is a QLS.

6. This is proved in [**3**, Theorem 1.3]. □

In Section 5 we prove that if Γ is a group of order p^j and $R = \oplus_i R(i) \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ is a coradically graded braided Hopf algebra of dimension p^{5-j} with $R(1) \simeq \mathbf{k}$, then R is generated by $R(1)$. With this and the previous lemma we can prove the following result.

THEOREM 3.2. *Let $A = \oplus_i A(i)$ be a coradically graded pointed Hopf algebra of dimension p^5. Let $\Gamma = G(A)$ be the group of group-likes of A. Let $R = \oplus_i R(i) = A^{\mathrm{co}A(0)} \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and let $V = R(1)$. Then R is generated by V (i.e. $A = \mathfrak{B}(V)\#\boldsymbol{k}\Gamma$) and $\mathfrak{B}(V)$ is one in the list below. By $B(\cdot)$ we denote the group of order p^4 in* [**5**, p. 145].

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Γ	$\dim \mathfrak{B}(V)$	Type	Conditions
$(C_p)^4$	1	QLS	
$(C_p)^2 \times C_{p^2}$	1	QLS	
$C_{p^2} \times C_{p^2}$	1	QLS	
$C_p \times C_{p^3}$	1	QLS	
C_{p^4}	1	QLS	
$B(vi)$	1	QLS	
$B(vii)$	1	QLS	
$B(viii)$	1	QLS	
$B(ix)$	1	QLS	
$B(x)$	1	QLS	
$B(xiv)$	1	QLS	
$(C_p)^3$	2	QLS	
$C_{p^2} \times C_p$	1	QLS	
	2	QLS	
C_{p^3}	1	QLS	
	2	QLS	
$(C_p)^2$	2	A_2	
	3	QLS	
C_{p^2}	2	A_2	$p = 3$ or $p \equiv 1 \bmod 3$
C_p	2	B_2	$p \equiv 1 \bmod 4$
	3	$A_2 \times A_1$	$p = 3$

Proof. For the groups of order p^4, the only condition for the existence of a QLS is the existence of a central element $g \in \Gamma$ and a character $\chi \in \hat{\Gamma}$ such that $\chi(g)$ has order p. This is possible if and only if $g \notin [\Gamma, \Gamma]$ where $[\Gamma, \Gamma]$ is the commutator subgroup of Γ. It follows by inspection of each case that the groups in the table are those Γ such that $Z(\Gamma) \not\subset [\Gamma, \Gamma]$.

We go now to $|\Gamma| = p^3$. It is clear that QLS of rank one exist for $\Gamma = C_{p^2} \times C_p$ and $\Gamma = C_{p^3}$, but not for $\Gamma = (C_p)^3$. The two non-abelian groups of order p^3 have centers included in their commutator subgroups, whence the 1-dimensional Yetter–Drinfeld modules give rise to infinite dimensional Nichols algebras. We prove now that for the three abelian groups there exist QLS of rank 2: let q_1, q_2, q_3 denote respectively (fixed) roots of unity of orders p, p^2, p^3. We denote the generators of $(C_p)^3$ by $\{g_1, g_2, g_3\}$ and the generators of $\widehat{(C_p)^3}$ by $\{\hat{g}_1, \hat{g}_2, \hat{g}_3\}$, where $\hat{g}_i(g_j) = q_1^{\delta_{ij}}$. We denote the generators of $C_{p^2} \times C_p$ by $\{g_1, g_2\}$ and the generators of $\widehat{C_{p^2} \times C_p}$ by $\{\hat{g}_1, \hat{g}_2\}$, where $\hat{g}_i(g_j) = q_{3-i}^{\delta_{ij}}$. We denote the generator of C_{p^3} by $\{g\}$ and the generator of $\widehat{C_{p^3}}$ by $\{\hat{g}\}$, where $\hat{g}(g) = q_3$. It is straightforward that the following Yetter–Drinfeld modules give QLS of dimension p^2:

$$\Gamma = (C_p)^3, \quad V = M(g_1, \hat{g}_1) \oplus M(g_2, \hat{g}_2),$$

$$\Gamma = C_{p^2} \times C_p, \quad V = M(g_1^p, \hat{g}_1) \oplus M(g_1^{-p}, \hat{g}_1),$$

$$\Gamma = C_{p^3}, \quad V = M(g^{p^2}, \hat{g}) \oplus M(g^{-p^2}, \hat{g}).$$

For the two non-abelian groups we should have V a Yetter–Drinfeld module of dimension 2. There are three possibilities.

