LINEAR OPERATORS WHICH COMMUTE WITH TRANSLATIONS

PART I: REPRESENTATION THEOREMS

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0. Introduction and preliminaries

In Part I of this paper we shall be concerned with the representation as convolutions of continuous linear operators which act on various function-spaces linked with a locally compact group and which commute with left — or right — translations; cf. the results in [12]. For completeness some known results are included whenever they follow from the general procedure. We have tried to follow simple general approaches as much as possible.

Nothing has been said about analogous problems for semigroups (the real positive semi-axis, for example); see [1], [9] and [38] and the references there cited.

Part II deals with some applications of the representation theorems to the study of averaging operators over groups, of normalizers of function-algebras over groups, and of some division problems in certain convolution algebras of functions, measures, and distributions.

For the sake of completeness and the reader's convenience, Part I contains the statements of several known representation theorems and references to the original proofs. On the other hand, neither Part I nor Part II contains any attempt at coverage of the theory of multipliers of abstract Banach algebras.

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In this section we set out some notations which remain standard throughout the paper, together with preliminary results which lie behind the general approach to the representation theorems.

0.1 X will always denote a Hausdorff locally compact group, e its neutral element, and dx its left Haar measure. If X is compact, dx is assumed

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to be normalized so that $\int dx = 1$. In general X is multiplicatively written, but when it is specialised to R^m ($R =$ the additive group of reals) or T^m ($T =$ the circle, usually regarded as the quotient $R/2\pi$) or a product of these, the additive notation is adopted.

The modular function Δ of X is the positive, continuous character of X defined by the integral identity

$$\int k(xa)dx = \Delta(a) \int k(x)dx \tag{0.1}$$

for continuous functions k which vanish off compact subsets of X; see [16], p. 250. (Note that our $\Delta(a)$ is Weil's $\Delta(a^{-1}) = \Delta(a)^{-1}$ ([35], p. 39).) One has also the formula ([35], p. 40; [16], p. 250)

$$\int k(x^{-1})dx = \int k(x)\Delta(x)dx. \tag{0.2}$$

X is unimodular if and only if $\Delta(a) = 1$ for all $a \in X$. This is the case if X is Abelian, or compact, or a semisimple Lie group ([35], p. 39; [16], p. 252).

$L^p = L^p(X)$, $1 \leqq p \leqq \infty$, is the usual Lebesgue space formed relative to dx ([16], Section 4.11). We do not usually distinguish notationally between a function and its class modulo neglibible (or, when $p = \infty$, locally negligible) functions. The usual norms are employed:

$$\begin{aligned} ||f||_{L^p} &= \left[\int |f(x)|^p dx\right]^{1/p} \quad \text{if} \quad p \neq \infty, \\ ||f||_{L^\infty} &= \text{loc. ess. sup } |f(x)|. \end{aligned} \tag{0.3}$$

0.2 $C = C(X)$ denotes the space of all continuous complex-valued functions on X, $C_0 = C_0(X)$ the subspace thereof formed of continuous functions which tend to zero at infinity, and $C_c = C_c(X)$ the even smaller subspace formed of continuous functions with compact supports. (The support of a continuous function f on X is the closure in X of the set of points $x \in X$ at which $f(x) \neq 0$. The support of f is denoted by supp f.)

Each of these spaces carries a "natural" topology: for C this is the topology of locally uniform convergence; for C_0 it is the topology of uniform convergence, defined by restricting to C_0 the norm $||\cdot||_{L^\infty}$; and for C_c it is the topology obtained by regarding C_c as the internal inductive limit of its subspaces

$$C_{c,K} = \{f \in C_c : \text{supp } f \subset K\},$$

K ranging over all (or over all members of a base for) the compact subsets of X, and each $C_{c,K}$ being regarded as a Banach space with the supremum norm; see [16], p. 430 (where C_c is denoted by $\mathscr{K}$).

When X is compact, C, C_0 and C_c, together with their natural topologies, become identical.

0.3 $M = M(X)$ denotes the space of all complex Radon measures on X; see [16], Chapter 4. It may and will be regarded as the dual of C_c, and the associated weak topology $\sigma(M, C_c)$ is the "vague topology of measures". (The σ-notation for weak topologies is as explained in [16], pp. 88—89, 500—501.)

If $\mu \in M$, the support of μ, denoted by $\operatorname{supp} \mu$, is the complement in X of the largest open subset U of X satisfying $|\mu|(U) = 0$. (This last requirement is equivalent to the demand that $\langle f, \mu \rangle = \int f d\mu$ shall vanish for any $f \in C_c$ having its support within U.)

$M_c = M_c(X)$ is the subspace of M formed of those measures having compact supports. M_c is viewed as the dual of C and carries an associated weak topology $\sigma(M_c, C)$; see [16], p. 203.

$M_{bd} = M_{bd}(X)$ is the subspace of M formed of those measures μ such that

$$(0.4) \qquad ||\mu|| = |\mu|(X) < +\infty.$$

M_{bd}, together with this norm, is identifiable with the dual of the Banach space C_0; see [16], Exercise 4.45.

Throughout, ε_x denotes the Dirac measure at the point x, and $\varepsilon \equiv \varepsilon_e$ ([16], p. 179). Then $\operatorname{supp} \varepsilon_x = \{x\}$.

0.4 For general groups X, $M(X)$ is the largest "function-space" we introduce, all others being subspaces of it in the following sense. The functions we have to deal with are invariably locally integrable, and thus belong to the space usually denoted by L^1_{loc}. Such a function, say f, is identified with the measure $f dx$ (see [16], p. 221), and the support of f is by definition that of the associated measure. (If f is continuous this agrees with the definition given in 0.2.)

In particular one obtains in this way a linear isometry of L^1 into M_{bd}.

We shall also consider the spaces L^p_c, composed of functions in L^p with compact supports. L^p_c is regarded as the inductive limit of the Banach spaces

$$L^p_{c,K} = \{f \in L^p : \operatorname{supp} f \subset K\}$$

with the norms induced by that of L^p.

0.5 If X is a Lie group, we sometimes consider the superspace $\mathscr{D}'(X)$ of $M(X)$ composed of Schwartz distributions on X. This is the dual of the space $C^\infty_c = C^\infty_c(X)$ of test functions (indefinitely differentiable and with compact supports) when the latter is regarded as the inductive limit of the Fréchet space $C^\infty_{c,K}$. For more details, see [32], [33], Chapter 1 of [22] and Chapter 5 of [16].

0.6 The left- and right-translation operators τ_a and ρ_a $(a \in X)$ are defined initially for functions by the formulae (cf. [16], p. 248; the right-translation operator R_a there defined differs from ρ_a by the numerical factor $\Delta(a)$):

$$\tau_a f(x) = f(a^{-1}x),$$
$$\rho_a f(x) = \Delta(a) f(xa^{-1}). \tag{0.5}$$

The factor $\Delta(a)$ is a convenience only. The definitions are then consistently extended to measures (or distributions) in the following way:

$$\int kd(\tau_a\mu) = \int \tau_{a^{-1}} k \cdot d\mu = \int k(ax)d\mu(x),$$
$$\int kd(\rho_a\mu) = \int \Delta(a)\rho_{a^{-1}} k \cdot d\mu = \int k(xa)d\mu(x) \tag{0.6}$$

for $k \in C_c$ (or C_c^∞).

It then turns out that

$$||\tau_a f||_{L^p} = ||f||_{L^p}, \quad ||\rho_a f||_{L^p} = \Delta(a)^{1/p'} ||f||_{L^p}, \tag{0.7}$$

where, as usual, p' is defined by $1/p + 1/p' = 1$. In particular,

$$||\tau_a f||_{L^1} = ||f||_{L_1} = ||\rho_a f||_{L^1}; \tag{0.8}$$

and more generally

$$||\tau_a \mu|| = ||\mu|| = ||\rho_a \mu|| \tag{0.9}$$

for any $\mu \in M_{bd}$.

All of the spaces of functions, measures, or distributions thus far introduced are invariant under both left- and right-translations.

0.7 Convolutions are essential in our discussions; for more details, see [16], Section 4.19. If λ and μ are positive measures, the convolution $\lambda * \mu$ is said to exist if and only if the following integrals (known to have a common value)

$$\int d\lambda(x) \int k(xy)d\mu(y), \quad \int d\mu(y) \int k(xy)d\lambda(x),$$
$$\int\int k(xy)d\lambda(x)d\mu(y)$$

are finite for each positive function $k \in C_c(X)$. Then $\lambda * \mu$ is the positive measure defined by setting $\int kd(\lambda * \mu)$ equal to this common value.

If λ and μ are complex measures, $\lambda * \mu$ is said to exist if and only if $|\lambda| * |\mu|$ exists in the preceding sense, in which case we can define $\lambda * \mu$ by the same expression as before (the integrals involved being absolutely convergent).

It is known that $\lambda * \mu$ exists if either of λ or μ has a compact support, or if both are bounded; in the latter case,

$$(0.10) \qquad ||\lambda * \mu|| \leqq ||\lambda|| \cdot ||\mu||.$$

Convolution is associative, provided all but at most one of the factors has a compact support, or provided all are bounded.

If $\lambda * \mu$ is defined, its support lies within $A \cdot B$, where A and B are the supports of λ and μ respectively.

Somewhat similar definitions and remarks apply in the case of distributions; see [32], [33], [22] and [16], Section 5.10. Except in §§ 4, 5 we shall meet distributional convolutions only for the case in which all but at most one of the factors have compact supports.

0.8 The definition of convolution given in 0.7 applies to (suitably restricted) functions; see [16], pp. 259—262. If, for example, λ is the function f (i.e., if $\lambda = fdx$) then formally $\lambda * \mu$ is a function, which we write as $f * \mu$, namely

$$(0.11) \qquad f * \mu(x) = \int f(xy^{-1})\Delta(y)d\mu(y).$$

Similarly

$$(0.12) \qquad \mu * f(x) = \int f(y^{-1}x)d\mu(y).$$

If f and g are functions, then

$$(0.13) \qquad f * g(x) = \int g(y^{-1}x)f(y)dy.$$

For arbitrary measurable f the pointwise interpretation of (0.11) and (0.12) meets with difficulties when μ has a component which is singular relative to Haar measure. One way out of the trouble is to replace f by a Borel measurable function equal l.a.e. to it. An alternative solution is to adopt the interpretation discussed in 0.9 below. Formula (0.13) likewise in general defines $f * g$ only a.e.

From the above formulae it is easy to infer that

$$(0.14) \qquad \begin{aligned} ||\mu * f||_{L^p} &\leqq ||\mu|| \cdot ||f||_{L^p}, \\ ||f * \mu||_{L^p} &\leqq \int \Delta^{1/p'} d|\mu| \cdot ||f||_{L^p}, \\ ||f * g||_{L^\infty} &\leqq ||f||_{L^p} \cdot ||\Delta^{1/p'} g||_{L^{p'}}. \end{aligned}$$

The first two equations here are taken to imply that $\mu * f$ and $f * \mu$ are functions in L^p whenever both factors on the right-hand side are finite.

The smoothing properties of convolution are important. For example, if $f \in C_c$ (or C_c^∞), then $f * \mu$ and $\mu * f \in C$ (or C^∞) whenever μ is a measure (or a distribution).

0.9 The formulae (0.11) and (0.12) suggest abstract definitions of the convolution, namely (cf. [16], p. 568)

$$(0.15) \qquad \lambda * \mu = \int \tau_a \mu \cdot d\lambda(a) = \int \rho_a \lambda \cdot d\mu(a).$$

These formulae do indeed hold if, for example, at least one of λ and μ has a compact support. The functions $a \to \tau_a \mu$ and $a \to \rho_a \lambda$ are regarded as continuous M-valued functions on X. If $\lambda \in M_c$, the existence of $\int \tau_a \mu \cdot d\lambda(a)$ is then evident. In the case of $\int \rho_a \lambda \cdot d\mu(a)$ we note that for any $k \in C_c$, the function $a \to \langle k, \rho_a \lambda \rangle$ is continuous and has a compact support, so that $\int \langle k, \rho_a \lambda \rangle d\mu(a)$ exists.

Formula (0.15) holds in other cases as well, notably if λ and μ are bounded measures. Likewise, if $f \in L^p$ and μ is a bounded measure, then

$$\mu * f = \int \tau_a f d\mu(a),$$

which exists even as a Bochner integral if $p \neq \infty$; and likewise with

$$f * \mu = \int \rho_a f d\mu(a)$$

if $f \in L^p$ and $\Delta^{1/p'}\mu$ is a bounded measure.

These representations in terms of integrals of vector-valued functions are most useful when considering the action of continuous linear operators which commute with translations.

