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ON GROUPS WITH FINITE CONJUGACY CLASSES IN A VERBAL SUBGROUP

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Abstract

Let w be a group-word. For a group G, let G_w denote the set of all w-values in G and let $w(G)$ denote the verbal subgroup of G corresponding to w. The group G is an $FC(w)$-group if the set of conjugates x^{G_w} is finite for all $x \in G$. It is known that if w is a concise word, then G is an $FC(w)$-group if and only if $w(G)$ is FC-embedded in G, that is, the conjugacy class $x^{w(G)}$ is finite for all $x \in G$. There are examples showing that this is no longer true if w is not concise. In the present paper, for an arbitrary word w, we show that if G is an $FC(w)$-group, then the commutator subgroup $w(G)'$ is FC-embedded in G. We also establish the analogous result for $BFC(w)$-groups, that is, groups in which the sets x^{G_w} are boundedly finite.

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1. Introduction

Let G be a group. For subsets X, Y of G, we denote by X^Y the set $\{x^y \mid x \in X, y \in Y\}$. The group G is called an FC-group if x^G is finite for all $x \in G$. The group G is said to be a BFC-group if x^G is finite for all $x \in G$ and the number of elements in x^G is bounded by a constant that does not depend on the choice of x. It was shown by Neumann that G is a BFC-group if and only if the commutator subgroup G' is finite [6]. The first explicit bound for the order of G' was found by Wiegold [10] and the best known bound was obtained in [5] (see also [7, 9]).

A subgroup H of G is said to be *FC-embedded* in G if x^H is finite for all $x \in G$. The subgroup H is *BFC-embedded* in G if x^H is finite for all $x \in G$ and the number of elements in x^H is bounded by a constant that does not depend on the choice of x.

Let $w = w(x_1, \ldots, x_n)$ be a group-word, that is, a nontrivial element of the free group freely generated by $x_1, x_2, \ldots$. We denote by G_w the (normal) set $\{w(g_1, \ldots, g_n) \mid g_i \in G\}$ of all w-values in G and by $w(G)$ the verbal subgroup of G corresponding to w, that is, the subgroup generated by G_w. The group G is an *$FC(w)$-group* if x^{G_w} is finite for

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all $x \in G$. The group G is a *$BFC(w)$-group* if x^{G_w} is finite for all $x \in G$ and the number of elements in x^{G_w} is bounded by a constant that does not depend on the choice of x.

Obvious examples of $FC(w)$-groups (respectively, $BFC(w)$-groups) are provided by groups G in which the verbal subgroup $w(G)$ is FC-embedded (respectively, BFC-embedded) in G. For certain group-words w, there are examples of $FC(w)$-groups which are not of that type (see the example in [1, Section 4]). However, it is the main result of the paper [4] that, for many group-words w, the groups G with FC-embedded verbal subgroups $w(G)$ are the only examples of $FC(w)$-groups. *If the word w is concise, then G is an $FC(w)$-group if and only if the verbal subgroup $w(G)$ is FC-embedded in G.* We recall that a group-word w is called concise if the finiteness of the set G_w always implies the finiteness of the verbal subgroup $w(G)$ (see [8, pages 119–121] for relevant results on concise words). It was shown in [2] that the order of $w(G)$ is bounded in terms of $|G_w|$ for each concise word w. Together with results from [1], this implies that, whenever w is a concise word, the group G is a $BFC(w)$-group if and only if $w(G)$ is BFC-embedded in G.

As the example in [1] shows, in the case where w is not concise, the verbal subgroup $w(G)$ of an $FC(w)$-group G need not be FC-embedded in G. The main goal of the present paper is to prove the following theorems. In the subsequent work, we write $w(G)'$ to denote the commutator subgroup of the verbal subgroup $w(G)$.