1. $V = M(h_1, \chi_1) \oplus M(h_2, \chi_2)$, where h_i are central and χ_i are characters; but by the same reason as in the rank one case, this would give infinite dimensional Nichols algebras.
2. $V = M(g, \chi)$, where χ is a character and $[\Gamma : \Gamma_g] = 2$; but this is impossible since $p \neq 2$ (this case arises when $p = 2$; see [**10**]).
3. $V = M(g, \rho)$, where g is central and ρ is an irreducible representation of Γ with $\deg \rho = 2$. Since $p \neq 2$, by the Frobenius theorem we find that this is impossible; (this case arises when $p = 2$; see [**10**]).

Let now $\Gamma = (C_p)^2$. It is immediate that there are no QLS of rank 1 nor 2, since otherwise there would be a character with a p^2-th root of unity in the image. The existence of a QLS of rank 3 is a consequence of [**2**, Lemma 4.1]. An explicit construction is as follows: let Γ have generators $\{g_1, g_2\}$ and $\hat{\Gamma}$ have generators $\{\hat{g}_1, \hat{g}_2\}$ where $\hat{g}_i(g_j) = q_1^{\delta_{ij}}$ (as before q_1 is a fixed p-th root of unity). Let $V = M(g_1, \hat{g}_1) \oplus M(g_1, \hat{g}_1{}^{-1}) \oplus M(g_2, \hat{g}_2)$. It is straightforward to see that V generates a QLS. For a construction of a Nichols algebra of type A_2, let $r = \frac{1}{2} \in \mathbb{Z}/p$ (the construction for $p = 2$ is slightly different; see [**10**]). Set $V = M(g_1, \hat{g}_1\hat{g}_2{}^{-r}) \oplus M(g_2, \hat{g}_1{}^{-r}\hat{g}_2)$. It is clear then that V has the matrix

$$(b_{ij}) = \begin{pmatrix} q_1 & q_1^{-r} \\ q_1^{-r} & q_1 \end{pmatrix}, \text{ whence } b_{ij}b_{ji} = b_{ii}^{a_{ij}} \text{ with } (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Let $\Gamma = C_{p^2}$. The non-existence of a p^3-dimensional QLS is a consequence of Lemma 4.1 below. Let $g, \hat{g}$ be respectively generators of Γ, $\hat{\Gamma}$, and let $q = \hat{g}(g)$. Suppose that $V \in {}_{\mathbf{k}\Gamma}^{\mathbf{k}\Gamma}\mathcal{YD}$ generates an algebra of type A_2. Let $V = M(g^{e_1}, \hat{g}^{f_1}) \oplus M(g^{e_2}, \hat{g}^{f_2})$. Since V has a matrix $b_{ij} = q^{e_i f_j}$ and b_{11}, b_{22} must have order p, then p divides e_1 and e_2, or p divides f_1 and f_2. Then the same arguments as in [**3**, Theorem 1.3] give the condition $p = 3$ or $p \equiv 1 \bmod 3$. Furthermore, let b be such that $b^2 + b + 1 \equiv 0 \bmod p$ (the condition on p is equivalent to the existence of such a b) and take $e_1 = p, f_1 = 1$, $e_2 = -p(b+1), f_2 = b$. It is straightforward to see that this gives a Cartan matrix of type A_2.

For $\Gamma = C_p$ it is proved in [**3**, Theorem 1.3] that there exists an algebra of type B_2 if and only if $p \equiv 1 \bmod 4$, and of type $A_2 \times A_1$ if and only if $p = 3$. □

4. Subsidiary results. The following lemma may be considered as an addendum to [**2**, Lemma 4.2].

LEMMA 4.1. *Let $\Gamma = C_{p^n}$ and $V \in {}_{k\Gamma}^{k\Gamma}\mathcal{YD}$ generate a finite dimensional QLS. Then V may be 1-dimensional (and hence $\dim \mathfrak{B}(V) = p^v$ with $1 \leq v \leq n$) or it may be 2-dimensional (and hence $\dim \mathfrak{B}(V) = p^{2v}$ with $1 \leq v \leq n$).*

Proof. The bound $\dim V \leq 2$ is the content of [**2**, Lemma 4.2]. Let Γ have a generator g and $\hat{\Gamma}$ have a generator $\hat{g}$. Let $q = \hat{g}(g)$, which is a primitive p^n-th root of unity. If V is 1-dimensional, the result is an easy consequence of Proposition 2.2.