1. Operators with range in the space of continuous functions

One of the crucial features of the cases dealt with in this section is that the range of the operator considered (which is to be continuous, linear, and commuting with translations) shall lie within the space C of all continuous complex-valued functions on X. We begin with a result, typical of numerous similar ones, showing that within wide limits the topology of the range space is not decisive.

1.1 PROPOSITION. *Suppose X is sigma-compact and that T is a linear operator from C_c into C which is continuous for the natural (inductive-limit) topology on C_c and some topology $\mathscr{T}$ on C with the following property:*

If $\{g_n\}$ is a sequence extracted from C such that $g_n \to g \in C$ locally uniformly whilst $g_n \to 0$ for $\mathscr{T}$, then $g = 0$.

Then T is continuous for the natural topology on C_c and the natural topology (of locally uniform convergence) on C.

PROOF. To prove that T is continuous for the natural topologies, it suffices to show that, for each compact set $K \subset X$, the restriction $T|C_{c,K}$ is continuous from the Banach space $C_{c,K}$ into C. X being σ-compact, C is a Fréchet space with its natural topology. Hence it suffices to show that $T|C_{c,K}$ has a graph closed in $C_{c,K} \times C$. But the assumed property of $\mathscr{T}$, combined with the assumed continuity of T with respect to the topology $\mathscr{T}$ on C, ensures that the said graph is closed.

Remark. As examples of suitable topologies $\mathscr{T}$ we indicate that of pointwise convergence; or $\sigma(M, C_c)|C$; or, if X is a Lie group, the distributional topology $\sigma(\mathscr{D}', C_c^\infty)|C$.

Our first representation theorem combines the results for four separate cases. Some of these are known, but others have not to our knowledge been previously formulated. The range functions being continuous, the proofs are more straightforward than those of most representation theorems of this type.

1.2 Theorem. (1) *Suppose that* $T\colon C_c \to C$ *is linear, continuous for the natural topology on* C_c *and the topology of pointwise convergence on* C, *and commutes with the* ρ_a [*resp. the* τ_a]. *Then there exists a* $\mu \in M$ *such that*

$$Tf = \mu * f \text{ [resp. } f * \mu] \tag{1.1}$$

for $f \in C_c$.

(2) *If* $T : C \to C$ *is linear, continuous for the natural topology on the domain space and that of simple* ($=$ *pointwise*) *convergence on the range space, and commutes with the* ρ_a [*resp.* τ_a], *then there exists some* $\mu \in M_c$ *such that* (1.1) *holds for* $f \in C$.

(3) *Suppose that* X *is* σ*-compact. Suppose too that* $T : C_c \to C_c$ *is linear and continuous for the inductive-limit topology on the domain space and for some topology* $\mathscr{T}$ *on the range space having the property that, if a sequence* $\{f_n\}$ *extracted from* C_c *converges to* $g \in C_c$ *for the inductive limit topology and to* 0 *for* $\mathscr{T}$, *then* $g = 0$. *Suppose finally that* T *commutes with the* ρ_a [*resp.* τ_a]. *Then there exists some* $\mu \in M_c$ *such that* (1.1) *holds for* $f \in C_c$.

(4) *If* $T : C_0 \to C_0$ *is linear and continuous* (*for the usual normed topology*) *and commutes with the* ρ_a [*resp.* τ_a], *then there exists a measure* μ *such that* $\mu \in M_{bd}$ [*resp.* $\Delta \cdot \mu \in M_{bd}$] *for which* (1.1) *holds for* $f \in C_0$.

Proof. (1) Linearity and continuity of T show that, if e denotes the neutral element of X, then $f \to Tf(e)$ is a Radon measure on X. Denote this measure by λ. If T commutes with the ρ_a one has

$$\begin{aligned} Tf(a) &= \Delta(a)\rho_{a^{-1}}Tf(e) = \Delta(a)T\rho_{a^{-1}}f(e) \\ &= \Delta(a)\int \rho_{a^{-1}}f \cdot d\lambda = \Delta(a)\int \Delta(a^{-1})f(xa)d\lambda(x) \\ &= \int f(xa)d\lambda(x) = \int f(x^{-1}a)d\check{\lambda}(x) \\ &= \check{\lambda} * f(a) = \mu * f(a), \end{aligned}$$

where $\mu = \check{\lambda} \in M$ is the measure defined by

$$\int k d\mu = \int k(x^{-1}) d\lambda(x)$$

for $k \in C_c$. Likewise, if T commutes with the τ_a,

$$\begin{aligned} Tf(a)\, \tau_{a^{-1}} Tf(e) = T\tau_{a^{-1}} f(e) &= \int \tau_{a^{-1}} f \cdot d\lambda \\ &= \int f(ax) d\lambda(x) = \int f(ax^{-1}) d\check{\lambda}(x) \\ &= \int f(ax^{-1}) \Delta(x) \cdot d\check{\lambda}(x)/\Delta(x) \\ &= f * \mu(a), \end{aligned}$$

where $\mu = \Delta^{-1} \cdot \check{\lambda} \in M$. Thus (1.1) is established.

(2) The topology on C inducing on C_c a topology weaker than the inductive-limit topology, (1) may be applied to $T|C_c$ to conclude that there exists a measure $\mu \in M$ such that (1.1) holds for $f \in C_c$. Besides this, the assumed continuity of T signifies that there exists a compact set $K \subset X$ and a number $k \geqq 0$ such that

$$|Tf(e)| \leqq k \cdot \mathrm{Sup}_K |f| \qquad (f \in C).$$

Combined with (1.1) this reads

$$\left| \int f(x^{-1}) d\mu(x) \right| \leqq k \cdot \mathrm{Sup}_K |f|$$
$$\left[\text{resp. } \left| \int f(x^{-1}) \Delta(x) d\mu(x) \right| \leqq k \cdot \mathrm{Sup}_K |f| \right]$$

for $f \in C_c$, which relation implies immediately that the support of μ is contained in the compact set K^{-1}. Thus $\mu \in M_c$. Accordingly, the mapping T' defined by $T'f = \mu * f$ [resp. $f * \mu$] is defined and continuous on C with values in C. Since T and T' coincide on C_c, and since C_c is dense in C, therefore T and T' coincide on the whole of C. In other words, (1.1) holds for all $f \in C$.

(3) One begins by showing, much as in Proposition 1.1, that T is indeed continuous for the natural topology on the range space. Then (1) applies and shows that (1.1) holds from some $\mu \in M$. It remains to show that, since $\mu * f$ [resp. $f * \mu$] belongs to C_c (and not merely to C) for each $f \in C_c$, μ must have a compact support.

To this end, fix some compact neighbourhood N of e in X, and consider the set

$$B = \{f \in C_c : \mathrm{supp}\, f \subset N,\ |f| \leqq 1\}.$$

B is a bounded subset of C_c. Hence $B' = T(B)$ is also a bounded subset of C_c. We will show that this entails the existence of a compact set $K \subset X$ such all functions in B' have their supports contained in K. Supposing this to have been established, (1.1) shows then that

$$\text{supp}\,(\mu * f) \subset K$$
$$[\text{resp. supp}\,(f * \mu) \subset K]$$

for every $f \in C_c$ with support inside N. One can then allow f to vary over a suitable directed family $\{f_i\}$ of functions in C_c with supports inside N and such that $\mu * f_i \to \mu$ [resp. $f_i * \mu \to \mu$] vaguely, so that the support of μ itself will be seen to lie within K.

To prove the existence of K, we argue by contradiction. If no such compact set K existed, there would exist (since X is σ-compact) a sequence $\{x_n\}$ extracted from X such that

(i) each compact subset of X contains x_n for at most a finite number of n,

(ii) for each n there exists $f_n \in B'$ such that

$$|f_n(x_n)| \neq 0.$$

Define, for $f \in C_c$,

$$p(f) = \text{Sup}_n\, |nf(x_n)/f_n(x_n)|.$$

By (i), $p(f) < +\infty$ for each $f \in C_c$. p is evidently a seminorm on C_c, and it is lower semicontinuous since for each n the mapping $f \to f(x_n)$ is a continuous linear functional on C_c. Since C_c is barrelled ([16], pp. 427—430), p is continuous ([16], p. 463). B' being bounded in C_c, $p(B')$ is bounded. But this is absurd since $f_n \in B'$ and

$$p(f_n) \geqq |nf_n(x_n)/f_n(x_n)| = n.$$

(4) In this case part (1) is directly applicable to show that a measure μ exists such that (1.1) is valid for $f \in C_c \subset C_0$. In addition, by virtue of the continuity of T from C_0 into itself, there is a number $k \geqq 0$ such that

$$|Tf(e)| \leqq k||f||_\infty$$

for $f \in C_0$. In conjunction with (1.1) this reads

$$\left|\int f(x^{-1})d\mu(x)\right| \leqq k\,||f||_\infty$$
$$\left[\text{resp.} \left|\int f(x^{-1})\varDelta(x)d\mu(x)\right| \leqq k\,||f||_\infty\right]$$

for $f \in C_c$. This entails that μ [resp. $\varDelta \cdot \mu$] is a bounded measure. From this point the argument proceeds to its conclusion as does the proof of (2).

1.3 Remarks. (1) In certain cases the assumption that X be σ-compact may be dropped from part (3) of Theorem 1.2.

(a) If the topology $\mathscr{T}$ involved is stronger than that induced on C_c by the vague topology $\sigma(M, C_c)$, then the said hypothesis may be dropped. This follows from Lemmas 2 and 3 of Edwards [10].

(b) The hypothesis may again be suppressed whenever T is assumed to be positive.

For in this case $\mu \geqq 0$ and has the property that $\mu * f$ [resp. $f * \mu$] lies in C_c whenever $f \in C_c$. This entails that μ has a compact support, as the following argument shows.

Suppose μ is a positive measure with support S. Then, if $s \in S$ and if $f \in C_c^+$ and $f(s) \neq 0$, it is clear that $\int f d\mu > 0$. Now

$$\mu * f(x) = \int f(y^{-1}x) d\mu(y) \ [f * \mu(x) = \int f(xy^{-1}) \Delta(y) d\mu(y)].$$

If $f \in C_c^+$ and $f(e) > 0$, and if $x \in S$, the function

$$y \to f(y^{-1}x) \ [\text{resp. } f(xy^{-1})\Delta(y)]$$

belongs to C_c^+ and takes for $y = x$ a non-zero value. By the opening remark above, therefore, $\mu * f(x) > 0$ [resp. $f * \mu(x) > 0$]. This shows that

$$S = \operatorname{supp} \mu \subset \operatorname{supp} (\mu * f) \cap \operatorname{supp} (f * \mu).$$

So S is compact whenever $\mu * f$ (or $f * \mu$) has a compact support.

(2) In each of the cases mentioned in Theorem 1.2, the converse statement is true (and rather trivial).

(3) Case (4) of Theorem 1.2 is discussed by Kelley [26].

(4) Helgason [20] considers continuous endomorphisms T of the space of uniformly almost periodic functions on an Abelian topological group G which commute with translations. To some extent this case is reducible to that dealt with in Theorem 1.2 by introducing the Bohr compactification X of G.

Linear maps into C, commuting with translations, of other spaces can be discussed in a similar fashion. We content ourselves with one theorem of this sort.

1.4 Theorem. *If T is a continuous linear map of L^p into C (the weak topology $\sigma(L^\infty, L^1)$ being involved if $p = \infty$) which commutes with the ρ_a [resp. τ_a], then* (1.1) *holds for some function μ such that $\Delta^{-1/p}\mu \in L^{p'}$ [resp. $\Delta^{1/p'}\mu \in L^{p'}$], where $1/p + 1/p' = 1$. Consequently $\Delta^{-1/p}Tf$ [resp. Tf] is bounded and continuous for each $f \in L^p$.*

Proof. Since $C_c \subset L^p$, the natural topology of C_c being stronger than the induced on it from L^p, case (1) of Theorem 1.2 applies to show that a measure μ exists so that (1.1) holds for $f \in C_c$. The continuity of T for the topology induced by that of L^p entails that μ is absolutely continuous with respect to Haar measure, *i.e.*, is a function. Moreover, since

$$|Tf(e)| \leqq k \cdot ||f||_p,$$

(1.1) yields

$$\left|\int f(x^{-1})\mu(x)dx\right| \leqq k \cdot ||f||_p$$
$$\left[\text{resp. } \left|\int f(x^{-1})\Delta(x)\mu(x)dx\right| \leqq k \cdot ||f||_p\right]$$

for $f \in C_c$. A little manipulation, combined with the converse of Hölder's inequality, leads from these inequalities to the stated properties of μ. The final statement of the theorem stems from Hölder's inequality itself.

1.5 By dualising, the preceding results yield a representation theorem for continuous linear maps T with values in various spaces of measures and which commute with the ρ_a [resp. τ_a]. Conversely, these latter results can be established *ab initio* and then used as a basis for proving results like Theorem 1.2.