THEOREM 1.1. *Let w be a group-word and let G be an $FC(w)$-group. Then $w(G)'$ is FC-embedded in G.*

THEOREM 1.2. *Let w be a group-word and let G be a $BFC(w)$-group. Then $w(G)'$ is BFC-embedded in G.*

In the course of proving the above theorems we establish the following facts that seem to be of independent interest (see Proposition 2.9 in the next section). *The group G is an $FC(w)$-group (respectively, $BFC(w)$-group) if and only if it is an $FC(w^{-1})$-group (respectively, $BFC(w^{-1})$-group).*

Throughout the paper, we use the term '$\{a, b, c, \ldots\}$-bounded' to mean 'bounded from above by some function depending only on the parameters $a, b, c, \ldots$'. Moreover, if X is a finite set, the 'order of X' means 'the number of elements in X'.

2. Preliminary results

We start with some lemmas concerning arbitrary groups. The first one is well known (see, for instance, [3, Proposition 1]).

LEMMA 2.1. *Let w be a group-word and let G be a group such that $|G_w| = m$. Then $w(G)'$ has finite m-bounded order.*

LEMMA 2.2. *Let w be a group-word, let x be an element of a group G and let A be a subset of G_w with $x^{G_w} = \{x^a \mid a \in A\}$. Then, for any $j \geq 1$ and $y_1, \ldots, y_j \in G_w$, there exist $a_1, \ldots, a_j \in A$ such that $x^{y_1 \cdots y_j} = x^{a_1 \cdots a_j}$.*

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PROOF. We argue by induction on j. The case $j = 1$ is clear. Let $j > 1$ and assume that $x^{y_1 \dots y_{j-1}} = x^{a_1 \dots a_{j-1}}$ with $a_1, \dots, a_{j-1} \in A$. Then

$$x^{y_1 \dots y_j} = x^{a_1 \dots a_{j-1} y_j} = x^{{y_j}^b a_1 \dots a_{j-1}},$$

where $b = (a_1 \dots a_{j-1})^{-1}$. Since ${y_j}^b \in G_w$, $x^{{y_j}^b} = x^{a_j}$ for some $a_j \in A$, and so $x^{y_1 \dots y_j} = x^{a_j a_1 \dots a_{j-1}}$. After renumbering the w-values a_i, we obtain the required result. □

Recall that a simple commutator of weight $k \geq 1$ in elements of a subset A of a group is a left-normed commutator

$$[b_1, b_2, b_3, \dots, b_k] = [\dots[[b_1, b_2], b_3], \dots, b_k],$$

where each $b_i \in A$.

LEMMA 2.3. *Let $x, b_1, \dots, b_j$ be elements of a group G. Then the simple commutator $[x, b_1, \dots, b_j]$ can be written as a product of 2^j conjugates $x^{\pm d_i}$, where each d_i is a product of at most j factors from the set $\{b_1, \dots, b_j\}$. Here the product of zero factors is understood as the trivial element.*

PROOF. This is an easy induction on j, using the fact that $[x, b] = x^{-1} x^b$. □

As usual, the centre of a group G is denoted by $Z(G)$.

LEMMA 2.4. *Let G be a group such that the commutator subgroup of $G/Z(G)$ has finite order m, and let A be a finite subset of G with $|A| = n$. Then the set of simple commutators in elements of A has finite $\{m, n\}$-bounded order. Moreover, any simple commutator in elements of A has weight at most $m + 1$.*

PROOF. Let X be the set of all simple commutators in elements of A. By hypothesis, X is finite modulo $Z(G)$. Let $\{x_1, \dots, x_l\}$ be a maximal subset of X consisting of commutators which are pairwise distinct modulo $Z(G)$. Clearly, $l \leq m + n$. Put

$$Y = A \cup \{[x_k, a] \mid a \in A, 1 \leq k \leq l\}.$$

Thus $|Y| \leq n(l + 1)$. We will show that $X \subseteq Y$. Let $x = [a_1, \dots, a_j] \in X$, with $a_i \in A$. If $j = 1$, then $x \in A \subseteq Y$. Assume that $j > 1$. Since $[a_1, \dots, a_{j-1}] = x_k z$ for some k with $1 \leq k \leq l$ and $z \in Z(G)$,

$$x = [[a_1, \dots, a_{j-1}], a_j] = [x_k z, a_j] = [x_k, a_j] \in Y.$$

Thus, indeed, $X \subseteq Y$. Obviously, $Y \subseteq X$ and so $X = Y$. It follows that X has at most $n(m + n + 1)$ elements.

