A NEW URN MODEL

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Abstract

In this paper, we propose a new urn model. A single urn contains b black balls and w white balls. For each observation, we randomly draw m balls and note their colors, say k black balls and $m - k$ white balls. We return the drawn balls to the urn with an additional ck black balls and $c(m - k)$ white balls. We repeat this procedure n times and denote by X_n the fraction of black balls after the nth draw. To investigate the asymptotic properties of X_n, we first perform some computational studies. We then show that $\{X_n\}$ forms a martingale, which converges almost surely to a random variable X. The distribution of X is then shown to be absolutely continuous.

Keywords: Urn model; martingale; symmetric polynomial; absolute continuity for a random variable

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1. Introduction

An urn model is constructed by imagining a number of urns, some or all containing balls of various colors. Sequences of experiments (trials) in which balls are drawn from and possibly returned to the urns according to certain rules are considered. These rules may include requirements for the addition to or removal of balls from certain urns at various stages of the experiment. Why have people investigated urn models? As explained by Kotz and Balakrishnan [10], there are three reasons. Firstly, urn models give an efficient way to describe the concept of 'random choice'. Secondly, urns and chance experiments can be compounded into new ones. Thirdly, as Pólya ably displayed over thirty years ago, an urn model allows a complex random (chance) process to be 'simulated'. Here, the term 'simulation' can be interpreted as a statistical equivalent to the mathematical concept of an *isomorphism*, which is essentially associated with urn models. As a result, an urn model plays a fundamental role in many problems.

The study of urn models has a long history. In 1657, Huygens proposed problems about urns in his treatise (see [11, pp. 48–55]), but he did not use the term 'urn'. James Bernoulli (1713) may have been the first person to mention problems in the language of urns (see [10]). Bernoulli discussed the problem, in the third book of his *Ars Conjectandi*, of drawing 'calculi' out of urns. The results on urn models up to 1977 were summarized in the book *Urn Models and*

Their Application [9, pp. 176–181]. This book stimulated many probabilists and statisticians to investigate different kinds of urn model. After two decades, Kotz and Balakrishnan published the survey paper [10]. This covered almost all kinds of urn model and their properties.

This paper is organized as follows. In Section 2, a brief survey on Pólya–Eggenberger models, taken from published material, is given. The model formulations and the asymptotic properties of the corresponding urn models are especially emphasized. In Section 3, a new urn model that draws m balls each time from an urn with black and white balls is proposed. Some computational studies are also conducted to compare the fitted beta distribution with the distribution of the fraction X_n of black balls in our models. These are illustrated in figures. In Section 4, $\{X_n\}$ is shown to form a martingale that converges almost surely (a.s.) to a random variable X. The absolute continuity of X is then proved through combinatorial and algebraic arguments. The complete characterization of X still remains to be resolved in the future. In the appendix, more figures are used to illustrate the effects of different initial numbers of black balls b and white balls w, and different numbers of balls drawn m.

2. The Pólya–Eggenberger urn model and its generalizations

In this section, we will sketch the Pólya–Eggenberger urn model and its generalizations and modifications.

Eggenberger and Pólya [3], in 1923, proposed an urn model with a single urn initially containing b black balls and w white balls. A ball is drawn at random and then replaced together with c balls of the same color. The procedure is repeated n times and we denote by X_n the proportion of black balls after the nth draw. It is known that $\{X_n\}$ is a martingale. Furthermore, $\lim_{n\to\infty} X_n = X$ exists, and X has a beta distribution with parameters b/c and w/c (see [4, p. 226]).

In 1980, Hill *et al.* [8] proposed a generalized Pólya–Eggenberger urn model. It was the same as the Pólya–Eggenberger urn model except that the probability of drawing balls, instead of being $B_n/(W_n + B_n)$, was $f(B_n/(W_n + B_n))$, where B_n and W_n denote the number of black balls and white balls, respectively, at the nth stage, and f is any function mapping $[0, 1]$ into itself. They showed that, under a condition on f at p_0, where $p_0 \in \{x : f(x) = x\}$, the fraction of black balls converges to p_0 with positive probability. Moreover, if p_0 is an unstable fixed point of f (i.e. $f(p_0) = p_0$ and, in some neighborhood of p_0, $f(x) < x$ for $x < p_0$ and $f(x) > x$ for $x > p_0$) then $P(B_n/(W_n + B_n) \to p_0) = 0$.