For example, suppose that T is linear, maps M_c into M, is continuous for the topologies $\sigma(M_c, C)$ and $\sigma(M, C_c)$, and commutes with the ρ_a [resp. τ_a]. Now if $f \in M_c$ one can write

$$f = \varepsilon * f = \int (\rho_x \varepsilon) df(x)$$
$$\left[\text{resp. } f = f * \varepsilon = \int (\tau_x \varepsilon) df(x)\right]$$

The stated continuity of T then ensures (cf. [16], pp. 562, 571) that it may be applied "under the integral sign", leading to

$$Tf = \int (T\rho_x \varepsilon) df(x) = \int (\rho_x T\varepsilon) df(x) = \mu * f$$
$$\left[\text{resp. } Tf = \int (T\tau_x \varepsilon) df(x) = \int (\tau_x T\varepsilon) df(x) = f * \mu\right],$$

where $\mu = T\varepsilon \in M$. Thus formula (1.1) is again true.

Besides this, if S is a continuous linear map of C_c into C which commutes with the ρ_a [resp. τ_a], its adjoint T maps M_c into M, commutes with the ρ_a [resp. τ_a], and is continuous for the weak topologies mentioned above. T is represented by (1.1), as we have just a verified. It is a simple computation to deduce that S is accordingly represented by (1.1) with $\check{\mu}$ in place of μ. Thus part (1) of Theorem 1.2 is recovered.

1.6 OPERATORS INTO L^∞. We begin with an observation due to D. A. Edwards [6]; see also some similar remarks in § 5 of R.E. Edwards [14]:

Let F be a vector subspace of M, stable under the ρ_a [resp. τ_a], and endowed with some topology. If $T : F \to L^\infty$ is continuous linear and commutes with ρ_a [resp. τ_a], and if $f \in F$ is such that $\lim_{a\to e} \rho_a f = f$ [resp. $\lim_{a\to e} \tau_a f = f$], then Tf belongs to C_{rbu} [resp. C_{lbu}] (the space of bounded right [resp. left] uniformly continuous complex-valued functions on X).

This applies in particular if $F = L^p$ with $p < \infty$ (see Theorem 1.4).

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Again, if $T : L^\infty \to L^\infty$, then $T(C_{rbu}) \subset C_{rbu}$ and $T(C_{lbu}) \subset C_{lbu}$. So, by Theorem 1.2 (1), there exists a measure μ such that

$$Tf = \mu * f \ [\text{resp. } f * \mu] \tag{1.1}$$

for $f \in C_c$. As in the proof of Theorem 1.2 (4), it appears that μ [resp. $\Delta \cdot \mu$] $\in M_{bd}$ and that (1.1) continues to hold for $f \in C_0$.

If also T is continuous for $\sigma(L^\infty, L^1)$ we may conclude that (1.1) holds for all $f \in L^\infty$. (To see this it suffices to approximate f in $\sigma(L^\infty, L^1)$ by functions $(\phi_N * f)u_i$, where the ϕ_N are as in Proposition 2.1 and the $u_i \in C_c^+$, $0 \leqq u_i \leqq 1$, and $u_i \to 1$ locally uniformly on X.)

2. Operators from L_c^1 and L^1

It is convenient to begin with two general propositions which will form the basis of several later arguments.

2.1 Proposition. *Let $\{N\}$ be a base of relatively compact neighbourhoods of e in X (which can all be assumed to lie within some fixed compact neighbourhood N_0 of e). For each N choose $\phi_N \in C_c$ such that $\phi_N \geqq 0$, supp $\phi_N \subset N$, $\int \phi_N(x)dx = 1$. Let G be a right [resp. left] translation-invariant vector subspace of M, and let T be a continuous linear map of C_c into G which commutes with the ρ_a [resp. τ_a]. Define $\mu_N = T\phi_N \in G \subset M$. Then*

(i) for each $f \in C_c$ the elements $\mu_N * f$ [resp. $f * \mu_N$] remain bounded in G (as N varies);

(ii) for each $f \in C_c$,

$$Tf = \lim \mu_N * f \ [\textit{resp.} \lim f * \mu_N] \tag{2.1}$$

in the sense of G. (The limits are taken with respect to the directed set $\{N\}$, partially ordered by set-inclusion reversed.)

Proof. If $f \in C_c$, the $\phi_N * f$ [resp. $f * \phi_N$] remain bounded in C_c as N varies. This is so because these functions are uniformly bounded and vanish outside a fixed compact subset of G; cf. [16], p. 432 (case $m = 0$). Moreover

$$\phi_N * f = \int (\rho_x \phi_N) f(x)dx \ \left[\text{resp. } f * \phi_N = \int (\tau_x \phi_N) f(x)dx\right],$$

the integrands being regarded as functions with values in C_c. The continuity of T, combined with its commutation properties, shows that

$$T(\phi_N * f) = \int (T\rho_x \phi_N) f(x)dx = \int (\rho_x T\phi_N) f(x)dx = \mu_N * f$$
$$\left[\text{resp. } T(f * \phi_N) = \int (T\tau_x \phi_N) f(x)dx = \int (\tau_x T\phi_N) f(x)dx = f * \mu_N\right];$$

observe that $\rho_x \mu_N$ [resp. $\tau_x \mu_N$] is a continuous M-valued function of x, so that the integrals

$$\int (\rho_x \mu_N) f(x) dx \left[\text{resp.} \int (\tau_x \mu_N) f(x) dx\right]$$

exist as elements of M ([16], p. 577). Thus

$$T(\phi_N * f) = \mu_N * f \ [\text{resp. } T(f * \phi_N) = f * \mu_N] \tag{2.2}$$

for each N and each $f \in C_c$.

From (2.2) and the boundedness of the $\phi_N * f$ [resp. $f * \phi_N$], (i) follows on account of the continuity of T.

Moreover since

$$\lim \phi_N * f = f \ [\text{resp.} \lim f * \phi_N = f]$$

in C_c, (2.1) follows from (2.2) and the continuity of T.

2.2 If X is a Lie group, one can replace M by $\mathscr{D}'$ and suppose furthermore that each $\phi_N \in C_c^\infty$.

2.3 PROPOSITION. *The assumptions and notations are as in Proposition* 2.1. *In addition assume that T is initially defined on some vector subspace F of M which contains C_c, which is separated locally convex and invariant under both right and left translations, and which fulfills the following conditions*:

(*a*) *The injection $C_c \to F$ is continuous and $x \to \tau_x f$ and $x \to \rho_x f$ are continuous from X into F for each $f \in F$.*

(*b*) *In F, the closed convex envelope of a compact set is compact* (*which is true whenever F is quasicomplete*; *see* [16], *p.* 651).

(*c*) *There exists a neighbourhood N_1 of e in X with the property that, given any neighbourhood V of 0 in F, there exists a neighbourhood V' of 0 in F such that the set*

$$\{\tau_x f : x \in N_1, f \in V'\} \ [\text{resp. } \{\rho_x f : x \in N_1, f \in V'\}]$$

is contained in V.

Let G be as in Proposition 2.1.

Then, if T is linear and continuous from F into G, and if T commutes with the ρ_a [*resp. τ_a*], *the set*

$$\{\mu_N * f : N \subset N_1, f \in B \cap C_c\}$$
$$[\textit{resp. } \{f * \mu_N : N \subset N_1, f \in B \cap C_c\}]$$

is bounded in G whenever B is a bounded subset of F.

PROOF. The second clause of (a) combines with (b) to show ([16], p. 561) that

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$$\phi_N * f = \int (\tau_x f)\phi_N(x)dx = \int (\rho_x \phi_N) f(x) dx$$
$$\left[\text{resp. } f * \phi_N = \int (\rho_x f)\phi_N(x)dx = \int (\tau_x \phi_N) f(x)dx\right]$$

for each N and each $f \in C_c$, the integrals being now taken in the sense of F. In particular, (2.2) remains true.

The first of these integral representations of $\phi_N * f$ [resp. $f * \phi_N$] shows, when combined with (c) and the bipolar theorem ([16], p. 503), that $\phi_N * f$ [resp. $f * \phi_N$] falls into the closed convex envelope in F of V, provided $N \subset N_1$ and $f \in V'$. As a consequence, the family of maps

$$f \to \phi_N * f \ \ [\text{resp. } f \to f * \phi_N],$$

obtained when N varies inside N_1, is equicontinuous from F into itself. Accordingly, the family of maps

$$f \to T(\phi_N * f) \ \ [\text{resp. } f \to T(f * \phi_N)],$$

obtained when N varies within N_1, is equicontinuous from F into G.

This signifies that, if W is a given neighbourhood of 0 in G, the set

$$A = \{f \in F : T(\phi_N * f) \in W \ \text{ for all } N \subset N_1\}$$
$$[\text{resp. } \{f \in F : T(f * \phi_N) \in W \ \text{ for all } N \subset N_1\}]$$

is a neighbourhood of 0 in F and therefore absorbs any preassigned bounded subset B of F. By virtue of (2.2), it follows that the set S of $\mu_N * f$ [resp. $f * \mu_N$], obtained when N varies within N_1 and f varies over $B \cap C_c$, is absorbed by W. This being true for any W, the set S is bounded in G.

2.4 The hypotheses (a)—(c) of Proposition 2.3 are satisfied if $F = L^p$ $(1 \leqq p < \infty)$ since $\|\tau_x f\|_{L^p} = \|f\|_{L^p}$ and $\|\rho_x f\|_{L^p} = \Delta(x)^{1/p'}\|f\|_{L^p}$. They are also satisfied if $f = C_{bd}(X)$, or $C_0(X)$, or $M_{bd}(X)$.

2.5 Theorem. *Let T be a continuous linear map of L^1_c into L^p $(1 \leqq p \leqq \infty)$ which commutes with the ρ_a [resp. τ_a]. Then there exists μ such that $\mu \in L^p$ if $1 < p \leqq \infty$, and $\mu \in M_{bd}$ if $p = 1$, for which*

$$Tf = \mu * f \ \ [\textit{resp. } f * \mu] \tag{1.1}$$

for $f \in L^1_c$.

Proof. We use Proposition 2.1, applied to $T|C_c$ and $G = L^p$. Notice that now the ϕ_N remain bounded in L^1_c, so that the μ_N are bounded in L^p.

If $1 < p \leqq \infty$, the bounded subsets of L^p are relatively compact for $\sigma(L^p, L^{p'})$, so that the directed family $\{\mu_N\}$ has a limiting point $\mu \in L^p$ for this topology. But then, if $f \in C_c$, $\mu * f$ [resp. $f * \mu$] is a limiting point (for the topology of locally uniform convergence) of the family $\{\mu_N * f\}$ [resp. $\{f * \mu_N\}$]. In view of (2.1) it follows that (1.1) holds for $f \in C_c$.

If $p = 1$, $\{\mu_N\}$ will have a limiting point $\mu \in M_{bd}$ for the topology $\sigma(M_{bd}, C_0)$, and a similar argument shows that again (1.1) holds for $f \in C_c$.

On the other hand if $\mu \in L^p$ then

$$(2.3)\quad \begin{cases} \|\mu * f\|_{L^p} = \left\| \int (\rho_x \mu) f(x) dx \right\|_{L^p} \leqq \|\mu\|_{L^p} \int \Delta(x)^{1/p'} |f(x)| dx \\ \text{and} \\ \|f * \mu\|_{L^p} = \left\| \int (\tau_x \mu) f(x) dx \right\|_{L^p} \leqq \|\mu\|_{L^p} \int |f(x)| dx. \end{cases}$$

These formulae show that the mapping $f \to \mu * f$ [resp. $f * \mu$] is continuous from L^1_c into L^p. Since C_c is dense in L^1_c, continuity shows that (1.1) continues to hold for all $f \in L^1_c$.

A similar argument applies if $p = 1$ and $\mu \in M_{bd}$.

2.6.1 Corollary. Suppose that T is a continuous linear map of L^1_c into itself which commutes with the ρ_a [resp. τ_a]. Then there exists a measure $\mu \in M_c$ such that

$$(1.1)\qquad Tf = \mu * f \text{ [resp. } f * \mu]$$

for $f \in L^1_c$.

Proof. The case $p = 1$ of the preceding theorem shows that (1.1) must hold with some $\mu \in M_{bd}$. Also, if $f \in C_c$, $\mu * f$ [resp. $f * \mu$] is continuous. Since it belongs to L^1_c, it must belong to C_c. We now appeal to Theorem 1.2 (3) and Remark 1.3 (1) in order to conclude that μ has a compact support.