We will now show that every element of X has weight at most $m + 1$. Since the commutator subgroup of $G/Z(G)$ has order m, the commutators $x_1, \dots, x_l$ can be chosen of weight at most m. On the other hand, by the above, any $x \in X \backslash A$ can be written as $[x_k, a]$ for some $k \in \{1, \dots, l\}$ and $a \in A$. Therefore x has weight at most $m + 1$. □

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Given an infinite subgroup H of a group G and an element $a \in G$, it may happen that $H^a < H$ and $H^a \neq H$. Indeed, let $n \geq 2$ be an integer and let α be the automorphism of the additive group of rational numbers $\mathbb{Q}$ sending every $x \in \mathbb{Q}$ to nx. Obviously, $\mathbb{Z}^\alpha = n\mathbb{Z}$ and so $\mathbb{Z}^\alpha < \mathbb{Z}$. Our next lemma gives a sufficient condition under which the containment $H^a \leq H$ implies the equality $H^a = H$.

Lemma 2.5. *Let H be a subgroup of a group G and N a normal subgroup of G such that the commutator subgroup of $N/Z(N)$ is finite. Suppose that $H^a \leq H$ for some $a \in N$. Then $H^a = H$.*

Proof. It is enough to prove that $H^{a^{-1}} \leq H$. Suppose, on the contrary, that there exists $h \in H$ such that $h^{a^{-1}} \notin H$. Since $[h, a] \in H$ and $[h, a^{-1}] \notin H$,

$$[a, [a, h]] = [h, a]^a [h, a]^{-1} \in H \cap N'$$

and

$$[a, [a, h]]^{a^{-1}} = [a, h, a^{-1}] = [h, a][a, h]^{a^{-1}} = [h, a][h, a^{-1}] \notin H \cap N'.$$

Note that $(H \cap N')^a \leq H \cap N'$ and N' is finite modulo $Z(N)$, so $(H \cap N')^a = H \cap N'$ modulo $Z(N)$. It follows that $[a, [a, h]]^{a^{-1}} = h_1 z$ for some $h_1 \in H \cap N'$ and $z \in Z(N)$. Hence $[a, [a, h]] = {h_1}^a z$. Since ${h_1}^a \in H \cap N'$, we conclude that $z \in H \cap N'$. In particular, $[a, [a, h]]^{a^{-1}} \in H \cap N'$, which is a contradiction. □

Let w be a group-word and let G be a group. A subgroup H of $w(G)$ is said to have *finite w-index* if the elements of G_w lie in finitely many right cosets of H in $w(G)$. The subgroup H has finite w-index m if there are exactly m right cosets of H in $w(G)$ containing elements of G_w. The centraliser $C_{w(G)}(x)$ has finite w-index m if and only if $|x^{G_w}| = m$. Thus, G is an $FC(w)$-group if and only if $C_{w(G)}(x)$ has finite w-index for all $x \in G$. Further, G is a $BFC(w)$-group if and only $C_{w(G)}(x)$ has finite w-index bounded by a constant which does not depend on the choice of $x \in G$.

The following lemma is taken from [4]. Its proof is straightforward.

Lemma 2.6. *Let w be a group-word, let G be a group and let $H_1, \dots, H_n$ be subgroups of $w(G)$ having finite w-indices $m_1, \dots, m_n$, respectively. Then $\bigcap_{i=1}^n H_i$ has finite w-index at most $m_1 \dots m_n$.*

Lemma 2.7. *Let w be a group-word.*

(i) *If G is a finitely generated $FC(w)$-group, then the set $(G/Z(G))_w$ is finite.*
(ii) *If G is an n-generator $BFC(w)$-group such that $|x^{G_w}| \leq m$ for all $x \in G$, then the set $(G/Z(G))_w$ has finite order at most m^n.*