In 1985, Bagchi and Pal [1] defined a tenable Pólya–Eggenberger urn model which described a single urn with b_0 black balls and $t_0 - b_0$ white balls, where $b_0 > 1$ and $t_0 > 1$. A ball is drawn at random. Its color is noted and the ball is returned to the urn. If the color is black, then a black balls and $b > 0$ white balls are added to the urn. Otherwise, $c > 0$ black balls and d white balls are added to the urn. Now, let $a + b = c + d \geq 1$, $a \neq c$. For the tenable urn process, Gouet [5] used martingale arguments to show that if $\max(b, c) > 0$, then the fraction of black balls converges almost surely to $c/(b + c)$. Four years later, by using martingale transforms and the functional central limit theorem for the tenable urn process, Gouet [6] provided weak convergence results for the sequence of processes $B_{k_n(t)} - c(B_{k_n(t)} + W_{k_n(t)})/(b + c)$, $n \in \mathbb{N}$, $t \geq 0$, where $\{k_n(t)\}$ is a sequence of deterministic time scales.

In 1990, Pemantle [12] generalized the Pólya–Eggenberger urn model by replacing c with a function of time. He showed that the proportions of black balls converges a.s. and the limit has no atoms except possibly at 0 or 1. Necessary and sufficient conditions for the limit to concentrate entirely on the set $\{0, 1\}$ were given.

In 2001, Schreiber [13] studied the asymptotic behavior of a class of generalized Pólya–Eggenberger urn model by using an ordinary differential equation (ODE) method. He also formulated a generalized Pólya–Eggenberger urn model called the replicator process.

> Two balls are selected at random with replacement from an urn and new balls are added or removed, at varying rates, according to probabilities that depend only on the colors of the chosen balls.

Schreiber proved that

(i) the number of balls increases asymptotically at a linear rate,

(ii) the distribution $x(n)$ of the strategies at the nth update is a noisy Cauchy–Euler approximation to the mean limit ODE of the process, and

(iii) the limit set of $x(n)$ is a.s. a connected, internally chain-recurrent set for the mean limit ODE.

3. A new urn model and some computational studies

In this section, we briefly introduce the Pólya–Eggenberger urn model and its generalizations and modifications. Drawing one ball at random each time is the common property of these models. We now introduce a new model in which more than one ball is drawn randomly each time.

Model M. A single urn contains b black balls and w white balls. We draw m balls at random, say k black balls and $m - k$ white balls, and their colors are noted. Return these balls to the urn with ck black balls and $c(m - k)$ white balls. Repeat the procedure n times.

To gain some insight into this new model, we perform some computational studies. Let Y_n be the number of black balls after the nth draw. Using the Markov property of $\{Y_n\}$ (see (4.1), below), the probability $P(Y_n = k)$ is computed recursively. The result is shown as solid curves in Figures 1 and 2. The dashed curves show the density of the fitted beta distribution. Here, the parameters of the beta distributions are estimated by the least-squares method assuming that $\{P(Y_n = k)(b + w + cmn)\}$ is the data set. All programs are executed using MAPLE®6.

The graphs for the fractions of black balls are obtained under the condition $c = 1$. In Figures 1 and 2, we denote our new model by $M(b, w, m)$ and a beta distribution with parameters α and β by beta(α, β). Figures 1 and 2 demonstrate the effects of different values of b, w, and m. More experiments are given in the appendix. Note that in Figures 1 and 2 the values of b are less than or equal to w. If we were to exchange the values of b and w, the new graphs would be symmetric with the old ones with respect to the line $x = 0.5$. Therefore, we do not consider the cases with $b > w$.

In Figure 1, we can see that the peak of the curve is higher and skews to the left if the value of w increases and $w \geq b$. This is reasonable since the probability of drawing white balls is getting larger and the fraction of black balls is getting smaller.