2.6.2 Corollary. (i) *Any continuous linear map T of L^1 into L^p $(1 \leqq p \leqq \infty)$ which commutes with the τ_a has the form*

$$Tf = f * \mu,$$

where $\mu \in L^p$ if $1 < p \leqq \infty$ and $\mu \in M_{bd}$ if $p = 1$; and conversely.

(*ii*) *If T is a continuous linear map of L^1 into L^p $(1 \leqq p \leqq \infty)$ which commutes with the ρ_a, and if either $p = 1$ or X is unimodular then*

$$Tf = \mu * f,$$

where $\mu \in L^p$ if $1 < p \leqq \infty$ and $\mu \in M_{bd}$ if $p = 1$; and conversely.

Proof. It suffices to apply Theorem 2.5, noting that under the stated hypothesis it follows from (2.3) that the map $f \to f * \mu$ [resp. $\mu * f$] is continuous from L^1 into L^p, whilst in all cases L^1_c is dense in L^1.

2.7 Remarks. (1) The Abelian case of Corollary 2.6.2 has been discussed in Edwards [7]; for $p = 1$ see also Wendel [36] and [37]. For a case involving certain subalgebras of L^1 when X is Abelian, see [27].

(2) The hypotheses in part (ii) of Corollary 2.6.2 are essential. In fact, if X is not unimodular and $p > 1$, there exists no non-trivial continuous linear map T of L^1 into L^p which commutes with the ρ_a. For suppose T were non-trivial and that $f \in L^1$ is chosen so that $g = Tf \neq 0$. Then

$$\begin{aligned}||\rho_x g||_{L^p} = ||\rho_x Tf||_{L^p} = ||T\rho_x f||_{L^p} &\leqq ||T||\,||\rho_x f||_{L^1} \\ &= ||T||\,||f||_{L^1}.\end{aligned}$$

But $||\rho_x g||_{L^p} = \Delta(x)^{1/p'}||g||_{L^p}$. It would follow that $\Delta^{1/p'}$ is bounded and so, since $p > 1$, that Δ itself is bounded. Since Δ is a character, it would be identically 1 and X would be unimodular, contrary to hypothesis.

2.8 Maps of L^1 into quotient spaces of $(L^1)^s$.

Here s is a natural number and $(L^1)^s$ is the product of s copies of L^1, normed in the usual way, *i.e.*, if $\boldsymbol{f} = (f_1, \cdots, f_s)$ is an element of $(L^1)^s$, then $||\boldsymbol{f}||$ is $\sum_{k=1}^s ||f_k||_{L^1}$.

We shall consider quotients $(L^1)^s/J$, where J is a closed vector subspace of $(L^1)^s$ which is stable under the ρ_a [resp. τ_a], it being understood that for $\boldsymbol{f} = (f_1, \cdots, f_s)$ one defines

$$\rho_a \boldsymbol{f} = (\rho_a f_1, \cdots, \rho_a f_s),$$

and analogously for $\tau_a \boldsymbol{f}$. It is equivalent to say that J is a closed vector subspace of $(L^1)^s$ which is stable under right [resp. left] convolution

$$\begin{gathered}\boldsymbol{f} \to \boldsymbol{f} * g = (f_1 * g, \cdots, f_s * g) \\ [\text{resp. } \boldsymbol{f} \to g * \boldsymbol{f} = (g * f_1, \cdots, g * f_s)].\end{gathered}$$

Our aim is to seek a representation formula, like (1.1), for continuous linear maps T of L^1 into $(L^1)^s/J$ which commute with the ρ_a [resp. τ_a] (or, equivalently, commute with right [resp. left] convolutions with elements of L^1). A natural conjecture is that any such T has an expression

$$Tf = (\boldsymbol{\mu} * f)_J \ [\text{resp. } (f * \boldsymbol{\mu})_J], \tag{2.4}$$

where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_s) \in (M_{bd})^s$,

$$\begin{aligned}\boldsymbol{\mu} * f &= (\mu_1 * f, \cdots, \mu_s * f), \\ f * \boldsymbol{\mu} &= (f * \mu_1, \cdots, f * \mu_s),\end{aligned}$$

and where $\boldsymbol{h}_J$ denotes the coset modulo J of $\boldsymbol{h} \in (L^1)^s$.

It will be shown that this conjecture proves to be correct, provided J fulfils an extra condition which amounts to assuming that it is closed in a stronger sense than that specified by the normed topology of $(L^1)^s$. This extra condition is satisfied in the applications we make in Part II. It is expressed as follows.

2.8.1 Definition. A closed vector subspace J of $(L^1)^s$ is said to satisfy condition (C) if, whenever $\{\boldsymbol{j}^{(\alpha)}\} = \{(j_1^{(\alpha)}, \cdots, j_s^{(\alpha)})\}$ is a norm-bounded directed family of elements of J such that

$$\lim_\alpha j_k^{(\alpha)} = j_k \qquad (k = 1, 2, \cdots, s)$$

for the topology $\sigma(L^1, C_0)$, then $\boldsymbol{j} = (j_1, \cdots, j_k)$ necessarily belongs to J.

2.8.2 In this subsection we collect a few statements concerning condition (C) which will prove useful later.

(a) *If X is compact, and if J is a closed vector subspace of $(L^1)^s$ which is stable under the ρ_a [resp. τ_a]* (i.e., *which is such that $J * L^1 \subset J$ [resp. $L^1 * J \subset J$]*), *then J satisfies condition (C). In fact, J is closed for the product of the topologies $\sigma(L^1, C_0)$.*

Proof. We may identify the dual of $(L^1)^s$ with $(L^\infty)^s$ in such a way that

$$\langle \boldsymbol{f}, \boldsymbol{h} \rangle = \textstyle\sum_{k=1}^s f_k * h_k(e)$$
$$[\text{resp. } \langle \boldsymbol{f}, \boldsymbol{h} \rangle = \textstyle\sum_{k=1}^s h_k * f_k(e)],$$

where $\boldsymbol{h} = (h_1, \cdots, h_s) \in (L^\infty)^s$. Suppose this identification to be made, and take any $\boldsymbol{h}$ in the annihilator of J. For any $g \in L^1$ we shall then have (by the assumed stability of J)

$$\textstyle\sum_{k=1}^s f_k^\alpha * g * h_k(e) = 0$$
$$[\text{resp. } \textstyle\sum_{k=1}^s h_k * g * f_k^\alpha(e) = 0]$$

whenever $\{\boldsymbol{f}^\alpha\}$ is a directed family extracted from J. Now $g * h_k$ [resp. $h_k * g] \in C$, which is here identical with C_0. Accordingly, if for each $k = 1, 2, \cdots, s$ one has $\lim_\alpha f_k^\alpha = f_k$ in the sense of the topology $\sigma(L^1, C_0)$, it follows that

$$\textstyle\sum_{k=1}^s f_k * g * h_k(e) = 0$$
$$[\text{resp. } \textstyle\sum_{k=1}^s h_k * g * f_k(e) = 0].$$

Since this is true for each $g \in L^1$, and since g can be made to vary so that $f_k * g \to f_k$ [resp. $g * f_k \to f_k$] in L^1 for each k, it follows that $\boldsymbol{f}$ belongs to the annihilator of the J. J being a closed vector subspace of $(L^1)^s$, the Hahn-Banach theorem ([16], Corollary 2.2.5) entails that $\boldsymbol{f} \in J$, which is what we had to show.

(a′) *The statement in (a) is false if X is non-compact and unimodular.*

Proof. Define J by the formula

$$J = \left\{\boldsymbol{f} \in (L^1)^s : \int f^1 dx = 0\right\}.$$

It is evident that J is a closed vector subspace of $(L^1)^s$ which is stable under both the ρ_a and the τ_a. We will show that, X being non-compact, J does not satisfy condition (C). Choose any $f_0 \in L^1$ with a compact support K and such that $\int f_0 dx = 1$. For each $a \in X$ define

$$\boldsymbol{f}_a = (f_0 - \tau_a f_0, 0, \cdots, 0).$$

Then it is evident that $\boldsymbol{f}_a \in J$ and that $\boldsymbol{f}_a$ has norm in $(L^1)^s$ at most 2. We will in a moment verify that $\tau_a f_0 \to 0$ for $\sigma(L^1, C_0)$ as $a \to \infty$. This, combined with the fact that $(f_0, 0, \cdots, 0)$ does not belong to J, will make it clear that J does not satisfy condition (C).

To verify the outstanding point, take any $u \in C_0$. Then

$$\begin{aligned}\left|\int (\tau_a f_0) u dx\right| &= \left|\int f_0(a^{-1}x) u(x) dx\right| \\ &= \left|\int f_0(y) u(ay) dy\right| \\ &= \left|\int_K f_0(y) u(ay) dy\right| \\ &\leqq \int |f_0| dx \cdot \mathrm{Sup}_{x \in aK} |u(x)|.\end{aligned}$$

Since K is compact and $u \in C_0$, this expression tends to 0 as $a \to \infty$. Indeed, for any $\varepsilon > 0$, the set $\{x \in X : |u(x)| \geqq \varepsilon\} = K_\varepsilon$ is compact. If a lies outside the compact set $K_\varepsilon K^{-1}$, therefore,

$$\mathrm{Sup}_{x \in aK} |u(x)| \leqq \varepsilon,$$

whence the desired conclusion.

(b) *The intersection of any collection of subsets of* $(L^1)^s$, *each satisfying condition* (C), *itself satisfies condition* (C).

PROOF. This is obvious.

(c) *Suppose that* $\lambda_k \in M_{bd}$ *for* $k = 1, 2, \cdots s$. *Then*

$$J = \{\boldsymbol{f} \in (L^1)^s : \textstyle\sum_{k=1}^s \lambda_k * f_k = 0\}$$
$$[\text{resp. } J = \{\boldsymbol{f} \in (L^1)^s : \textstyle\sum_{k=1}^s f_k * \lambda_k = 0\}]$$

is stable under the ρ_a [*resp.* τ_a] *and is closed for the product of the topologies* $\sigma(L^1, C_0)$, *hence satisfies condition* (C).

PROOF. Stability is clear, as also is the fact that J is a closed vector subspace of $(L^1)^s$.

Suppose that $\{\boldsymbol{j}^\alpha\}$ is a directed family extracted from J such that $\lim_\alpha j_k^\alpha = j_k$ exists in the sense of $\sigma(L^1, C_0)$ for each k. For each α,

$$\textstyle\sum_{k=1}^s \lambda_k * j_k^\alpha = 0$$
$$[\text{resp. } \textstyle\sum_{k=1}^s j_k^\alpha * \lambda_k = 0].$$

Reference to the lemma immediately below shows that the same is true when, for each k, j_k^α is replaced by j_k, so that $\boldsymbol{j} \in J$.

2.8.3 LEMMA. *Suppose that* μ_α, μ *and* λ *belong to* M_{bd}, α *ranging over some directed set, and that*

$$\lim_\alpha \mu_\alpha = \mu$$

for $\sigma(M_{bd}, C_0)$. *Then*

$$\lim_\alpha \mu_\alpha * \lambda = \mu * \lambda \;\; [\textit{resp.} \lim_\alpha \lambda * \mu_\alpha = \lambda * \mu]$$

in the same sense.

PROOF. If $u \in C_0$, the Fubini-Tonelli theorem ([16], pp. 244—245) shows that

$$\int ud(\mu_\alpha * \lambda) = \int v d\mu_\alpha$$

$$\left[\text{resp.} \int ud(\lambda * \mu_\alpha) = \int w d\mu_\alpha\right]$$

where

$$v(x) = \int u(xy)d\lambda(y)$$

$$\left[\text{resp.}\ w(x) = \int u(yx)d\lambda(y)\right].$$

It therefore suffices to verify that v [resp. w] belongs to C_0. Now, given any $\varepsilon > 0$, there exists a compact set $K \subset X$ such that $|\lambda|(X\backslash K) \leqq \varepsilon$. Then, uniformly with respect to $x \in X$,

$$\left|v(x) - \int_K u(xy)d\lambda(y)\right| \leqq ||u||_\infty \cdot \varepsilon$$

$$\left[\text{resp.} \left|w(x) - \int_K u(yx)d\lambda(y)\right| \leqq ||u||_\infty \cdot \varepsilon\right].$$

So it suffices to verify that

$$\int_K u(xy)d\lambda(y) \;\; \left[\text{resp.} \int_K u(yx)d\lambda(y)\right]$$

belongs to C_0. However, there exists a compact set $K' \subset X$ such that $|u(z)| \leqq \varepsilon$ for $z \in X\backslash K'$. Then, if x lies outside $K'K^{-1}$ [resp. $K^{-1}K'$],

$$\left|\int_K u(xy)d\lambda(y)\right| \leqq \varepsilon \cdot \int_K d|\lambda|$$

$$\left[\text{resp.} \left|\int_K u(yx)d\lambda(y)\right| \leqq \varepsilon \cdot \int_K d|\lambda|\right],$$

which, since $K'K^{-1}$ [resp. $K^{-1}K'$] is compact, establishes what is required.