Proof. Write $G = \langle x_1, \dots, x_n \rangle$. Since G is an $FC(w)$-group, for every $i = 1, \dots, n$ the subgroup $C_{w(G)}(x_i)$ has finite w-index, say, m_i. It is clear that

$$w(G) \cap Z(G) = \bigcap_{1 \leq i \leq n} C_{w(G)}(x_i).$$

Thus, by Lemma 2.6, $w(G) \cap Z(G)$ has finite w-index at most $m_0 = m_1 m_2 \dots m_n$. It follows that $(G/Z(G))_w$ has at most m_0 elements. Finally, if $|x^{G_w}| \leq m$ for all $x \in G$, then $m_i \leq m$ for all $i = 1, \dots, n$. Hence $m_0 \leq m^n$. □

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Note that, for any element g of a group G, $g \in G_w$ if and only if $g^{-1} \in G_{w^{-1}}$. Thus $w(G) = w^{-1}(G)$ and so $C_{w(G)}(x) = C_{w^{-1}(G)}(x)$ for all $x \in G$. We do not know whether the condition that $C_{w(G)}(x)$ has finite w-index necessarily implies that $C_{w(G)}(x)$ also has finite w^{-1}-index. Our next goal is to show that this is true in the case where G is an $FC(w)$-group.

Lemma 2.8. *Let w be a group-word and let x be an element of a group G such that $C_{w(G)}(x)$ has finite w-index m. Suppose that $C_{w(G)}(x)$ contains a subgroup N of finite index l which is normal in $w(G)$. Then the w^{-1}-index of $C_{w(G)}(x)$ is at most ml.*

Proof. Put $C = C_{w(G)}(x)$. We have $G_w \subseteq \bigcup_{i=1}^{m} Cg_i$ and $C = \bigcup_{j=1}^{l} c_jN$, for some $g_i \in G_w$ and $c_j \in C$. Then $G_w \subseteq \bigcup_{i,j} c_jNg_i$ and so $G_{w^{-1}} \subseteq \bigcup_{i,j} g_i^{-1}Nc_j^{-1}$. Since N is normal in $w(G)$, we get $G_{w^{-1}} \subseteq \bigcup_{i,j} Cg_i^{-1}c_j^{-1}$. □

The next proposition provides the main technical tool for the proof of our main results.

Proposition 2.9. *Let $w = w(x_1, \ldots, x_n)$ be a group-word.*

(i) *The group G is an $FC(w)$-group if and only if it is an $FC(w^{-1})$-group.*
(ii) *The group G is a $BFC(w)$-group if and only if it is a $BFC(w^{-1})$-group. More precisely, if G is a $BFC(w)$-group such that $C_{w(G)}(x)$ has w-index at most m for all $x \in G$, then $C_{w(G)}(x)$ has finite $\{m, n\}$-bounded w^{-1}-index.*

Proof. We will deal only with the statement (ii), since the proof of (i) can be obtained in the same way by simply forgetting the bounds. More precisely, assuming that G is a $BFC(w)$-group such that $C_{w(G)}(x)$ has finite w-index at most m for all $x \in G$, we will prove that $C_{w(G)}(x)$ has finite $\{m, n\}$-bounded w^{-1}-index for all $x \in G$.

Let M be the monoid generated by G_w, that is, the set of all finite products of w-values in G (here, and in the subsequent work, the empty product stands for the element 1). Take any $x \in G$ and put $H = \langle x^M \rangle$. Choose elements $a_1, \ldots, a_m \in G_w$ such that $x^{G_w} = \{x^{a_1}, \ldots, x^{a_m}\}$. Let $A = \{a_1, \ldots, a_m\}$ and denote by A_0 the set of all simple commutators of the form $[x, b_1, \ldots, b_j]$, with $j \geq 1$ and $b_1, \ldots, b_j \in A$. Note that $[x, b_1, \ldots, b_j] \in H$ by Lemma 2.3. Hence $\langle x, A_0 \rangle \leq H$. We claim that $H = \langle x, A_0 \rangle$. For any $j \geq 1$, $x^{b_1 \cdots b_j} \in \langle x, A_0 \rangle$ for any $b_1, \ldots, b_j \in A$. In fact, $x^{b_1} = x[x, b_1] \in \langle x, A_0 \rangle$ and, if $j > 1$, the induction hypothesis implies that $x^{b_1 \cdots b_j} = (x^{b_1 \cdots b_{j-1}})^{b_j} \in \langle x, A_0 \rangle^{b_j} \leq \langle x, A_0 \rangle$. On the other hand, for any $j \geq 1$ and $y_1, \ldots, y_j \in G_w$, by Lemma 2.2, $x^{y_1 \cdots y_j} = x^{b_1 \cdots b_j}$ for some $b_1, \ldots, b_j \in A$. Thus $x^{y_1 \cdots y_j} \in \langle x, A_0 \rangle$ and therefore $H = \langle x, A_0 \rangle$, as claimed.