If we compare Figure 1(a) with Figure 2, it is easy to see that the peaks get higher and higher as b increases. Moreover, if $b = w = m$ then $M(b, b, b)$ is close to beta(α, α) for some α. It
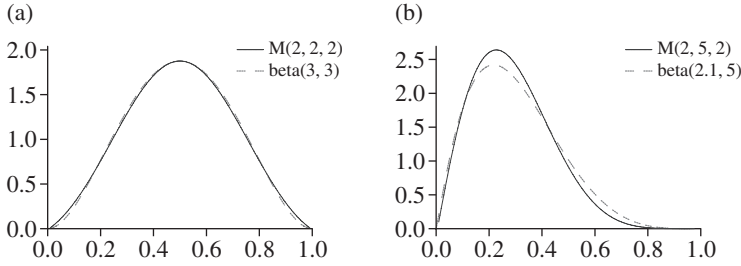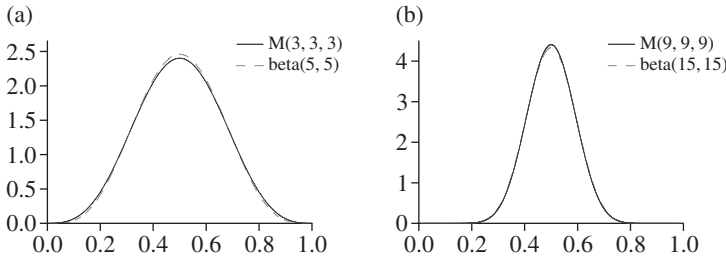
FIGURE 1: Two examples with $n = 100$.

FIGURE 2: Two examples with $n = 100$.

would be of interest to see the relationship between b and α. Figures 1 and 2 seem to indicate that the general limiting distributions for $m > 1$ may not be beta distributed, as they are for $m = 1$. Although they cannot help us in guessing the exact behavior of the limiting distribution at this moment, they do provide some insight. For example, they indicate the existence of the limiting distributions and the densities. These will be shown in Section 4.

4. Martingale property and absolute continuity of the limit

In this section, we show that in Model M the fraction of black balls forms a martingale. This martingale converges a.s. to a limit X. We then prove that X is absolutely continuous.

4.1. Martingale

Recall our new urn model defined in Section 3. Let Y_n be the number of black balls after n draws and let $X_n = Y_n/\gamma_n$ be the fraction of black balls, where $\gamma_n = b + w + cmn$ is the total number of balls after n draws. Also, let $\mathcal{F}_n$ be the σ-field generated by $Y_1, Y_2, \ldots, Y_n$.

Theorem 4.1. *For Model M, $\{X_n, \mathcal{F}_n, \ n \geq 1\}$ is a martingale, i.e. $\mathrm{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$. Furthermore, X_n converges a.s. to a random variable X with $0 \leq X \leq 1$.*

Proof. It is clear that

$$P(Y_{n+1} = j \mid Y_n = k, \ Y_{n-1} = k_{n-1}, \ldots, Y_0 = b) = P(Y_{n+1} = j \mid Y_n = k). \tag{4.1}$$

Thus, $\{Y_n\}$ is a discrete-time Markov process, and so is $\{X_n\}$. Since $\{X_n\}$ is a discrete-time Markov process, $\mathrm{E}[X_{n+1} \mid \mathcal{F}_n] = \mathrm{E}[X_{n+1} \mid Y_n]$. Since $0 \leq X_n \leq 1$, $\{X_n, \mathcal{F}_n, \ n \geq 1\}$ is a martingale if we can prove that

$$\mathrm{E}[X_{n+1} \mathbf{1}_{\{Y_n=k\}}] = \mathrm{E}[X_n \mathbf{1}_{\{Y_n=k\}}],$$

for $k = b, b+1, \ldots, b + cmn$, where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. By definition,

$$
\begin{aligned}
\mathrm{E}[X_{n+1} \mathbf{1}_{\{Y_n=k\}}] &= \sum_{\ell=0}^{m} \frac{k + \ell c}{\gamma_{n+1}} \mathrm{P}(Y_{n+1} = k + \ell c \mid Y_n = k)\, \mathrm{P}(Y_n = k) \\
&= \sum_{\ell=0}^{m} \frac{k + \ell c}{\gamma_{n+1}} \binom{k}{\ell} \binom{\gamma_n - k}{m - \ell} \binom{\gamma_n}{m}^{-1} \mathrm{P}(Y_n = k),
\end{aligned}
\tag{4.2}
$$

where

$$
\binom{n}{m} = \begin{cases} \dfrac{n!}{m!\,(n-m)!} & \text{if } n \geq m, \\[2mm] 0 & \text{if } n < m. \end{cases}
$$