Suppose that $V = M(g^{e'_1}, \hat{g}^{f'_1}) \oplus M(g^{e'_2}, \hat{g}^{f'_2})$. Let $e'_i = p^r e_i$ such that e_1, e_2 are not both divisible by p, $f'_i = p^s f_i$ such that f_1, f_2 are not both divisible by p. Then V has a

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matrix given by $b_{ij} = q^{e'_i f'_j} = q^{e_i f_j p^{r+s}}$. Since $\mathfrak{B}(V)$ is finite dimensional, $r + s < n$ (for if not $b_{11} = b_{22} = 1$). Let $u = n - r - s$. Suppose that $p \nmid e_1$ (if $p \nmid e_2$ it is analogous). Suppose first that $p \nmid f_2$; then b_{12} has order p^u. Since V generates a QLS, $b_{21} b_{12}^{-1}$ also has order p^u and thus $p \nmid e_2$, $p \nmid f_1$. This implies the result with $v = u$. Suppose next that $p | f_2$. Then $p \nmid f_1$. Let $f_2 = p^t b$, $e_2 = p^y a$ with $p \nmid b$, $p \nmid a$. We prove that $t = y$: we have $t < u$ since if not $b_{22} = 1$. Now, b_{12} has order p^{u-t}, whence b_{21} has order p^{u-t}. Since $p \nmid f_1$ we have $p^t | e_2$, whence $y \geq t$. By similar considerations $y \leq t$. This implies the result with $v = u - t$. □

We shall make use of the following important tool for Nichols algebras.

Definition 4.2. Let $V \in {}^H_H\mathcal{YD}$ and $c = c_{V,V}$. For $i + j = n$, we denote by $\Delta_{i,j} : \mathfrak{B}^n(V) \to \mathfrak{B}^i(V) \otimes \mathfrak{B}^j(V)$ the (i, j)-component of the comultiplication of $\mathfrak{B}(V)$.

It is proved in [**14**] (or see [**1**, Definition 3.2.10]) that $\Delta_{i,j}$ is injective, $\forall i, j$. Let $\{x_1, \ldots, x_\theta\}$ be a basis of V and let $\{x_1^*, \ldots, x_\theta^*\}$ be its dual basis. We denote by ∂_{x_i} the differential operator on $\mathfrak{B}(V)$ given by

$$\partial_{x_i}(z) = (\mathrm{id} \otimes x_i^*)\Delta_{n-1,1}(z), \quad \text{if } z \in \mathfrak{B}^n(V), \; n > 0, \quad \text{and } \partial_{x_i}(1) = 0.$$

By the injectivity of $\Delta_{i,j}$ it is immediate that for $z \in \mathfrak{B}^n(V)$ $(n > 0)$ we have $z = 0$ if and only if $\partial_{x_i}(z) = 0$, for all $i = 1, \ldots, \theta$. Suppose now that $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and ∂_{x_i} is such that there exists $g \in \Gamma$ with $\partial_{x_i}(v) = 0$ if $\delta(v) = h \otimes v$ and $h \neq g$; (this happens for instance if $\delta(x_j) = g_j \otimes x_j$, $j = 1, \ldots, \theta$ and $g = g_i$). Then it is easy to see that ∂_{x_i} satisfies the Leibniz rule

$$\partial_{x_i}(z_1 z_2) = \partial_{x_i}(z_1)(g \rightharpoonup z_2) + z_1 \partial_{x_i}(z_2).$$

The following theorem is proved in [**2**, Theorem 0.2] in the case in which Γ is an abelian group.

Theorem 4.3. *Let Γ be a finite group. Let $V \in {}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ be such that* $\dim \mathfrak{B}(V) = p^2$, *where p is the smallest prime number dividing $|\Gamma|$. Then* $\dim V \leq 2$ *and $\mathfrak{B}(V)$ is a QLS over some subgroup $\Gamma' \subset \Gamma$. Furthermore, if $p > 2$ then $V = M(g, \chi)$ with g central, χ is a character such that $\chi(g)$ has order p^2 and hence $\mathfrak{B}(V)$ is a QL over Γ, or $V = M(g_1, \chi_1) \oplus M(g_2, \chi_2)$ where g_i is central, χ_i is a character $(i = 1, 2)$ such that $\chi_i(g_i)$ has order p and hence $\mathfrak{B}(V)$ is a QP over Γ.*

Proof. Let $V = \oplus_{i=1}^{\theta} M(g_i, \rho_i)$. It can be shown that $\dim \mathfrak{B}(V) \geq \dim \mathfrak{B}(M(g_i, \rho_i))$, $\forall i$. Let $I = [\Gamma : \Gamma_{g_1}]$ and $d = \deg(\rho_1)$. We have $\dim M(g_1, \rho_1) = dI$. We have $d = 1$ or $d \geq p$, and $I = 1$ or $I \geq p$. Since $\dim \mathfrak{B}(V) \geq 1 + \dim V$ we have $\dim V < p^2$.