Put $C = C_{w(G)}(x)$. Since w depends on n variables and C has finite w-index m, we can choose a subgroup J generated by at most $mn + 1$ elements of G with $x \in J$ and $a_1, \ldots, a_m \in J_w$. By (ii) of Lemma 2.7, $(J/Z(J))_w$ has finite $\{m, n\}$-bounded order. Then, by Lemma 2.1, $w(J)'$ has finite $\{m, n\}$-bounded order modulo $Z(J)$. Applying Lemma 2.5, we get $H^a = H$ for all $a \in A$, from which it follows that $H^y = H$ for all $y \in G_w$. Indeed, for any $y \in G_w$, there exists $a \in A$ such that $y \in Ca$. Of course, both C and a normalise H. Hence $y \in N_G(H)$, as required. Thus H is normal in $w(G)$.

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Let B be the subgroup generated by A and all commutators $[x, a]$ with $a \in A$. Since $B \leq w(J)$, the commutator subgroup of $B/Z(B)$ has finite $\{m, n\}$-bounded order. By Lemma 2.4 used with B in place of G, there are only $\{m, n\}$-boundedly many commutators of the form $[x, b_1, \ldots, b_j]$, with $j \geq 1$ and $b_1, \ldots, b_j \in A$. Moreover, the weight of these commutators is at most some $\{m, n\}$-bounded number k. Since $H = \langle x, A_0\rangle$, it follows from Lemma 2.3 that

$$H = \langle x^d \mid d \in D\rangle,$$

where D is the set of all products of at most k factors from A. Obviously, D is finite with $\{m, n\}$-boundedly many elements. Let S be the set of all right cosets of C in $w(G)$ containing products of at most k factors from G_w. By Lemma 2.2, S is precisely the set of all right cosets of C containing products of at most k factors from A. Write $S = \{Cd_1, \ldots, Cd_s\}$, where $d_i \in D$ and s is $\{m, n\}$-bounded. Clearly, C acts by conjugation on the set S. Denote by K the kernel of this action. Then the index $|C : N|$ is finite and depends only on s, so it is $\{m, n\}$-bounded.

We claim that $K \leq C_{w(G)}(H)$. For any $c \in K$, $Cd_i^c = Cd_i$ for all $i = 1, \ldots, s$. Therefore $d_i d_i^{-c} \in C$ and thus $c \in C^{d_i}$ for all i, and hence $K \leq C \cap C^{d_1} \cap \cdots \cap C^{d_s}$. Now let $c \in C \cap C^{d_1} \cap \cdots \cap C^{d_s}$. For any $d \in D$, there exists $i \in \{1, \ldots, s\}$ such that $Cd = Cd_i$, so $x^d = x^{d_i}$ and $[c, x^d] = 1$. Hence $[c, H] = 1$, which proves that $C \cap C^{d_1} \cap \cdots \cap C^{d_s} \leq C_{w(G)}(H)$. Consequently, $K \leq C \cap C^{d_1} \cap \cdots \cap C^{d_s} \leq C_{w(G)}(H)$.