To simplify (4.2), we need the following two identities:

$$
\sum_{\ell=0}^{m} \binom{k}{\ell} \binom{\gamma_n - k}{m - \ell} = \binom{\gamma_n}{m},
\tag{4.3}
$$

$$
\sum_{\ell=0}^{m} \ell \binom{k}{\ell} \binom{\gamma_n - k}{m - \ell} = k \binom{\gamma_n - 1}{m - 1}.
\tag{4.4}
$$

Here, (4.3) can be derived by comparing the coefficients of the term x^m in the expressions of both sides of the identity $(1 + x)^k (1 + x)^{\gamma_n - k} = (1 + x)^{\gamma_n}$. Identity (4.4) can then be derived from (4.3). If we simplify (4.2) using (4.3) and (4.4), then we obtain

$$
\begin{aligned}
\mathrm{E}[X_{n+1} \mathbf{1}_{\{Y_n=k\}}] &= \mathrm{P}(Y_n = k) \left[k \binom{\gamma_n}{m} + kc \binom{\gamma_n - 1}{m - 1} \right] \left[\gamma_{n+1} \binom{\gamma_n}{m} \right]^{-1} \\
&= \frac{\mathrm{P}(Y_n = k) k}{\gamma_n} \\
&= \mathrm{E}[X_n \mathbf{1}_{\{Y_n=k\}}].
\end{aligned}
$$

The last identity holds since $X_n = k/\gamma_n$ when $Y_n = k$. Since $0 \leq X_n \leq 1$, $n = 1, 2, \ldots$, we can apply the martingale convergence theorem (see [7, p. 18]) to claim that there exists an X, $0 \leq X \leq 1$, such that $X_n \xrightarrow{\text{a.s.}} X$ as $n \to \infty$.

Remark 4.1. When $m = 1$, our model becomes the classical Pólya–Eggenberger urn model. It has been proved that $X_n \xrightarrow{\text{a.s.}} X$ and X has a beta distribution with parameters b/c and w/c (see [9]). Theorem 4.1 can be viewed as a generalization of this result.

Remark 4.2. Let Model M be modified as follows. 'A single urn contains balls of r different colors. For each color i, we initially have b_i balls. We draw m balls at random, say k_i balls of color i, and their colors are noted. Return these balls to the urn with ck_i balls of color i. Repeat the procedure n times.' Then Theorem 4.1 can be viewed as results about the fraction of balls of the first color for an urn model with two colors. Similarly, if we replace Y_n by $Y_{n,i}$ and X_n by $X_{n,i}$ in the proof of Theorem 4.1, we find that, for all $i = 1, \ldots, r$, $\{X_{n,i}, \mathscr{F}_n, n \geq 1\}$ is a martingale and converges a.s. Therefore, $X_n = (X_{n,1}, \ldots, X_{n,r})$ is also a martingale and converges a.s. to a random variable $X = (X_1, \ldots, X_r)$, with $0 \leq X_i \leq 1$ a.s. for $i = 1, \ldots, r$.

4.2. Absolute continuity

To understand the asymptotic performance of Model M, the distribution of X is definitely of interest. Unfortunately, we are not able to characterize the distribution of X at this moment. Instead, we will prove that X is absolutely continuous. That is, we will prove that the distribution of X has a density. This is the content of the following theorem.

Theorem 4.2. *For fixed b, w, c, and m, X is absolutely continuous.*

The analytical complexity of proving the absolute continuity is somewhat greater than that of proving the continuity of the distribution (see [15]). We adopt the following approach. Let $(\Omega, \mathcal{F}, P)$ be the probability space. We first find a sequence of events Ω_ℓ such that $\Omega_\ell \subset \Omega_{\ell+1}$ and $P(\bigcup_{\ell=1}^\infty \Omega_\ell) = 1$. Then, by restricting X to Ω_ℓ, we show that it has a density f_ℓ. Finally, we prove that $f = \lim_{\ell \to \infty} f_\ell$ exists and that f is the density of X.