Suppose first that $d \geq p$. This implies that $I = 1$, whence g_1 is central in Γ. By 2.6 we have $p^2 = \dim \mathfrak{B}(V) \geq \dim \mathfrak{B}(M(g_1, \rho_1)) \geq N^d$ with N the order of q, where $q\mathrm{id} = \rho_1(g_1)$. Since $\mathfrak{B}(V)$ is finite dimensional, we have $q \neq 1$ and hence $N \geq p$. If $p > 2$ we have a contradiction. If $p = 2$ we must have $\theta = 1$, $d = 2$, $N = 2$. The condition $d = 2$ implies that V comes from the abelian case, as explained after Definition 2.4. The condition on N tells us that $q = -1$. Furthermore, by Proposition 2.2, $\mathfrak{B}(V)$ is a QLS, and it is shown in [**1**] that the matrix of c is $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$.

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Suppose then that $I \geq p$. This implies that $d = 1$, whence ρ_1 is a character of Γ_{g_1}. Let $q = \rho_1(g_1)$ and let N be the order of q. Let x be a generator of the space affording ρ_1, and let $\{h_1 = 1, h_2, \dots, h_I\}$ be a set of representatives of the cosets of Γ/Γ_{g_1}. Then $M(g_1, \rho_1)$ has as basis the elements $\{h_1 \rightharpoonup x, \dots, h_I \rightharpoonup x\}$ and we have

$$\begin{aligned} c(h_i \rightharpoonup x \otimes h_i \rightharpoonup x) &= h_i g_1 h_i^{-1} h_i \rightharpoonup x \otimes h_i \rightharpoonup x \\ &= h_i g_1 \rightharpoonup x \otimes h_i \rightharpoonup x = q(h_i \rightharpoonup x \otimes h_i \rightharpoonup x). \end{aligned} \tag{4.4}$$

It is straightforward to see using derivations that the elements

$$\{1, (h_i \rightharpoonup x)^r \mid 1 < r < N, \ i = 1, \dots, I\}$$

are linearly independent, whence

$$p^2 = \dim \mathfrak{B}(V) \geq 1 + I(N - 1). \tag{4.5}$$

Thus, $N \leq p$. On the other hand, $q \neq 1$ for if not it is easy to see using derivations that the elements $\{x^r \mid r \geq 0\}$ would be linearly independent and $\mathfrak{B}(V)$ would be infinite dimensional; (note that we have not proved at present that $\mathbf{k}x$ is a sub-YD-module nor that $M(g_1, \rho_1)$ comes from the abelian case, and hence Proposition 2.2 cannot be used.) We have thus proved that $N = p$. Suppose for a moment that $I > p$. It is clear that if $p > 2$ then $I \geq p + 2$, but then (4.5) tells us that this is a contradiction. If $p = 2$, then $I = 3$ but, by [**1**, Proposition 3.2.2], $\dim \mathfrak{B}(M(g_1, \rho_1)) \geq 5$, also a contradiction. Hence, we have that $I = p$ and then Γ_g, having index the smallest prime dividing $|\Gamma|$, is invariant in Γ. As stated after Definition 2.4, this implies that $\mathfrak{B}(M(g_1, \rho_1))$ comes from the abelian case, but then Proposition 2.2 applies and (4.4) tells us that $\dim \mathfrak{B}(M(g_1, \rho_1)) \geq p^p$. This is a contradiction if $p > 2$. If $p = 2$, then $\theta = 1$, $q = -1$ and it is proved in [**1**] that the matrix of c is

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Suppose finally that $I = d = 1$. Then g_1 is central and ρ_1 is a character. Let $q = \rho_1(g_1)$ and let N be its order. Then $\dim \mathfrak{B}(V) \geq \dim \mathfrak{B}(M(g_1, \rho_1)) = N$ implies that $N \leq p^2$. If $N = p^2$, then $\theta = 1$ and the result follows at once. If $N < p^2$, then $N \geq p$ and N is prime. Since $\dim \mathfrak{B}(M(g_1, \rho_1)) = N$, we have $\theta > 1$. Since $\dim \mathfrak{B}(M(g_2, \rho_2)) \leq p^2 - 1$ (because if x is a generator of $M(g_1, \rho_1)$ then x does not belong to $\mathfrak{B}(M(g_2, \rho_2))$) by the same arguments as above applied to $M(g_2, \rho_2)$ we have necessarily that g_2 is central and ρ_2 is a character. Let N_2 be the order of $\rho_2(g_2)$. Thus $N_2 < p^2$, and since g_1, g_2 are both central, $M(g_1, \rho_1) \oplus M(g_2, \rho_2)$ comes from the abelian case, whence by Proposition 2.2, $\mathfrak{B}(M(g_1, \rho_1) \oplus M(g_2, \rho_2))$ has dimension at least NN_2. This implies that $N = N_2 = p$, $\theta = 2$ and $\mathfrak{B}(V)$ is a QLS over Γ.