After these preliminaries we pass to the statement and proof of the representation theorem.

2.9 THEOREM. *Suppose that* J *satisfies condition* (C) *and is stable under the* ρ_a [*resp.* τ_a]. *Let* T *be a continuous linear map of* L^1 *into* $(L^1)^s/J$ *which commutes with the* ρ_a [*resp.* τ_a]. *Then* T *is expressible in the form* (2.4) *for some* $\boldsymbol{\mu} \in (M_{bd})^s$.

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PROOF. This will utilise the system of functions ϕ_N introduced in Proposition 2.1 Put

$$T\phi_N = (\boldsymbol{f}^N)_J \in (L^1)^s/J.$$

Then $\lim_N \phi_N * g = g = \lim_N g * \phi_N$ in L^1 for each $g \in L^1$. Since T commutes with right [resp. left] convolutions by elements of L^1, it follows that

$$Tg = \lim_N (\boldsymbol{f}^N)_J * g \quad [\text{resp.} \lim_N g * (\boldsymbol{f}^N)_J] \tag{2.5}$$

in $(L^1)^s/J$. Here the family $\{\boldsymbol{f}^N_J\}$ is bounded in $(L^1)^s/J$, T being continuous and the family $\{\phi_N\}$ being bounded in L^1. So, by adding suitable elements of J to the $\boldsymbol{f}^N$, we can arrange that the latter are bounded in $(L^1)^s$. Then, for each k, the f^N_k are bounded in L^1. Regarding L^1 as imbedded in M_{bd}, it follows that by passage to a cofinal subfamily it can be arranged (cf. [16], p. 205) that

$$\lim_N f^N_k = \mu_k$$

exists in the sense of the topology $\sigma(M_{bd}, C_0)$ for each $k = 1, 2, \cdots, s$. Lemma 2.8.3 shows then that

$$\lim_N f^N_k * g = \mu_k * g \quad [\text{resp.} \lim_N g * f^N_k = g * \mu_K] \tag{2.6}$$

for $\sigma(L^1, C_0)$.

On the other hand, from (2.5), we may choose elements $\boldsymbol{j}^N$ of J such that, on putting

$$Tg = \boldsymbol{g}_J = (g_1, \cdots, g_s)_J,$$

one has

$$\begin{gathered} \lim_N \|g_k - f^N_k * g + j^N_k\|_{L^1} = 0 \\ [\text{resp.} \lim_N \|g_k - g * f^N_k + j^N_k\|_{L^1} = 0]. \end{gathered} \tag{2.7}$$

From (2.6) and (2.7) it appears that the j^N_k remain bounded in L^1 and that

$$\lim_N j^N_k = \mu_k * g - g_k \quad [\text{resp.}\ g * \mu_k - g_k] \tag{2.8}$$

for $\sigma(L^1, C_0)$. Since J satisfies condition (C), (2.8) shows that $\boldsymbol{\mu} * g - \boldsymbol{g} \in J$ [resp. $g * \boldsymbol{\mu} - \boldsymbol{g} \in J$]. Hence

$$Tg = \boldsymbol{g}_J = (\boldsymbol{\mu} * g)_J \quad [\text{resp.}\ (g * \boldsymbol{\mu})_J],$$

which establishes (2.4).

3. Positive operators

In this section we concentrate on those linear maps T which commute with right (or left) translations and which are further assumed to be positive. Here the positivity of T means that Tf is a positive measure (or function) whenever f is a positive measure (or function) in the domain of T. This assumption is pretty severe: if X is a Lie group and the range of T lies in the space of distributions on X, Tf, if positive, is necessarily a measure; so, if each element of the domain of T is expressible as a difference of positive elements, T must map its domain into the set of measures. The fact that a positive distribution is necessarily a measure ([32], pp. 28—29) is one of the main reasons for the possible simplifications when T is positive.

We start with a sort of "blanket" theorem.

3.1 THEOREM. *Suppose that T is a positive linear map of C_c into M which commutes with the ρ_a [resp. τ_a]. Then there exists a positive measure $\mu \in M$ such that*

$$Tf = \mu * f \text{ [resp. } f * \mu] \tag{3.1}$$

for $f \in C_c$.

PROOF. We shall use Proposition 2.1, taking there $G = M$ with the topology $\sigma(M, C_c)$. The first step is to show (as will be true in most cases of practical interest) that positivity of T implies continuity. In the present case it suffices to show that if, for some compact set $K \subset X$, a sequence $\{f_n\}$ extracted from $C_{c,K}$ converges uniformly to zero, then the Tf_n converge to zero vaguely.

For this, choose a positive $u \in C_c$ majorising the characteristic function of K. Then positive numbers α_n exist such that $|f_n| \leqq \alpha_n u$. By positivity of T, $|Tf_n| \leqq \alpha_n \cdot |Tu|$, which majorisation is more than enough to show that $Tf_n \to 0$ vaguely.

With the notation of Proposition 2.1, the μ_N are now positive measures. By (i) of that proposition, the $\mu_N * f$ [resp. $f * \mu_N$] are vaguely bounded for each $f \in C_c$. Taking f and g to be positive elements of C_c, it follows that the numbers $\langle g, \mu_N * f\rangle$ [resp. $\langle g, f * \mu_N\rangle$] remain bounded. Written explicitly this signifies that

$$\operatorname{Sup}_N \iint g(xy)f(y)d\mu_N(x)dy < +\infty$$

$$\left[\text{resp. } \operatorname{Sup}_N \iint g(xy)f(x)d\mu_N(y)dx < +\infty\right],$$

or that

$$\operatorname{Sup}_N \int h d\mu_N < \infty,$$

where

$$h(x) = \int g(xy)f(y)dy \quad \left[\text{resp. } \int g(yx)f(y)dy\right].$$

Now, given any compact set $H \subset X$, it is evident that f and g may be chosen from C_c so as to be positive and to make the corresponding function h a majorant of the characteristic function of H. So it appears that the μ_N

themselves are vaguely bounded. But then, by Theorems 1.11.4 and 7.1.1 of [16], the directed family $\{\mu_N\}$ has a vague limiting point μ, itself a positive measure. Then, for each $f \in C_c$, the family $\{\mu_N * f\}$ [resp. $\{f * \mu_N\}$] has $\mu * f$ [resp. $f * \mu$] as a limiting point for the topology of locally uniform convergence. Comparison with (2.1) shows that (1.1) holds.

3.2 REMARKS. (i) If in Theorem 3.1 we replace positivity of T by continuity, at the same time removing the restriction that μ be positive in the conclusion, the statement so obtained is false, at any rate if X is Abelian and non-discrete. For in this case any compact subset K of X having interior points fails to be a Helson set (see Rudin [31], Chapter 5 and especially Theorem 5.6.10). As a consequence (see Edwards [15]), K carries a pseudomeasure s (see § 4 *infra*) which is not a measure. Then $T : f \to s * f$ maps L^2 (and L^2_c) continuously into itself and commutes with translations. Reference [15] contains also a more constructive approach to the existence of pseudomeasures s of this sort.

(ii) Concerning not-necessarily-positive maps T, see Theorem 5.1 (d) below.

3.3 THEOREM. *If T is a positive linear map of C_c into M_{bd} which commutes with the ρ_a [resp. τ_a], then there exists a positive measure $\mu \in M_{bd}$ such that* (1.1) *holds for $f \in C_c$. Consequently the following assertions are true:*

(1) *A positive linear map T of C_c into M_{bd} which commutes with the ρ_a can be continuously extended into a map of C_0 into C_0, C_{bd} into C_{bd}, L^p into L^p $(1 \leqq p \leqq \infty)$ and M_{bd} into M_{bd}.*

(2) *If X is unimodular, statement* (1) *remains true of any positive linear map T of C_c into M_{bd} which commutes with the τ_a.*

(3) *In any case, if T is linear and positive from C_c into M_{bd} and commutes with the τ_a, then it maps C_c into C and can be continuously extended so as to map L^1 into L^1 and M_{bd} into M_{bd}; moreover, if $f \in C_c$ vanishes outside a compact set $K \subset X$, then*

$$|Tf(x)| \leqq m_K \cdot ||f||_\infty \cdot \Delta(x),$$

the number m_K being independent of f.

PROOF. This is virtually a repetition of that of Theorem 3.1. On this occasion we apply Proposition 2.1, taking $G = M_{bd}$; the argument used in the proof of Theorem 3.1 shows that again positivity of T implies its continuity. The positive measures μ_N are now such that the $\mu_N * f$ [resp. $f * \mu_N$] are bounded in M_{bd} for each $f \in C_c$. Applying this condition with f chosen to be positive and such that $\int f dx > 0$, it is readily seen that the μ_N are themselves bounded in M_{bd}. The family $\{\mu_N\}$ therefore ([16], p. 205) has a limiting point μ for the topology $\sigma(M_{bd}, C_0)$, which point is necessarily positive. Use of (2.1) leads as before to (1.1).

Assertion (1) is now immediate from the relation

$$\mu * f = \int (\tau_a f) d\mu(a),$$

together with the fact that $\tau_a f$ is bounded with respect to $a \in X$ in any one of the spaces C_0, C_{bd}, L^p and M_{bd} (each taken with its natural norm).

The same argument yields (2), when X is unimodular, since

$$f * \mu = \int (\rho_a f) d\mu(a)$$

and the $\rho_a f$ are bounded with respect to $a \in X$ because $\Delta \equiv 1$.

As for (3) we observe that $\rho_a f$ is bounded in L^1 and in M_{bd}, whether or not X is unimodular. Moreover, if f vanishes outside a compact set $K \subset X$,

$$\begin{aligned} |f * \mu(x)| &= \left| \int f(xa^{-1}) \Delta(a) d\mu(a) \right| \\ &\leqq ||f||_\infty \cdot \int_{K^{-1}x} \Delta(a) d\mu(a) \\ &\leqq ||f||_\infty \cdot \Delta(x) \cdot \int_X \mathrm{Sup}_{K^{-1}} \Delta d\mu(a), \end{aligned}$$

and we may take

$$m_K = \int_X \mathrm{Sup}_{K^{-1}} \Delta d\mu(a).$$

REMARK. For not-necessarily-positive maps T, see Theorem 5.1 (c) *infra*.

3.4 POSITIVE MAPS OF L^p INTO L^q.

The remainder of this section will be concerned with positive linear maps T of L^p into L^q ($1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$) which commute with the ρ_a [resp. τ_a]. Theorems 3.1 and 3.2 each contribute towards the solution of this problem. The discussion is facilitated by introducing certain sets of measures on X.

3.4.1 DEFINITION. $M_L^{p,q}$ [resp. $M_R^{p,q}$] denotes the set of measures $\mu \in M$ such that

$$(3.1) \qquad \begin{gathered} ||\mu * f||_{L^q} \leqq \text{const. } ||f||_{L^p} \\ [\text{resp. } ||f * \mu||_{L^q} \leqq \text{const. } ||f||_{L^p}] \end{gathered}$$

for $f \in C_c$. If X is Abelian we write simply $M^{p,q}$.

3.4.2 If $\mu \in M_L^{p,q}$ [resp. $M_R^{p,q}$] and $1 \leqq p < \infty$, the mapping $f \to \mu * f$ [resp. $f * \mu$] can be extended by continuity into a mapping of L^p into L^q, the relation (3.1) remaining valid for $f \in L^p$. This extended mapping commutes with the ρ_a [resp. τ_a] and is continuous.

Notice that in any case $M_L^{p,p} \supset M_{bd}$; and if X is unimodular one has likewise $M_R^{p,p} \supset M_{bd}$. If X is compact,

$$M_L^{p,q} = M,\ M_R^{p,q} = M$$

whenever $p \geqq q$.

Young's inequality, which is a special case of Theorem 9.5.1 (a) of [16],

$$||\mu * f||_{L^q} \leqq ||\mu||_{L^r}\, ||f||_{L^p},$$

valid when $p \geqq 1$, $r \geqq 1$, $1/p+1/r \geqq 1$ and q is determined by the relation $1/q = 1/p+1/r-1$, shows that in these circumstances $M_L^{p,q} \supset L^r$ and $M_R^{p,q} \supset L^r$.

3.4.3 The argument in 2.7 (2) shows that, if X is not unimodular, and if $p \neq q$, there exists no non-trivial continuous linear map T of L^p into L^q which commutes with the ρ_a; in particular, $M_L^{p,q} = \{0\}$. On the other hand, an argument due to Hörmander ([23], Theorem 1.1) shows that if $\infty > p > q$, there exists no non-trivial continuous linear map T of L^p into L^q which commutes with the τ_a: in particular $M_R^{p,q} = \{0\}$.