We now show that our approach is valid using the following propositions.

Proposition 4.1. *Let Ω_ℓ be a sequence of increasing events such that $P(\bigcup_{\ell=1}^\infty \Omega_\ell) = 1$. If there exist nonnegative Borel measurable functions $\{f_\ell\}_{\ell \geq 1}$ such that $P(\Omega_\ell \cap X^{-1}(B)) = \int_B f_\ell(x)\, dx$ for all Borel sets B, then $f = \lim_{\ell \to \infty} f_\ell$ exists almost everywhere (a.e.) and f is the density of X.*

Proof. Since $\Omega_{\ell+1} \supset \Omega_\ell$, for all Borel sets B we obtain

$$\int_B f_{\ell+1}(x)\, dx = P(\Omega_{\ell+1} \cap X^{-1}(B)) \geq P(\Omega_\ell \cap X^{-1}(B)) = \int_B f_\ell(x)\, dx.$$

This implies that $f_{\ell+1} \geq f_\ell \geq 0$ a.e. Let $f = \lim_{\ell \to \infty} f_\ell$, where $0 \leq f \leq \infty$. The monotone convergence theorem (see [16]) ensures that $\int_B f(x)\, dx = \lim_{\ell \to \infty} \int_B f_\ell(x)\, dx$ for all Borel sets B. Moreover, since $P(\bigcup_{\ell=1}^\infty \Omega_\ell) = 1$,

$$\int_\mathbb{R} f(x)\, dx = \lim_{\ell \to \infty} \int_\mathbb{R} f_\ell(x)\, dx = \lim_{\ell \to \infty} P(\Omega_\ell) = 1.$$

Therefore, $f < \infty$ a.e. and X is absolutely continuous with density f.

The following proposition provides a way to construct the events Ω_ℓ.

Proposition 4.2. *For fixed b, w, c, and m, let*

$$\Omega_\ell = \{\omega \colon Y_\ell(\omega) \geq cm \ \text{ and } \ \gamma_\ell - Y_\ell(\omega) \geq cm\},$$

where $\gamma_n = b + w + cmn$ and $\ell \geq 1$. Then $\Omega_{\ell+1} \supset \Omega_\ell$ and $P(\bigcup_{\ell=1}^\infty \Omega_\ell) = 1$.

Proof. The assertion $\Omega_{\ell+1} \supset \Omega_\ell$ is obvious from the fact that the number of black balls Y_ℓ and the number of white balls $\gamma_\ell - Y_\ell$ are increasing with ℓ. For the second assertion, it is equivalent to show that $P(Y_\ell \geq cm$ and $\gamma_\ell - Y_\ell \geq cm$ for some $\ell) = 1$. Observe that it is sufficient to prove that $P(Y_\ell < cm$ for all $\ell) = 0$ and $P(\gamma_\ell - Y_\ell < cm$ for all $\ell) = 0$. In fact, we only have to prove that

$$P(Y_\ell < cm \text{ for all } \ell) = 0.$$

This is because, by exchanging the roles of black balls and white balls, the second statement can be proved similarly.

Consider the events E_n, 'at least one black ball is drawn in the sample of size m at time n', and F_n, 'the first ball drawn in the sample of size m at time n is black'. Then

$$P(Y_\ell < cm \text{ for all } \ell) \leq 1 - P(E_n \text{ infinitely often}).$$

From the conditional Borel–Cantelli lemma (see [7, p. 32]), we have $P(E_n \text{ infinitely often}) = 1$ if $\sum_n P(E_{n+1} \mid \mathcal{F}_n) = \infty$ a.s. Since

$$P(E_{n+1} \mid \mathcal{F}_n) \geq P(F_{n+1} \mid \mathcal{F}_n) = \frac{Y_n}{b + w + cmn} \geq \frac{b}{b + w + cmn} \quad \text{a.s.}$$

and $\sum_n (b/(b + w + cmn)) = \infty$, we have $\sum_n P(E_{n+1} \mid \mathcal{F}_n) = \infty$ a.s. Consequently, the claim is proved.