Remark 4.6. It is proved in [**8**] in a different way that if $\dim \mathfrak{B}(V) = p$, where p is the smallest prime number dividing Γ, then $\dim V = 1$ and $\mathfrak{B}(V)$ is a QLS. It is proved also, with the same ideas as here, in [**3**, Proposition 7.5].

Remark 4.7. We note that the proof of Theorem 4.3 above says that there are no V in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ such that $\dim \mathfrak{B}(V) = \pi^2$ if π is a prime number smaller than every prime dividing $|\Gamma|$.

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The previous theorem implies the following result.

COROLLARY 4.8. *Let A be a pointed Hopf algebra of dimension m whose coradical has dimension m/p^2, where p is the smallest prime number dividing m. Then p^3 divides m and* $\dim A_1 = (r+1)m/p^2$, *where $r = 1$ or 2.*

Proof. Consider the coradical filtration of A and let $H = \oplus_i H(i)$ be the associated graded algebra. Then H is pointed and $H(0) \simeq \mathbf{k}\Gamma$, where Γ is the group of group-likes of A that has order m/p^2. Let $R = H^{\mathrm{co}H(0)} \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and let $R' \subset R$ be the algebra generated by $R(1)$; ($R' = R$ if and only if R is a Nichols algebra). Thus $\dim R = p^2$ and by the Nichols–Zoeller theorem, $\dim R' = p^j$ with $0 \le j \le 2$. The case $j = 0$ would imply that $\dim R(1) = 0$, which is impossible. The case $j = 1$ is also impossible, for in that case Remark 4.6 says that $\dim R(1) = \dim R'(1) = 1$, and [**2**, Theorem 3.2] says that R is a Nichols algebra. Then $R' = R$. Remark 4.7 says that p divides $|\Gamma|$ (whence p^3 divides m) and Theorem 4.3 says that $r = \dim R(1)$ may be 1 or 2, whence $\dim H(1) = r|\Gamma|$ and

$$\dim A_1 = \dim H(0) + \dim H(1) = (r+1)|\Gamma| = (r+1)m/p^2.$$

□

LEMMA 4.9. *Let Γ be a p-group and V a 2-dimensional module in ${}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ such that* $\dim \mathfrak{B}(V) = p^3$. *Recall that under this assumption c has a matrix (b_{ij}) with respect to some basis $\{x, y\}$. Let q be a primitive p^2-th root of unity, and suppose that $b_{ij} = q^{c_{ij}}$. If p divides c_{11} and c_{22}, then p divides $c_{12} + c_{21}$.*

Proof. We have $x^p = y^p = 0$. Let $z = \mathrm{Ad}_x(y) = xy - b_{12}yx$ and $\sigma = 1 - b_{12}b_{21}$. We have

$$\partial_x(z) = b_{12}y - b_{12}y = 0, \qquad \partial_y(z) = x - b_{12}b_{21}x = \sigma x \Rightarrow \partial_x\partial_y(z) = \sigma.$$

Furthermore,

$$\partial_x(x^r y^s z^t) = (r)_{b_{11}} b_{11}^t b_{12}^{s+t} x^{r-1} y^s z^t,$$
$$\partial_y(y^s z^t) = (s)_{b_{22}} b_{21}^t b_{22}^t y^{s-1} z^t + \sum_{i=0}^{t-1} \sigma b_{21}^i b_{22}^i y^s z^{t-1-i} x z^i,$$
$$\partial_x\partial_y(y^s z^t) = \sigma (t)_{(b_{11}b_{12}b_{21}b_{22})} y^s z^{t-1}.$$

Thus, if $p \nmid c_{12} + c_{21}$, the order of $(b_{11}b_{12}b_{21}b_{22})$ is p^2, whence the set $\{z^t \mid 0 \le t < p^2\}$ is linearly independent. This implies inductively that the set $\{y^s z^t \mid 0 \le s < p, 0 \le t < p^2\}$ is linearly independent, and then that the set $\{x^r y^s z^t \mid 0 \le r, s < p, 0 \le t < p^2\}$ is linearly independent, so that $\dim \mathfrak{B}(V) \ge p^4$. □

The following result is a consequence of [**3**, Corollary 1.2]. We give a direct proof here.