Finally, put $N = C_{w(G)}(H)$. Since H is normal in $w(G)$, the subgroup N is normal in $w(G)$. Also, by the above, $K \leq N$. It follows that $|C : N| \leq |C : K|$ is $\{m, n\}$-bounded. By Lemma 2.8, we conclude that $C_{w(G)}(x)$ has finite $\{m, n\}$-bounded w^{-1}-index, as required. □

3. Proofs of Theorems 1.1 and 1.2

Lemma 3.1. *Let $w = w(x_1, \ldots, x_n)$ be an arbitrary group-word and let*

$$v = [w(x_1, \ldots, x_n), w(x_{n+1}, \ldots, x_{2n})].$$

Let G be a BFC(w)-group such that $|x^{G_w}| \leq m$ for all $x \in G$. Then G is a BFC(v)-group such that x^{G_v} has only $\{m, n\}$-boundedly many elements for all $x \in G$.

Proof. Let $y \in G_v$. We have $y = zt$, where $z = w(g_1, \ldots, g_n)^{-1} \in (G_w)^{-1} = G_{w^{-1}}$ and $t = w(g_1, \ldots, g_n)^{w(g_{n+1}, \ldots, g_{2n})} \in G_w$, for some $g_i \in G$. Given $x \in G$, put $C = C_{w(G)}(x)$ and let $a_1, \ldots, a_m \in G_w$ be such that $x^{G_w} = \{x^{a_1}, \ldots, x^{a_m}\}$. Thus $G_w \subseteq \bigcup_{1=1}^{m} Ca_i$. Furthermore, by Proposition 2.9(ii), C has finite $\{m, n\}$-bounded w^{-1}-index, say, m'. So there exist $b_1, \ldots, b_{m'} \in G_{w^{-1}}$ such that $G_{w^{-1}} \subseteq \bigcup_{j=1}^{m'} Cb_j$. It follows that $z = c_1 b_j$ and $t = c_2 a_i$ for some $c_1, c_2 \in C$ and i, j. Hence

$$x^y = x^{c_1 b_j c_2 a_i} = x^{b_j^{c_2} a_i} = x^{b_k a_i},$$

where $x^{b_j^{c_2}} = x^{b_k}$, $1 \leq k \leq m'$ and $1 \leq i \leq m$. This proves the result. □

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Lemma 3.2. *Let $w = w(x_1, \dots, x_n)$ be an arbitrary group-word and let*

$$v = [w(x_1, \dots, x_n), w(x_{n+1}, \dots, x_{2n})].$$

Let G be a $BFC(w)$-group such that $|x^{G_w}| \le m$ for all $x \in G$. There exists an $\{m, n\}$-bounded positive integer e such that $y^e \in Z(G)$ for any $y \in G_v$.

Proof. Take any $y \in G_v$. Then there exist $y_1, \dots, y_{2n} \in G$ such that

$$y = [w(y_1, \dots, y_n), w(y_{n+1}, \dots, y_{2n})].$$

Let y_0 be an arbitrary element of G, and put $J = \langle y_i \mid 0 \le i \le 2n \rangle$. Of course, $y \in v(J)$, and J is a finitely generated $BFC(w)$-group such that $|x^{J_w}| \le m$, for all $x \in J$. By Lemma 2.7(ii), the set $(J/Z(J))_w$ has finite order at most m^{2n+1}. Hence, by Lemma 2.1, the commutator subgroup of $w(J/Z(J))$ has finite $\{m, n\}$-bounded order. In particular, $v(J)$ has finite $\{m, n\}$-bounded order modulo $Z(J)$, say, e. Therefore $y^e \in Z(J)$ and $[y^e, y_0] = 1$. □

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Recall that $w = w(x_1, \dots, x_n)$ is a group-word and that G is a $BFC(w)$-group. We will prove that $w(G)'$ is BFC-embedded in G.

Set $v = [w(x_1, \dots, x_n), w(x_{n+1}, \dots, x_{2n})]$ and note that $v(G) = w(G)'$. Assume that $|x^{G_w}| \le m_0$ for all $x \in G$. By Lemma 3.1, G is a $BFC(v)$-group such that x^{G_v} has $\{m_0, n\}$-boundedly many elements, say, at most m, for all $x \in G$. Let x be an arbitrary element of G and choose $a_1, \dots, a_m \in G_v$ such that $x^{G_v} = \{x^{a_1}, \dots, x^{a_m}\}$. Define an order $<$ on the set of all (formal) products of the form $a_{i_1} \dots a_{i_j}$, with $1 \le i_k \le m$ and $j \ge 1$, as follows. Put

$$a_{i_1} \dots a_{i_j} < a_{i'_1} \dots a_{i'_{j'}} \tag{*}$$

if and only if one of the following conditions is satisfied: $j < j'$, or $j = j'$ and there is a positive integer $l \le j$ such that $i_l < i'_l$ and $i_k = i'_k$ for all $k > l$.