Next, we will show that by restricting X to Ω_ℓ, X has a density. It is clear that this will hold if we can show that the restriction of X to $\Omega_{\ell j} = \{\omega : Y_\ell(\omega) = j\}$ has a density for each j, with $cm \leq j \leq \gamma_{\ell-1}$. For this, we first need the following lemmas.

Lemma 4.1. *For $cm \leq j \leq \gamma_{\ell-1}$, $n \geq \ell$, and $k \leq m(n + 1)$,*

$$\sum_{i=0}^{m} \binom{j + c(k - i)}{i} \binom{\gamma_n - j - c(k - i)}{m - i} = \frac{\gamma_n^m}{m!} + \frac{(1 - m - 2c)\gamma_n^{m-1}}{2(m - 1)!} + \cdots,$$

which is a polynomial in γ_n of degree m with coefficients depending on b, w, m, and c only.

Proof. Since the proof is a little complex, we use two steps.

Step 1. For $x, y \in \mathbb{R}$, define

$$C_i^x = \begin{cases} 1 & \text{if } i = 0, \\ x(x - 1) \cdots \dfrac{(x - i + 1)}{i!} & \text{if } i > 0, \end{cases}$$

and

$$f_m(x, y) = \sum_{i=0}^{m} C_i^{x-ci} C_{m-i}^{y-c(m-i)}. \tag{4.5}$$

Note that $f_m(x, y)$ is a polynomial symmetric in x and y. We claim that, for $m, c \in \mathbb{N}$,

$$f_m(x, y) \in \mathbb{Z}[x + y], \tag{4.6}$$

where $\mathbb{Z}[x + y]$ is the ring of polynomials over $\mathbb{Z}$ in the indeterminate $x + y$ (see [14] for a definition). The remaining part of this step is to prove (4.6).

Now we proceed to prove (4.6) by induction. For $m = 1$, the claim is true since $\sum_{i=0}^{1} C_i^{x-ci} C_{1-i}^{y-c(1-i)} = x + y - 2c$. Suppose that (4.6) holds for $m = n - 1$, where $n \in \mathbb{N}$. We use the identity

$$C_i^z = C_{i-1}^{z-1} + C_i^{z-1} = C_{i-1}^{z-1} + C_{i-1}^{z-2} + C_i^{z-2} = C_{i-1}^{z-1} + C_{i-1}^{z-2} + \cdots + C_{i-1}^{z-c} + C_i^{z-c},$$

which holds for $z \in \mathbb{R}$, to obtain

$$
\begin{aligned}
f_n(x, y) &= \sum_{i=0}^{n} C_i^{x-ci} C_{n-i}^{y-c(n-i)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{c} C_{i-1}^{x-ci-j} C_{n-i}^{y-c(n-i)} + \sum_{i=0}^{n} C_i^{(x-c)-ci} C_{n-i}^{y-c(n-i)} \\
&= \sum_{j=1}^{c} f_{n-1}(x - c - j, y) + f_n(x - c, y).
\end{aligned}
\tag{4.7}
$$

Replacing y by $y + c$ in (4.7), we obtain

$$
f_n(x, y + c) = \sum_{j=1}^{c} f_{n-1}(x - c - j, y + c) + f_n(x - c, y + c).
$$

Then

$$
f_n(x - c, y + c) = f_n(x, y + c) - \sum_{j=1}^{c} f_{n-1}(x - c - j, y + c).
$$

Since $f_n(x, y) = f_n(y, x)$, we obtain

$$
f_n(x - c, y + c) = f_n(y + c, x) - \sum_{j=1}^{c} f_{n-1}(x - c - j, y + c).
\tag{4.8}
$$

By replacing x by $y + c$ and y by x in (4.7), we have

$$
f_n(y + c, x) = \sum_{j=1}^{c} f_{n-1}(y - j, x) + f_n(y, x).
\tag{4.9}
$$

Therefore, by (4.8) and (4.9),

$$
f_n(x - c, y + c) = f_n(y, x) + \sum_{j=1}^{c} (f_{n-1}(y - j, x) - f_{n-1}(x - c - j, y + c)).
\tag{4.10}
$$