3.5 THEOREM. *If T is a positive linear map of L^p into L^q which commutes with the ρ_a [resp. τ_a], then there exists a positive $\mu \in M_L^{p,q}$ [resp. $M_R^{p,q}$] such that*

$$Tf = \mu * f \ \ [\textit{resp. } f * \mu]$$

for $f \in C_c$; if $p < \infty$, (1.1) holds for $f \in L^p$. And conversely.

PROOF. In view of Theorem 3.1 it remains only to show that a positive linear map T of L^p into L^q is continuous. The proof is simple. If T were not continuous, there would exist $f_n \in L^p$ $(n = 1, 2, \cdots)$ such that $||f_n||_{L^p} \leqq 1$ and $||Tf_n||_{L^q} \geqq n^3$. T being positive, $|Tf_n| \leqq T|f_n|$, and we may therefore suppose that $f_n \geqq 0$. The series $f = \sum_{n=1}^{\infty} n^{-2} f_n$ converges in L^p and $f_n \leqq n^2 f$. Hence $0 \leqq Tf_n \leqq n^2 Tf$ and so

$$n^2 ||Tf||_{L^q} \geqq ||Tf_n||_{L^q} \geqq n^3,$$

which is absurd if n is sufficiently large.

3.5.1 REMARKS. (i) It is well-known that positivity implies continuity (for linear maps) in much more general circumstances; see, for example, [16], p. 612 and the references cited there.

(ii) If X is compact the definition of (Radon) measures shows that any positive μ in $M_L^{p,q}$ of $M_R^{p,q}$ is necessarily bounded. We shall now comment briefly on this matter for groups X which are not necessarily compact, as a result of which the above inference will be justified on less direct grounds.

3.5.2 Let us denote by $C_L^{p,q}$ [resp. $C_R^{p,q}$] the following hypothesis concerning X — there exists a number $c < 1$ with the property that to each

K belonging to a base for the compact subsets of X corresponds a positive function f_K such that

(i) $\operatorname{Sup}_K \|f_K\|_{L^p} < +\infty$;
(ii) $\|f_K\|_{L^q} = 1$ for all K;
(iii) $\|\tau_a f_K - f_K\|_{L^q} \leqq c$ [resp. $\|\rho_a f_K - f_K\|_{L^q} \leqq c$] for all $a \in K$.

If X is Abelian we write simply $C^{p,q}$.

3.5.3 It follows from Reiter's work [29] that if X is Abelian we can fulfill conditions (ii) and (iii) for any q. Dieudonné's work [5] shows that (ii) and (iii) can be fulfilled for left translates, if X is a nilpotent Lie group; and from more recent work of Reiter [30] it appears that the same is true whenever X is a solvable Lie group. Notice that if we can choose $f_K \leqq 1$, then (ii) implies (i) for any $p \geqq q$. This is the case if $X = R^m$, in which case we can take $f_K = m(K')^{-1}\chi_{K'}$, where K' is a sufficiently large hypercube (depending upon K).

3.5.4 If X is compact, $C_L^{p,q}$ and $C_R^{p,q}$ are true — simply take $f_K = 1$.

3.5.5 If X is unimodular $\|f_K\|_{L^q} = \|\check{f}_K\|_{L^q}$ and

$$\|\rho_a f_K - f_K\|_{L^q} = \|\tau_{a^{-1}}\check{f}_K - \check{f}_K\|_{L^q},$$

for any q, so that $C_L^{p,q} \Leftrightarrow C_R^{p,q}$.

3.5.6 Proposition. *Suppose that X satisfies $C_L^{p,q}$ [resp. $C_R^{p,q}$]. Then any positive $\mu \in M_L^{p,q}$ [resp. $M_R^{p,q}$] is bounded.*

Proof. Suppose μ is a positive measure in $M_L^{p,q}$ [resp. $M_R^{p,q}$]. Then (3.1) and (i) of $C_L^{p,q}$ [resp. $C_R^{p,q}$] shows that there is a number k, independent of K, such that

$$\|\mu * f_K\|_{L^q} \leqq k \text{ [resp. } \|f_K * \mu\|_{L^q} \leqq k].$$

A fortiori since $f_K \geqq 0$ and $\mu \geqq 0$,

$$\|\mu_K * f_K\|_{L^q} \leqq k \text{ [resp. } \|f_K * \mu_K\|_{L^q} \leqq k], \tag{3.2}$$

where $\mu_K = \mu|K$. Putting $m_K = \int d\mu_K$, we have

$$\mu_K * f_K - m_K f_K = \int (\tau_a f_K - f_K)\, d\mu_K(a)$$

$$\left[\text{resp. } f_K * \mu_K - m_K f_K = \int (\rho_a f_K - f_K)\, d\mu_K(a)\right].$$

So, applying (ii) and (iii) of $C_L^{p,q}$ [resp. $C_R^{p,q}$] we obtain

$$m_K = \|m_K f_K\|_{L^q} = \|\mu_K * f_K - \int (\tau_a f_K - f_K)\, d\mu_K(a)|_{L^q}$$
$$\left[\text{resp. } \|f_K * \mu_K - \int (\rho_a f_K - f_K)\, d\mu_K(a)\|_{L^q}\right]$$
$$\leqq k + c m_K.$$

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Hence

$$m_K \leqq k(1-c)^{-1}.$$

Letting K expand, it follows that μ is bounded.

3.5.7 We shall in § 4 consider continuous (not necessarily positive) endomorphisms of L^p for the case in which X is locally compact Abelian.

4. Representation of multipliers of L^p

4.0 Throughout this section X is assumed to be locally compact Abelian. We denote by $\hat{X} = \{\xi\}$ the character group of X and assume the Haar measures on X and $\hat{X}$ to be adjusted so that the Fourier transformation is an isometry of $L^2(X)$ onto $L^2(\hat{X})$.

By a multiplier of $L^p = L^p(X)$ we understand a continuous linear operator T mapping L^p into itself which commutes with translations. (Some justification for the term "multiplier" will appear in the course of 4.5 and 4.6, especially equation (4.5.2).) The multipliers of L^1 have been fully identified in Corollary 2.6.2; and in 1.6 it has been seen that if T is a multiplier of L^∞, then the restriction $T|C_0$ is represented by convolution with a bounded measure on X. When $1 < p < \infty$, the situation is in general quite different and obscure: for explicit examples, see the proof of Theorem 4.13 below. If $X = R^n = \hat{X}$, a detailed study has been made by Hörmander [23].

In this section we aim to show in detail that any multiplier of L^p is expressible as convolution with a suitable type of pseudomeasure on X. It follows from Theorem 5.1 (d) that likewise any continuous linear operator T from L^p into L^q which commutes with translations is representable as convolution with a suitable type of quasimeasure on X.

4.1 The space $P(X)$ of pseudomeasures on X.

We write $A = A(X)$ for the set of functions u on X of the form

$$u(x) = \int_{\hat{X}} v(\xi)\xi(x)d\xi \tag{4.1.1}$$

for some $v \in L^1(\hat{X})$; v is uniquely determined by u and we define on A the norm

$$\|u\|_A = \int_{\hat{X}} |v(\xi)|d\xi. \tag{4.1.2}$$

It is evident that A is a Banach algebra under pointwise operations. It is moreover a dense subset of C_0, the injection map of A into C_0 being continuous.

We write $P = P(X)$ for the topological dual of A. Elements of P are

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termed pseudomeasures on X, the name being suggested by the fact that any element of $M_{bd}(X)$ defines, and is uniquely determined by, its restriction to $A(X)$, this restriction being a pseudomeasure on X. If X is non-discrete, $P(X)$ will contain elements with compact support which are not measures; see Remark 3.2 (i) and the references cited there. Moreover, $P(X)$ will in general contain Radon measures on X which are not bounded measures. For example, suppose $X = R$ and let $(c_n)_{N=1}^{\infty}$ be any sequence which decreases monotonically to zero and is such that nc_n is bounded. Then (see, for example, [39], Vol. 1, pp. 182—183) the series $\sum_{n=1}^{\infty} c_n \sin nx$ is boundedly convergent for $x \in R$ and therefore weakly convergent in $L^{\infty}(R)$. This last means precisely that the series $\sum_{n=1}^{\infty} c_n(\varepsilon_n - \varepsilon_{-n})$ of measures is weakly convergent in $P(R)$. The sum of this series is at the same time a measure which, if $\sum_{n=1}^{\infty} |c_n| = \infty$, is not a bounded measure.

On the other hand, it is quite easy to show that any positive pseudomeasure belongs to $M_{bd}(X)$.

P is a vector space and also a module over A. By using partitions of unity whose elements belong to A, pseudomeasures can be localised; in particular one can define the support of a pseudomeasure s as the complement of the largest open subset U of G having the property that $s(u) = 0$ for every $u \in A$ satisfying $\operatorname{supp} u \subset U$.

It is possible to extend the Fourier transformation to pseudomeasures in such a way as to realise a linear isometry of $P(X)$ onto $L^{\infty}(X)$ (the norm of $\phi \in L^{\infty}(\hat{X})$ being the local essential supremum of $|\phi|$). The extension is made in such a way that the relation $\hat{s} = \phi$ signifies exactly that $\phi \in L^{\infty}(\hat{X})$ and $s \in P(X)$ are related by the formula

$$s(u) = \int_{\hat{X}} v(\xi)\phi(-\xi)d\xi \tag{4.1.3}$$

whenever $u \in A$ is given by (4.1.1). The isometric nature of this extended Fourier transformation is an immediate consequence of the definition of the natural norm on P as the dual of A (see [16], Section 1.10.6).

If X is a finite product of lines or circles, the pseudomeasures on X can be thought of as comprising exactly the Schwartz distributions on X whose Schwartz-Fourier transforms belong to $L^{\infty}(\hat{X})$.

In terms of the Fourier transform one can now make P into an algebra under convolution, $s_1 * s_2$ being that pseudomeasure whose transform is $\hat{s}_1 \cdot \hat{s}_2$.

4.2 Multipliers defined by pseudomeasures. If $s \in P$ and $f \in L^2$, then $\hat{s} \cdot \hat{f} \in L^2(\hat{X})$ is the Plancherel-Fourier transform of some uniquely determined element of L^2: this element we denote by $s * f$. The Parseval formula shows that

$$||s * f||_{L^2} \leqq ||s||_P \cdot ||f||_{L^2}, \tag{4.2.1}$$

where $||s||_P$ denotes the norm of s *qua* element of the dual of A, which is identical with $||\hat{s}||_\infty$.

It is simple to verify that the formula

$$Tf = s * f \tag{4.2.2}$$

defines T as a multiplier of L^2 for which $||T|| = ||s||_P$. The converse statement will form part of our main theorem yet to be stated and proved.

4.3 The spaces $P^p(X)$. For any exponent p satisfying $1 \leqq p \leqq \infty$ we define $P^p = P^p(X)$ to be the set of pseudomeasures $s \in P(X)$ such that

$$||s * f||_{L^p} \leqq \text{const.}\ ||f||_{L^p} \tag{4.3.1}$$

for each $f \in C_c$, the constant depending upon s. The remarks in **4.0** referring to Corollary 2.6.2 may be interpreted as meaning that $P^1 = P^\infty = M_{bd}$. It is easily seen furthermore that

$$P^p = P^{p'} \qquad (1/p + 1/p' = 1) \tag{4.3.2}$$

and that always

$$M_{bd} \subset P^p \subset P. \tag{4.3.3}$$

In 4.2 it has appeared that

$$P^2 = P. \tag{4.3.4}$$

If $s \in P^p$ we can define $s * f \in L^p$ for each $f \in L^p$ (or each $f \in C_0$, if $p = \infty$) by continuity; then (4.3.1) continues to hold for each $f \in L^p$ (or each $f \in C_0$, if $p = \infty$). As a consequence, if $p \neq \infty$ and $s \in P^p$, the formula (4.2.2) defines T as a multiplier of L^p.

We can now state the main theorem of this section.

4.4 Theorem. *Suppose that* $1 \leqq p < \infty$ *and that* T *is a multiplier of* L^p. *Then there exists a pseudomeasure* $s \in P^p$ *such that*

$$Tf = s * f \tag{4.2.2}$$

for $f \in L^p$. *Conversely, if* $s \in P^p$, *then* (4.2.2) *defines* T *as a multiplier of* L^p.

The converse assertion is evident from the substance of 4.2 and 4.3. The direct assertion will be established in two stages, taking the special case $p = 2$ first.