Let y be an arbitrary element of $v(G)$. Then $y = y_1 \dots y_j$, where each $y_i \in G_v \cup G_v^{-1}$. It is easy to see that the word v has the property that $G_v^{-1} = G_v$ and so each $y_i \in G_v$. Lemma 2.2 tells us that

$$x^y = x^{a_{i_1} \dots a_{i_j}},$$

with $1 \le i_k \le m$. Clearly, we can choose $a_{i_1} \dots a_{i_j}$ to be the smallest (in the sense of the order $<$) product of elements from $\{a_1, \dots, a_m\}$ such that $x^y = x^{a_{i_1} \dots a_{i_j}}$. Let us now show that $i_1 \ge i_2 \ge \dots \ge i_j$. Suppose that $i_k < i_{k+1}$ for some k. Then

$$x^y = x^{a_{i_1} \dots a_{i_{k-1}} a_{i_k} a_{i_{k+1}} a_{i_{k+2}} \dots a_{i_j}} = x^{a_{i_1} \dots a_{i_{k-1}} b a_{i_k} a_{i_{k+2}} \dots a_{i_j}},$$

where $b = a_{i_k} a_{i_{k+1}} a_{i_k}^{-1} \in G_v$. In view of Lemma 2.2,

$$x^{a_{i_1} \dots a_{i_{k-1}} b} = x^{a_{i'_1} \dots a_{i'_{k-1}} a_{i'_{k+1}}}$$

for some $1 \le i'_1, \dots, i'_{k-1}, i'_{k+1} \le m$, so that

$$x^y = x^{a_{i'_1} \dots a_{i'_{k-1}} a_{i'_{k+1}} a_{i_k} a_{i_{k+2}} \dots a_{i_j}}.$$

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This contradicts the choice of the product $a_{i_1} \dots a_{i_j}$ because

$$a_{i_1} \dots a_{i_{k-1}} a_{i_k} a_{i_{k+1}} a_{i_{k+2}} \dots a_{i_j} > a_{i'_1} \dots a_{i'_{k-1}} a_{i'_{k+1}} a_{i_k} a_{i_{k+2}} \dots a_{i_j}.$$

Thus $x^y = x^{a_{i_1} \dots a_{i_j}}$ with $i_1 \geq i_2 \geq \dots \geq i_j$ or, equivalently,

$$x^y = x^{a_m{}^{e_m} \dots a_1{}^{e_1}}$$

for some nonnegative integers $e_m, \dots, e_1$.

Finally, by Lemma 3.2, there exists an $\{m_0, n\}$-bounded positive integer e such that $a_i{}^e \in Z(G)$, for all i. Thus, we may assume that $e_i < e$ for all i. Hence $|x^{v(G)}| \leq e^m$, and $v(G)$ is BFC-embedded in G. □

The following two results are the analogues of Lemmas 3.1 and 3.2 for $FC(w)$-groups.

Lemma 3.3. *Let $w = w(x_1, \dots, x_n)$ be an arbitrary group-word and let*

$$v = [w(x_1, \dots, x_n), w(x_{n+1}, \dots, x_{2n})].$$

If G is an $FC(w)$-group, then G is an $FC(v)$-group.

Proof. The proof is similar to that of Lemma 3.1. The modifications required are evident and therefore we omit the details. □

Lemma 3.4. *Let $w = w(x_1, \dots, x_n)$ be an arbitrary group-word and let*

$$v = [w(x_1, \dots, x_n), w(x_{n+1}, \dots, x_{2n})].$$

Let $A = \{a_1, \dots, a_m\}$ be a finite subset of G_v. If G is an $FC(w)$-group, then, for any $x \in G$, there exists a positive integer e such that $a_i{}^e \in Z(\langle x, A\rangle)$ for all $a_i \in A$.