LEMMA 4.10. *Let $V = (x, y)$ be a 2-dimensional module in ${}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ such that* $\dim \mathfrak{B}(V) = p^3$. *Let V have a matrix (b_{ij}) and suppose that $b_{ij}^p = 1$, for all i, j. Then b_{ij} is a Cartan matrix of type A_2.*

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Proof. Let $q = b_{11}$ and c_{ij} be given by $b_{ij} = q^{c_{ij}}$. We may suppose as above that $b_{12} = b_{21}$. Let $b_{12} = q^a$, $b_{22} = q^c$. Take $z = \mathrm{Ad}_x(y) = xy - q^a yx$, and let $\sigma = 1 - q^{2a}$; thus $\sigma \neq 0$, since otherwise $\mathfrak{B}(V)$ would be a QLS and $\dim \mathfrak{B}(V)$ would be p^2. As before

$$\partial_x(z) = 0, \qquad \partial_y(z) = \sigma x \Rightarrow \partial_x\partial_y(z) = \sigma,$$

whence

$$\begin{aligned}\partial_x(x^r y^s z^t) &= (r)_q q^{t+a(s+t)} x^{r-1} y^s z^t,\\ \partial_y(y^s z^t) &= (s)_{q^c} q^{as+ct} y^{s-1} z^t + \sum_{i=0}^{t-1} \sigma q^{(a+c)i} y^s z^{t-1-i} x z^i,\\ \partial_x\partial_y(y^s z^t) &= \sigma (t)_{q^{1+2a+c}} y^s z^{t-1}.\end{aligned}$$

As before, the set $\{x^r y^s z^t \mid 0 \le r, s, t < p\}$ is linearly independent; (as a remark, note that we must have $1 + 2a + c \not\equiv 0 \pmod p$ since if not $\mathfrak{B}(V)$ would be infinite dimensional). Now let $w = \mathrm{Ad}_x(z) = xz - q^{1+a} zx$. We have

$$\partial_x(w) = 0, \qquad \partial_y(w) = \sigma x^2 - \sigma q^{1+a} q^a x^2 = \sigma(1 - q^{2a+1}) x^2.$$

The (x, y)-bidegree of w is $(2, 1)$, whence the set $\{x^2 y, xz, w\}$ must be linearly dependent in order for $\mathfrak{B}(V)$ to be p^3-dimensional. This implies $2a + 1 = 0$, which means that $b_{12} b_{21} = b_{11}^{-1}$.

With the same reasoning, we must have $b_{12} b_{21} = b_{22}^{-1}$, and thus $b_{ij} b_{ji} = b_{ii}^{c_{ij}}$ with

$$c_{ij} \equiv \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \bmod p,$$

and b_{ij} is a Cartan matrix of type A_2. As a remark, note that $b_{11} = b_{22}$. □

5. The classification is complete. We have to prove that Theorem 3.2 lists all the coradically graded pointed Hopf algebras of dimension p^5. This amounts to proving that a coradically graded pointed Hopf algebra is generated by its homogeneous component of degree 1, which in turn is equivalent to proving that if $A = \oplus_i A(i)$ is a coradically graded pointed Hopf algebra and $R = A^{\mathrm{co}A_0}$ is its algebra of coinvariants then R is a Nichols algebra. As in [**3**, §8], let $S = R^*$ be its dual. Then S is a graded braided Hopf algebra in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$, $S = \oplus_i S(i)$ and is generated by $S(0) \oplus S(1)$. Furthermore, we have a surjection $S \twoheadrightarrow S'$, $S' = \mathfrak{B}(S(1))$. We have to prove that S is coradically graded; i.e. that $\mathcal{P}(S) = S(1)$. This is the same as saying that $S' = S$. Now, [**2**, Theorem 3.2] plus Remark 4.6 solve the problem for the cases in which Γ has order p^4 or p^3, and [**3**, Theorem 8.2] and [**3**, Lemma 8.5] solve the problem for the case in which Γ has order p or $\Gamma = C_p \times C_p$. The following theorem solves the pending case.

THEOREM 5.1. *Let Γ be a finite group and p the smallest prime number dividing $|\Gamma|$. Let $S = \oplus_i S(i)$ in ${}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ be a graded braided Hopf algebra of dimension p^3 such that $S(0) = \boldsymbol{k}$ and S is generated by $S(1)$. Suppose that $S(1)$ comes from the abelian*

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case; i.e. there exists an abelian subgroup $\Gamma' \subset \Gamma$ *such that* $S(1)$ *is a YD-module over* Γ'. *Then* S *is a Nichols algebra.*

Proof. We prove the statement for $p > 2$, the case $p = 2$ being treated in [**10**]. Let $S' = \mathfrak{B}(S(1))$, and consider the canonical projection $S \twoheadrightarrow S'$. We must prove that this is an isomorphism. If $\dim S(1) = 3$ then, by Proposition 2.2, we have $\dim S' \geq p^3$; but this implies that $S' = S$ and S is a Nichols algebra. If $\dim S(1) = 1$, then [**2**, Theorem 3.2] shows that S is a Nichols algebra. Hence we are led to consider the case $\dim S(1) = 2$. We have $\dim S' \leq p^3$, and we suppose that $\dim S' < p^3$. Then by Proposition 2.2 we have $\dim S' \geq p^2$, whence $\dim S' = p^2$. Now Theorem 4.3 says that S' is a QLS over Γ', $S'(1)$ has a basis $\{x, y\}$ and the braiding c has a matrix (b_{ij}) in this basis, where b_{ii} are primitive p-th roots of unity and $b_{12}b_{21} = 1$. Furthermore, the linear spans $\mathbf{k}x$ and $\mathbf{k}y$ are sub-YD-modules over Γ'. Let $z = x_1x_2 - b_{12}x_2x_1 \in S$. If we prove that $z = 0$ in S, then $\dim S = p^2$, but this would be a contradiction and we would be done.