4.5 Proof when $p = 2$. Let T be a multiplier of L^2. Then, as at the outset of the proof of Proposition 2.1, we have $Tf * g = T(f * g) = f * Tg$ whenever, $f, g \in C_c$. Hence

$$(Tf)^\wedge \cdot \hat{g} = \hat{f} \cdot (Tg)^\wedge \text{ a.e.} \tag{4.5.1}$$

whenever $f, g \in C_c$. Now, according to [2], p. 6, Proposition 4, or [16],

p. 229, $\hat{X}$ is the disjoint union of a locally negligible set N and a disjoint locally countable family $\{K_i\}$ of compact subsets of $\hat{X}$. For each index i we may choose and fix a function $g_i \in C_c$ such that g_i is non-vanishing on K_i. Define the function ϕ on X by setting it equal to $(Tg_i)^\wedge/g_i$ on K_i for each i and to 0 on N. Then, if $f \in C_c$ we have by (4.5.1) the relation $(Tf)^\wedge = \phi \cdot \hat{f}$ a.e. on K_i. The family $\{K_i\}$ being locally countable, whilst $\hat{f}$ vanishes outside a σ-finite subset of X, it follows that

$$(Tf)^\wedge = \phi \cdot \hat{f} \text{ a.e. on } \hat{X}. \tag{4.5.2}$$

It is evident that ϕ is measurable. Moreover, the continuity of T shows that (4.5.2) continuous to hold for each $f \in L^2$. Parseval's formula and (4.5.2) combine to show that ϕ is locally essentially bounded and that $||T|| = ||\phi||_\infty$.

We pause here to remark that it is the function ϕ appearing in (4.5.2), rather than T itself, which should be spoken of as a "multiplier" of L^2 — at least if one wishes to retain the terminology which is customary in the study of ordinary Fourier series.

To complete the proof, introduce the pseudomeasure s on X for which $\hat{s} = \phi$ (see 4.1). According to the definition of $s * f$ (see 4.2), we have

$$(s * f)^\wedge = \phi \cdot \hat{f} \text{ a.e.}$$

and comparison with (4.5.2) shows that $Tf = s * f$ *qua* elements of L^2.

4.6 Proof when $p \neq 2$. Let T' be the adjoint (see [16], p. 515) of the given multiplier T of L^p, so that

$$Tf * f'(0) = f * T'f'(0) \tag{4.6.1}$$

for $f \in L^p$ and $f' \in L^{p'}$. By the basic properties of convolution (see Section 4.19.15 of [16]) T' commutes with translations. Thus T' is a multiplier of $L^{p'}$.

If we suppose that $f, f' \in C_c \subset L^p \cap L^{p'}$, then we know that $Tf * f' = f * Tf'$, and comparison with (4.6.1) shows that T and T' coincide on C_c. Therefore we have for $f \in C_c$ both

$$||Tf||_{L^p} \leqq ||T|| \cdot ||f||_{L^p}$$

and

$$||Tf||_{L^{p'}} \leqq ||T'|| \cdot ||f||_{L^{p'}}.$$

At this point the Riesz-Thorin convexity theorem ([39], Vol. II, p. 95) may be applied to conclude that

$$||Tf||_{L^2} \leqq \text{const } ||f||_{L^2}$$

for $f \in C_c$. Accordingly $T|C_c$ can be extended into a multiplier of L^2. By (4.5), therefore, there exists a pseudomeasure $s \in P$ such that (4.2.2) holds for $f \in C_c$. This in itself entails that $s \in P^p$; and then (4.2.2) must, by continuity, continue to hold for all $f \in L^p$. The proof is thus complete.

4.7 REMARKS. We have noted that $P^1 = P^\infty = M_{bd}$ which, if X is non-discrete, is a proper subset of P. (M_{bd} can exhaust P only if X is a Helson set, which is certainly false if X is non-discrete; see [31], p. 119.)

If $X = R^n$, $P^p \neq P$ whenever $p \neq 2$: this follows from Theorem 1.12 of Hörmander [23] (taking $F \equiv 1$ therein).

If X is non-discrete compact Abelian, then Theorem 4.13 shows that $M_{bd} \neq P^p$ whenever $1 < p < \infty$. In this case we have also $P^p \neq P$ whenever $p \neq 2$. In verifying this one may, by (4.3.2), assume that $p < 2$. If P^p were to coincide with P, then for any bounded function β on $\hat{X}$ and any $f \in L^p$, $\beta \cdot \hat{f}$ is the Fourier transform of some function in L^p. Then (see Helgason [21] and Edwards [11]) $\hat{f} \in l^2(\hat{X})$, and so $f \in L^2$, whenever $f \in L^p$. Since $p < 2$, this would entail that X is discrete.

Presumably $P^p \neq P$ for all $p \neq 2$ and all infinite X, but we do not know of a proof of this. The relation is established for any locally compact Abelian X containing an infinite discrete subgroup in G. I. Gaudry "Multipliers of type (p, q)" (to appear Pacific J. Math.).

4.8 Suppose that T is a multiplier of L^p, that E is a closed subset of X, and that either of the following (actually equivalent) conditions is fulfilled:

(i) $\operatorname{supp} Tf \subset E + \operatorname{supp} f$ for $f \in C_c$;
(ii) for each $f \in L^p$ and each neighbourhood U of E, Tf is the limit in L^p of linear combinations of translates $\tau_a f$ with $a \in U$.

It is then easily seen that, in the representation (4.2.2), $\operatorname{supp} s \subset E$. Consequently, if $E \cap K$ is a Helson set for each compact set $K \subset X$, then s is necessarily a measure with support contained in E.

For example, if $\{a_n\}$ is a finite or infinite sequence of points of X which has no limiting point in E, and if (i) or (ii) holds with $E = \{a_n\}$, then T is expressible in the form

$$Tf = \sum c_n \cdot \tau_{a_n} f,$$

where the series converges weakly in L^2 for each $f \in L^2 \cap L^p$.

4.9 A type of representation theorem somewhat different from Theorem 4.4 *supra* has been announced by Figà-Talamanca [17], his assertion being that the multipliers of L^p can be put into a linear isometric correspondence with the elements of the dual of a certain Banach function-space A_p. This space A_p is the space of functions h on X admitting at least one expression of the form

$$h = \sum_{n=1}^{\infty} f_n * g_n, \tag{4.9.1}$$

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wherein

$$(4.9.2)\qquad f_n \in L^p,\ g_n \in L^{p'},\ \textstyle\sum_{n=1}^{\infty} \|f_n\|_{L^p} \cdot \|g_n\|_{L^{p'}} < \infty.$$

The norm on A_p is defined to be

$$(4.9.3)\qquad \|h\|_{A_p} = \operatorname{Inf} \textstyle\sum_{n=1}^{\infty} \|f_n\|_{L^p} \cdot \|g_n\|_{L^{p'}},$$

the infimum being taken over all expressions (4.9.1) for h. (The space A of 4.1 is none other than A_2.) In the said correspondence the multiplier T is associated with the functional t on A_p for which

$$(4.9.4)\qquad t(f * g) = Tf * g(0),$$

and the crucial step in establishing the correspondence lies in showing that any relation of the form

$$(4.9.5)\quad f_n \in L^p,\ \ g_n \in L^{p'},\ \ \textstyle\sum_{n=1}^{\infty} \|f_n\|_{L^p} \cdot \|g_n\|_{L^{p'}} < \infty,\ \ \sum_{n=1}^{\infty} f_n * g_n = 0$$

implies that

$$(4.9.6)\qquad \textstyle\sum_{n=1}^{\infty} Tf_n * g_n(0) = 0.$$

One way of achieving this object is to establish the following approximation theorem for multipliers T.

4.10 THEOREM. *Let T be a multiplier of L^p $(1 < p < \infty)$. Then there exists a net $\{T_\alpha\}$ of multipliers of L^p satisfying the following conditions*

(*i*) $\|T_\alpha\| \leqq \|T\|$
(*ii*) $\lim_\alpha T_\alpha f * g(0) = Tf * g(0)$ $(f \in L^p,\ g \in L^{p'})$
(*iii*) *each T_α is defined by convolution with a function in C_c.*

In the presence of (i), (ii) is equivalent to saying that $T = \lim_\alpha T_\alpha$ in the strong operator topology; or again that $Tf = \lim_\alpha T_\alpha f$, uniformly when f ranges over precompact subsets of L^p.

4.11 In order to establish 4.10 it suffices to show that multipliers T_α exist satisfying (iii) and

$$(\mathrm{i}')\qquad |T_\alpha f * g(0)| \leqq \|T\|\, \|f\|_{L^p} \|g\|_{L^{p'}} \quad (f,\ g \in C_c),$$

and such that (ii) holds for $f,\ g \in C_c$. Such a net $\{T_\alpha\}$ may be obtained by setting

$$T_\alpha f = [H_\alpha (k_\alpha * s)] * f = j_\alpha * f$$

where

s is the pseudomeasure defined by T (see Theorem 4.4); the $k_\alpha \in C_c$ and form an approximate identity in L^1 chosen so that $\hat{k}_\alpha \in L^1(\hat{X})$; and H_α is the inverse Fourier transform of h_α, $\{h_\alpha\}$ forming an approximation identity in $L^1(\hat{X})$ for which $H_\alpha \in C_c$.

It is evident that these conditions ensure that $j_\alpha \in C_c$ for each α.

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4.12 Granted 4.10, it is easy to show that (4.9.5) implies (4.9.6). Thus, by (i) and (ii) and dominated convergence of the series,

$$\textstyle\sum_{n=1}^{\infty} Tf_n * g_n(0) = \lim_\alpha \sum_{n=1}^{\infty} T_\alpha f_n * g_n(0).$$

By (iii),

$$\begin{aligned} T_\alpha f_n * g_n(0) &= (j_\alpha * f_n) * g_n(0) = j_\alpha * (f_n * g_n)(0) \\ &= \int j_\alpha(-x) \cdot f_n * g_n(x) dx, \end{aligned}$$

and, since $j_\alpha \in L^1$ and $\sum_{n=1}^{\infty} f_n * g_n(x)$ is uniformly convergent we thus obtain

$$\begin{aligned} \textstyle\sum_{n=1}^{\infty} T_\alpha f_n * g_n(0) &= \int f_\alpha(-x) \cdot \textstyle\sum_{n=1}^{\infty} f_n * g_n(x) dx \\ &= \int j_\alpha(-x) \cdot 0 \, dx = 0. \end{aligned}$$

4.13 Remarks. (i) As mentioned in 4.0, by using the results in [19] one can show likewise that any multiplier T of L^p into L^q ($1 \leqq p < \infty$, $1 \leqq q < \infty$) is representable in the form (4.4.2), where now s is a quasi-measure on X (a concept defined in [19]) such that

$$||s * f||_{L^q} \leqq \text{const. } ||f||_{L^p}$$

for $f \in C_c$. Moreover (see 2.6.2 (i) and 4.14 (b) *infra*) only the case $1 \leqq p < q < \infty$ needs further consideration. For this case one can show that analogues of 4.9—4.12 subsist, p' being replaced in (4.9.2) by q' and the series (4.9.1) being now convergent in L^r, where $1/r = 1/p + 1/q' - 1$. Extensions, to the case of multipliers of L^p into L^q, of the results in 4.9 and 4.10 are discussed in [18].

(ii) The concept of pseudomeasure can be satisfactorily extended to all unimodular locally compact groups X, but it has yet to be seen to what extent the preceding proofs (seemingly dependent on use of the Fourier transform) can be carried over in detail.

(iii) Hörmander ([23], Theorem 1.9) shows that if $X = R^n$ and $p < 2 < q$, then there exist distributions μ such that $f \to \mu * f$ is a continuous linear map of L^p into L^q (which evidently commutes with translations) and yet $\hat{\mu}$ is not a measure, so that μ is certainly not a bounded measure.

4.14 It has been noted in 3.4.3 that

(a) if X is not unimodular, and if $p \neq q$, there exists no non-trivial continuous linear map of L^p into L^q which commutes with right translations; and that

(b) if $X = R^n$, and if $q < p < \infty$, there exist no non-trivial continuous linear maps of L^p into L^q commuting with translations. (Hörmander's proof of this, given for $X = R^n$, in fact obviously extends to the case of left translations on any locally compact group X.)

We shall now show that no such collapse occurs when X is compact Abelian. Indeed, if $p = 1$ the existence of an abundance of non-trivial maps of the desired sort of L^1 into L^q follows from Corollary 2.6.2. (This does not depend on the special nature of X.) On the other hand, if X is infinite compact Abelian and $1 < p \leqq \infty$, $1 \leqq q < \infty$, the following theorem shows that there exist many maps of the desired type which, moreover, are not expressible by convolution with any (necessarily bounded) measure.