Proof. Let x be an arbitrary element of G. For any $a_i \in A$, there exist $g_{i,1}, \dots, g_{i,2n} \in G$ such that

$$a_i = [w(g_{i,1}, \dots, g_{i,n}), w(g_{i,n+1}, \dots, g_{i,2n})].$$

Put $J = \langle x, g_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq 2n\rangle$. Of course, each $a_i \in v(J)$ and, by Lemma 2.7(i), the set $(J/Z(J))_w$ is finite. Thus, by Lemma 2.1, the commutator subgroup of $w(J/Z(J))$ is finite. It follows that $v(J)$ has finite order modulo $Z(J)$, say, e. Therefore $a_i{}^e \in Z(J)$ for all i. As $\langle x, A\rangle \leq J$, the result follows. □

Proof of Theorem 1.1. Recall that w is a group-word and that G is an $FC(w)$-group. We need to prove that $w(G)'$ is FC-embedded in G.

Set $v = [w(x_1, \dots, x_n), w(x_{n+1}, \dots, x_{2n})]$. Clearly, $v(G) = w(G)'$ and, by Lemma 3.3, G is an $FC(v)$-group. Let x be an arbitrary element of G and choose $a_1, \dots, a_m \in G_v$ such that $x^{G_v} = \{x^{a_1}, \dots, x^{a_m}\}$. Define the order $<$ on the set of all (formal) products of the form $a_{i_1} \dots a_{i_j}$, with $1 \leq i_k \leq m$ and $j \geq 1$, as in (*) in the proof of Theorem 1.2.

Let y be an arbitrary element of $v(G)$. Arguing as in the proof of Theorem 1.2, write $x^y = x^{a_m{}^{e_m} \dots a_1{}^{e_1}}$ for some nonnegative integers $e_m, \dots, e_1$. If $A = \{a_1, \dots, a_m\}$, by Lemma 3.4, there exists a positive integer e such that $a_i{}^e \in Z(\langle x, A\rangle)$ for all i. Hence we may assume that $e_i < e$ for all i, and so $|x^{v(G)}| \leq e^m$. Thus $x^{v(G)}$ is finite for all $x \in G$. We conclude, therefore, that $v(G)$ is FC-embedded in G. □

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References

[1] S. Brazil, A. Krasilnikov and P. Shumyatsky, 'Groups with bounded verbal conjugacy classes', *J. Group Theory* **9**(1) (2006), 127–137.

[2] G. A. Fernández-Alcober and M. Morigi, 'Outer commutator words are uniformly concise', *J. Lond. Math. Soc. (2)* **82**(3) (2010), 581–595.

[3] G. A. Fernández-Alcober, M. Morigi and G. Traustason, 'A note on conciseness of Engel words', *Comm. Algebra* **40**(7) (2012), 2570–2576.

[4] S. Franciosi, F. de Giovanni and P. Shumyatsky, 'On groups with finite verbal conjugacy classes', *Houston J. Math.* **28**(4) (2002), 683–689.

[5] R. M. Guralnick and A. Maróti, 'Average dimension of fixed point spaces with applications', *Adv. Math.* **226**(1) (2011), 298–308.

[6] B. H. Neumann, 'Groups with finite classes of conjugate elements', *Proc. Lond. Math. Soc. (3)* **1** (1951), 178–187.

[7] P. M. Neumann and M. R. Vaughan-Lee, 'An essay on BFC groups', *Proc. Lond. Math. Soc. (3)* **35**(2) (1977), 213–237.

[8] D. J. R. Robinson, *Finiteness Conditions and Generalized Soluble Groups, Part 1* (Springer, New York, Berlin, 1972).

[9] D. Segal and A. Shalev, 'On groups with bounded conjugacy classes', *Quart. J. Math. Oxford Ser. (2)* **50** (1999), 505–516.

[10] J. Wiegold, 'Groups with boundedly finite classes of conjugate elements', *Proc. R. Soc. Lond. A* **238** (1957), 389–401.

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