Suppose that $z \neq 0$. Now, it is immediate that z is primitive in S. Consider the coradical filtration of S and let $T = \oplus_i T(i)$ be the associated graded algebra. We have $x, y, z \in S_1$. Consider $\bar{x}, \bar{y}, \bar{z} \in T(1)$. It is easy to see that these elements are linearly independent. We compute the matrix of c for $\{\bar{x}, \bar{y}, \bar{z}\}$. It is given by

$$(b'_{ij}) = \begin{pmatrix} b_{11} & b_{12} & b_{11}b_{12} \\ b_{21} & b_{22} & b_{21}b_{22} \\ b_{11}b_{21} & b_{12}b_{22} & b_{11}b_{12}b_{21}b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{11}b_{12} \\ b_{21} & b_{22} & b_{21}b_{22} \\ b_{11}b_{21} & b_{12}b_{22} & b_{11}b_{22} \end{pmatrix}.$$

Consider now the canonical projection $T \twoheadrightarrow T' = \mathfrak{B}(T(1))$. Since $\mathbf{k}x$, $\mathbf{k}y$ and $\mathbf{k}z$ are sub-YD-modules of S over Γ', then $\mathbf{k}\bar{x}$, $\mathbf{k}\bar{y}$ and $\mathbf{k}\bar{z}$ are sub-YD-modules of T over Γ'. Thus, if $W = (\bar{x}, \bar{y}, \bar{z})$ we have $\dim \mathfrak{B}(W) \leq \dim T' \leq \dim T = p^3$. Now Proposition 2.2 applies; (notice that b'_{ii} has order p, $\forall i$, since $b_{11}b_{22} = 1$ would imply that $\dim \mathfrak{B}(W) = \infty$). Hence $\mathfrak{B}(W)$ is a QLS, but this implies that $1 = b_{11}^2 b_{12}b_{21} = b_{11}^2$, which is impossible since $p \neq 2$. Hence, $z = 0$ in S and the theorem is proved. □

6. Final remarks. In order to give a complete classification of the pointed Hopf algebras of dimension p^5, the following steps should be taken.

1. For each Nichols algebra R in Theorem 3.2, give all the modules M in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ such that $\mathfrak{B}(M) \simeq R$.
2. Classify the isomorphism classes of the bosonizations of the Nichols algebras in the previous step; (note that there exist non isomorphic Nichols algebras which give isomorphic algebras after bosonization).
3. For each coradically graded p^5-dimensional Hopf algebra in the previous step, classify all the liftings.

These steps are highly non trivial. For instance let $\Gamma = C_{p^n}$, where $n > 0$ and $p \neq 2$, and let $0 \leq s \leq n$. The number of QLS of rank 1 over Γ with dimension p^s is given by

$$\sum_{\substack{i+j-n=s \\ i,j \leq n}} \phi(i, j),$$

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while the number of isomorphisms classes of these QLS after bosonization is given by

$$\sum_{\substack{i+j-n=s\\ i,j\leq n}} \frac{\phi(i,j)}{I(i,j)},$$

where

$$\begin{aligned}\phi(i) &= p^{i-1}(p-1),\\ \phi(i,j) &= \phi(i)\phi(j),\\ I(k_1,\dots,k_r) &= \phi(\max\{k_1,\dots,k_r\}).\end{aligned}$$

Furthermore, the number of QLS of rank 2 over Γ with dimension p^s, (where s is even, by Lemma 4.1), is given by

$$\frac{1}{2}\sum_{i_1,j_1,i_2,j_2} \frac{\phi(i_1,j_1)\phi(i_2,j_2)}{L_n(i_1,j_2)},$$

while the number of isomorphism classes of these QLS after bosonization is given by

$$\frac{1}{2}\sum_{i_1,j_1,i_2,j_2} \frac{\phi(i_1,j_1)\phi(i_2,j_2)}{L_n(i_1,j_2)I(i_1,i_2,j_1,j_2)},$$

where

$$L_n(i,j) = \phi(i+j-n),$$

and the sum is over the tuples such that

$$\begin{gathered} i_1+j_1-n = s_1 \geq 1, \quad i_2+j_2-n = s_2 \geq 1,\\ s_1+s_2 = s, \quad i_1,j_1,i_2,j_2 \leq n, \quad i_1+j_2 = i_2+j_1. \end{gathered}$$

As a result of this, the number of coradically graded non isomorphic Hopf algebras of dimension p^5 with coradical C_{p^4}, C_{p^3} is, respectively, $2(p^2-1)$ and $p(p-1)[2+\frac{p(p-1)(p+2)}{2}]$.