4.15 THEOREM. *Let X be infinite compact Abelian, $1 < p \leqq \infty$, and $1 \leqq q < \infty$. Then there exist continuous linear maps T of L^p into L^q which commute with translations and yet which are not expressible in the form $Tf = \mu * f$ for any measure μ on X.*

The proof depends on the existence and properties of infinite Sidon subsets S of the character groups $\hat{X}$: see § 5.7 of Rudin [31].

Before starting the proof of the theorem we collect some facts about these Sidon sets.

LEMMA 1. *Let X be as in the above theorem, let S be a Sidon subset of $\hat{X}$, and let $1 < p \leqq \infty$. Then for each $f \in L^p$ we have*

$$[\textstyle\sum_{\xi \in S} |\hat{f}(\xi)|^2]^{\frac{1}{2}} \leqq \text{const. } ||f||_{L^p}.$$

PROOF. See [31], p. 130.

LEMMA 2. *Let X and S be as in Lemma 1, and let μ be any Radon measure on X such that $\hat{\mu}(\xi) = 0$ for $\xi \in \hat{X} \backslash S$. Then $\mu \in L^r$ for $1 \leqq r < \infty$.*

PROOF. Take a directed family (t_i) of trigonometric polynomials on X such that $||t_i||_1 \leqq 1$ and $t_i * \mu \to \mu$ vaguely. By [31], Theorem 5.7.7 it follows that

$$||t_i * \mu||_{L^r} \leqq \text{const. } ||t_i * \mu||_{L^1} \leqq \text{const.}$$

Assuming (as we may) that $r > 1$, it follows that the net $\{t_i * \mu\}$ has a weak limiting point in L^r. By vague convergence, this limiting point must be μ, so that $\mu \in L^r$.

LEMMA 3. *Let X and S be as in Lemma 1, let $1 < p \leqq \infty$, $1 \leqq q < \infty$, and let β be any bounded complex-valued function on X which vanishes on $\hat{X} \backslash S$. Then there exists a continuous linear map T of L^p into L^q which commutes with translations and for which $(Tf)\hat{} = \beta\hat{f}$.*

PROOF. Immediate from Lemmas 1 and 2.

PROOF OF THEOREM 4.15. Let S be any infinite Sidon subset of $\hat{X}$ (see [31], p. 126), choose β as in Lemma 3 and such that furthermore $\beta \notin l^2(\hat{X})$, and then appeal to Lemma 2 once more.

4.16 A RESULT OF STEIN. Throughout this section X will denote the circle group, so that $\hat{X}$ is identifiable with the additive group Z of integers. Stein ([34], Theorem 9) has given a result which comes rather close to providing *a* necessary and sufficient in order that a bounded function ϕ on Z shall define a multiplier T of L^p *via* the relation

$$(Tf)^\wedge = \phi \cdot \hat{f}. \tag{4.16.1}$$

This result is expressed in terms of the function Φ, defined a.e. on X by the formula

$$\Phi(x) = \textstyle\sum_{n\neq 0} (in)^{-1}\phi(n)e^{inx}, \tag{4.16.2}$$

the series being convergent in L^2 (for example).

Given an exponent q satisfying $1 \leqq q \leqq \infty$, Φ is said to belong to V_q if and only if $\Phi \in L^q$ and

$$\mathrm{Sup}\, ||\textstyle\sum_{k=1}^{r} \{\tau_{b_k}\Phi - \tau_{a_k}\Phi\}|| < \infty,$$

the supremum being taken relative to all finite sequences $\{[a_k, b_k]\}_{k=1}^{r}$ of non-overlapping subintervals $[a_k, b_k]$ of $[0,2\pi)$.

Prior to stating Stein's result we remark that from (4.3.2) and the Riesz-Thorin convexity theorem ([39], Vol. II, p. 95) it follows that if ϕ defines a multiplier of L^p, then it also defines a multiplier of L^r for any r lying in the closed interval spanned by r and r'. In particular, it suffices to discuss the multipliers of L^p for $1 \leqq p \leqq 2$. Stein's theorem asserts that, if $1 < p < 2$, then:

(a) if ϕ defines a multiplier of L^p (and so of L^r whenever $p \leqq r \leqq p'$), then $\Phi \in V_{p'}$;

(b) if $\Phi \in V_{p'}$, then ϕ defines a multiplier of L^r whenever $p < r < p'$.

As Stein remarks, $\Phi \in V_\infty$ if and only if Φ is (equal a.e. to) a function of bounded variation. This, in conjunction with the case $p=1$ of Corollary 2.6.2, shows that ϕ defines a multiplier of L^1 if and only if $\Phi \in V_\infty$. Similarly, $\Phi \in V_2$ if and only if ϕ is bounded; and so, by (4.3.4) and Theorem 4.4, one may say that ϕ defines a multiplier of L^2 if and only if $\Phi \in V_2$.

4.17 RESULTS FOR THE GROUPS R^n. A number of results are known which give sufficient conditions on a function ϕ on R^n (identified as the dual of R^n) ensuring that (4.5.2) defines T as a multiplier of L^p into L^q, in which case ϕ may itself be described as a (p, q)-multiplier. For these results we refer the reader to Hörmander [23], Theorems 1.11, 2.4 and 2.5. Of these the first and second have known analogous for the circle group; see [39], Vol. II, p. 127. The third, like the results in [28], involve generalised derivatives of ϕ; we know of no published analogous for the circle group.

De Leeuw [4] establishes relations between bounded functions ϕ which (the are (p, p)-multipliers on R with those which are (p, p)-multipliers on R_d line with its discrete topology), together with relations between multipliers on R, on the circle and on Z.

5. Further results for Abelian groups

Throughout this section we assume again that X is Abelian. In terms of the pseudomeasures introduced in § 4 we can extend Theorems 3.1 and 3.3 to not-necessarily-positive operators T. The results are published elsewhere, but we record them here for convenience.

We write $P_c = P_c(X)$ for the set of pseudomeasures with compact supports.

5.1 Theorem. (*a*) *If T is a continuous linear map from C_c into M_c which commutes with translations, there exists a pseudomeasure $s \in P_c$ such that*

$$Tf = s * f;$$

and conversely.

(*b*) *As* (*a*), *save that C and M replace C_c and M_c, respectively.*

(*c*) *If T is a continuous linear map from C_c into M_{bd} which commutes with translations, there exists a pseudomeasure $s \in P$ such that*

$$Tf = s * f.$$

(*d*) *If T is a continuous linear map from C_c into M which commutes with translations, there exists a quasimeasure q on X such that*

$$Tf = q * f.$$

Proof. That of (a) appears in Edwards [**13**], Theorem 2; (b) follows from (a) by considering the adjoint of T. Assertion (c) is Theorem 1 of [**13**]. Gaudry [**19**] gives the definition of quasimeasures (which can be regarded as locally finite sums of pseudomeasures) and the proof of (d).

Remarks. Concerning the nature of pseudomeasures, see again Remark 3.2(i). It is interesting to note that the stated representation formulae entail that T actually maps L^2_c into L^2_{loc} in all cases; L^2_c into L^2_c and L^2 into L^2 in cases (a) and (b); and L^2 into L^2 in case (c).

Again for the reader's convenience we record some known results for the case $X = R^n$ or T^n or a finite product of such groups.

5.2 Theorem. *Let X be of the form $R^m \times T^n$.*

(*a*) *Any continuous linear map T of C^∞_c into $\mathscr{D}'$ which commutes with translations has the form*

$$Tf = \mu * f$$

for some $\mu \in \mathscr{D}'$; and conversely.

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(*b*) *Any continuous linear map T of C^∞ into $\mathscr{D}'$ which commutes with translations has the form*

$$Tf = \mu * f$$

for some $\mu \in \mathscr{D}'_c$; and conversely.

(*c*) *Any continuous linear map T of C_c^∞ into $\mathscr{D}'_c$ (the latter with the topology $\sigma(\mathscr{D}'_c, C^\infty)$) which commutes with translation has the form*

$$Tf = \mu * f$$

for some $\mu \in \mathscr{D}'_c$; and conversely.

PROOF. Statements (a) and (b) are proved by Schwartz [**33**], pp. 53—54 and pp. 18—19 respectively. It is not difficult to derive statement (c) by applying (b) to the adjoint of T.

6. Continuity questions

We have frequently used the fact that, under quite mild restrictions on the function-spaces F and G, a continuous linear map T of F into G which commutes with translations also commutes with convolutions, and vice versa. At root this is because in most cases convolution is a limit of linear combinations of translations, whilst reciprocally translation is a limit of convolutions with arbitrarily smooth functions.

It will be shown in 6.2 and 6.3 that, under mild conditions on F and G, a linear map T of F into G which commutes with convolutions is necessarily continuous. These results refine and extend Theorem 3.1 of de Leeuw [3].

A detailed analysis of similar continuity questions for a somewhat different class of operators is given by B. E. Johnson [25].

We begin with a definition.

6.1 DEFINITION. Given a non-void subset Σ of M (or of $\mathscr{D}'$, if X is a Lie group), by a left [resp. right] Σ-space is meant a vector subspace H of M (or of $\mathscr{D}'$) with the following properties:

(i) H is stable under convolution on the left [resp. right] by elements of Σ;

(ii) H is a separated topological vector space such that, for each $s \in \Sigma$, the map $h \to s * h$ [resp. $h \to h * s$] is a continuous endomorphism of H;

(iii) if $h \in H$ satisfies $s * h = 0$ [resp. $h * s = 0$] for all $s \in \Sigma$, then $h = 0$.

The following result sharpens de Leeuw's Theorem 3.1 in so far as T is not assumed to commute with translations.

6.2 THEOREM. *Suppose that X is compact and that Σ denotes the set of all trigonometric polynomials on X. Let F and G be two left [resp. right] Σ-spaces, and suppose that the closed graph theorem applies to linear maps of F into G. The conclusion is that if T is any linear map of F into G such that*

$$T(s * f) = s * Tf \text{ [resp. } T(f * s) = Tf * s] \tag{6.1}$$

for all $f \in F$ and all trigonometric polynomials s, then T is continuous.

REMARK. We recall that a trigonometric polynomial on X is a finite linear combination of co-ordinates of finite-dimensional, continuous, unitary representations of X. If X is Abelian, the trigonometric polynomials are therefore simply the finite linear combinations of continuous characters of X. In this case, therefore, (6.1) is equivalent to condition 4 in de Leeuw's Theorem 3.1.

PROOF. We consider the left-handed case, which is typical. It suffices to show that the graph of T is closed in $F \times G$. Suppose that $\{f_n\}$ is a net which converges to 0 in F and for which, moreover, $\{Tf_n\}$ is convergent in G to g. It must be shown that $g = 0$.

Choose any $s \in \Sigma$. Then $F_s = s * F$ is a finite-dimensional vector subspace of F. As U ranges over a base at 0 in F, the sets $s * U$ define a base at 0 for a vector space topology on F_s which, since F is separated and the map $f \to s * f$ is continuous by 6.1 (ii), is separated. Consequently, $T|F_s$ is continuous for this topology. The definition of the topology on F_s is arranged so that $f \to s * f$ is continuous from F into F_s. In particular, therefore, $T(s * f_n) \to 0$ in G. On the other hand, (6.1) shows that $T(s * f_n) = s * Tf_n$, which converges in G to $s * g$. It follows that $s * g = 0$, and this for each $s \in \Sigma$. Hence, by (iii) of 6.1, $g = 0$.

6.3 There are analogues of Theorem 6.2 for certain other interesting cases. Suppose for example, that F is a separated topological vector space which is also a commutative algebra under convolution, the latter being separately continuous on $F \times F$. Suppose further that G is a left [resp. right] F-space in the sense of Definition 6.1. It then follows by a similar argument that any linear map T of F into G, satisfying

$$T(f * f') = f * Tf' \text{ [resp. } = Tf * f'] \tag{6.2}$$

for $f, f' \in F$, has a graph closed in $F \times G$.

These conditions are fulfilled, and lead to continuity of T, if X is Abelian and if, for example,

$$F = C_c, L_c^1, \text{ or } L^1$$

and

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$$G = L^p \ (1 \leqq p \leqq \infty) \text{ or } M_{bd},$$

as well as in numerous other cases.

6.4 We do not know whether there are valid analogues of Theorems 6.2 and 6.3 in which the main hypothesis, that T commutes with convolutions, is replaced by the assumption that T commutes with translations. We note, however, the following germane result, due to Iwahori [24]:

Let X be a compact group, Σ the set of trigonometric polynomials on X, and T an endomorphism of Σ such that

(1) T commutes with the τ_a $(a \in X)$;

(2) $T1 = 1$;

(3) $\overline{Tf} = T\bar{f}$ for all $f \in \Sigma$;

(4) $T(fg) = Tf \cdot Tg$ for all $f, g \in \Sigma$.

Then there exists $a \in X$ such that $Tf = \rho_a f$ for all $f \in \Sigma$.

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