See also the discussion in [**3**, §9] for the first step, [**3**, §6] for the second. In particular, a necessary and sufficient condition for two YD-modules to give isomorphic algebras after bosonization is given in [**3**, Proposition 6.3].

As an example of the last step, let A be a pointed Hopf algebra of dimension p^5 with coradical Γ of order p^4. Let H be the associated graded Hopf algebra and R its invariants as in (2.1). Then $H = R\#\mathbf{k}\Gamma$, where $R = \mathfrak{B}(V)$. We have then $V = M(h,\chi)$ where h is central and χ is a character such that $\chi(h)$ has order p. Let x be a generator of V. Then, by [**10**, Proposition 2.0.17], x can be lifted to $a \in A$ such that $\Delta(a) = h\otimes a + a\otimes 1$ and $gag^{-1} = \chi(g)a\ \forall g\in\Gamma$. Since the elements $\{x^i\#g \mid 0\leq i<p,\ g\in\Gamma\}$ are a basis of H, the elements $\{a^ig \mid 0\leq i<p,\ g\in\Gamma\}$ are a basis of A. The lifting A is then determined by the element a^p, the case $a^p=0$ being the bosonization

$A = R\#\mathbf{k}\Gamma$. It is easy to see that a^p is a skew-primitive and $\Delta(a^p) = h^p \otimes a^p + a^p \otimes 1$. Looking at the space of skew-primitives, this implies that

$$a^p = \lambda(h^p - 1), \quad \lambda \in \mathbf{k}.$$

Taking a suitable scalar multiple of a we may suppose that $\lambda \in \{0, 1\}$. Hence there are no more than 2 liftings. In some of the cases we must have $a^p = 0$. These cases are given by the diamond lemma

$$\begin{aligned} ga^p &= g\lambda(h^p - 1) = \lambda(h^p - 1)g, \\ ga^p &= \chi(g)aga^{p-1} = \ldots = \chi^p(g)a^p g = \chi^p(g)\lambda(h^p - 1)g, \end{aligned}$$

whence $\lambda(\chi^p - 1) = 0$ for A to be p^5-dimensional. This tells us that over the group $B(vi)$ any pointed Hopf algebra of dimension p^5 is coradically graded.

On the other hand, it is clear that if $h^p = 1$ then $a^p = 0$. This tells us that over the groups $B(viii)$, $B(ix)$, $B(x)$ and $B(xiv)$ any pointed Hopf algebra of dimension p^5 is coradically graded.

As a corollary we note that a pointed Hopf algebra of dimension p^5 and non abelian coradical is coradically graded, unless its coradical is isomorphic to $\mathbf{k}B(vii)$. We classify all the liftings in this case: $B(vii)$ can be presented with generators X, Y, Z and relations

$$X^{p^2} = Y^p = Z^p = 1, \quad [Z, Y] = X^p, \quad [X, Y] = [X, Z] = 1.$$

Hence $Z(B(vii)) = (X)$ while $[B(vii), B(vii)] = (X^p)$. Let q be a (fixed) p-th root of unity. The Yetter–Drinfeld modules generating Nichols algebras of dimension p are then

$$V = M(X^i, \chi) \text{ such that } p \nmid i, \ \chi(X) = q^a \ (p \nmid a), \ \chi(Y) = q^b, \ \chi(Z) = q^c.$$

However, it can be shown that most of them give isomorphic algebras after bosonization. We are led to consider two modules:

$$\begin{aligned} V_i = M(X, \chi_i) \ (i = 1, 2), \quad \chi_1(X) = q, \quad \chi_1(Y) = \chi_1(Z) = 1, \\ \chi_2(X) = \chi_2(Y) = q, \quad \chi_2(Z) = 1. \end{aligned}$$

Hence we have two pointed Hopf algebras of dimension p^5 with non abelian coradical that are not coradically graded:

$A^{(1)}$ generated by a, X, Y, Z and the relations of $B(vii)$ together with $Xa = qaX, Ya = aY, Za = aZ, a^p = X^p - 1, \Delta(a) = X \otimes a + a \otimes 1$,

$A^{(2)}$ generated by a, X, Y, Z and the relations of $B(vii)$ together with $Xa = qaX, Ya = qaY, Za = aZ, a^p = X^p - 1, \Delta(a) = X \otimes a + a \otimes 1$.

A description of the liftings of QLS (respectively of the algebras of type A_2 over groups of exponent p) is made in [**2**] (respectively [**4**]).

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