Now, by the induction hypothesis, (4.6) holds for $m = n - 1$, i.e. there exists a polynomial g_{n-1} (in a single variable) such that $g_{n-1}(x + y) = f_{n-1}(x, y)$. Hence, for $1 \le j \le c$,

$$
f_{n-1}(y - j, x) = g_{n-1}(x + y - j) = f_{n-1}(x - c - j, y + c).
\tag{4.11}
$$

In view of (4.10) and (4.11), we have

$$
f_n(x - c, y + c) = f_n(y, x) = f_n(x, y).
\tag{4.12}
$$

Let $h_n(x, y) = f_n(x, y) - f_n(cm, x + y - cm)$. To finish the induction proof, we are going to show that, as a polynomial in x and y, $h_n(x, y) = 0$.

Note that, for a fixed x, $h_n(x, y)$ is a polynomial in y with degree at most n. Hence, we can write $h_n(x, y) = a_n^n(x)y^n + a_n^{n-1}(x)y^{n-1} + \cdots + a_n^0(x)$, where $a_n^i(x)$ is a polynomial

in x with a degree of at most n, $0 \leq i \leq n$. Let x_0 be a multiple of c. Then, by (4.12), $f_n(x_0, y) = f_n(cm, x_0 + y - cm)$ and, consequently, $h_n(x_0, y) = 0$. This in turn implies that $a_n^n(x_0) = 0, a_n^{n-1}(x_0) = 0, \ldots, a_n^0(x_0) = 0$ for all x_0. Since there are infinitely many such x_0s,

$$a_n^n(x) = 0, \ a_n^{n-1}(x) = 0, \ldots, \ a_n^0(x) = 0,$$

for all $x \in \mathbb{R}$. This implies that $h_n(x, y)$ is a zero polynomial.

Step 2. We substitute x by $j + ck$ and y by $\gamma_n - j - ck + cm$ in (4.5). Then

$$\sum_{i=0}^m C_i^{j+c(k-i)} C_{m-i}^{\gamma_n-j-c(k-i)} = a_m \gamma_{n+1}^m + a_{m-1}\gamma_{n+1}^{m-1} + \cdots + a_0 := b_m \gamma_n^m + b_{m-1}\gamma_n^{m-1} + \cdots + b_0.$$

In order to obtain the coefficients b_m and b_{m-1}, let $k = 0$ in $\sum_{i=0}^m C_i^{j+c(k-i)} C_{m-i}^{\gamma_n-j-c(k-i)}$. Then

$$\sum_{i=0}^m C_i^{j+c(k-i)} C_{m-i}^{\gamma_n-j-c(k-i)} = C_m^{\gamma_n-j} + C_1^{j-c} C_{m-1}^{\gamma_n-j+c} + \cdots$$

$$= \frac{\gamma_n^m}{m!} + \frac{(1 - m - 2c)\gamma_n^{m-1}}{2(m-1)!} + \cdots.$$

Note that $j + c(k - i) \geq 0$ and $\gamma_n - j - c(k - i) \geq 0$, since $cm \leq j \leq \gamma_{\ell-1}$, $n \geq \ell$, and $k \leq m(n + 1)$. Therefore,

$$\sum_{i=0}^m C_i^{j+c(k-i)} C_{m-i}^{\gamma_n-j-c(k-i)} = \sum_{i=0}^m \binom{j + c(k - i)}{i}\binom{\gamma_n - j - c(k - i)}{m - i}.$$

Hence, Lemma 4.1 holds.

Lemma 4.2. *For a fixed ℓ, there exists a positive constant c' such that, for all $n \geq \ell$, j with $cm \leq j \leq \gamma_{\ell-1}$, and $k \leq m(n + 1)$, the following inequality holds:*

$$\sum_{i=0}^m P(Y_{n+1} = j + ck \mid Y_n = j + c(k - i)) \leq 1 - \frac{1}{n} + \frac{c'}{n^2}. \tag{4.13}$$

Proof. By Lemma 4.1, we have

$$(\sum_{i=0}^m \binom{j + c(k - i)}{i}\binom{\gamma_n - j - c(k - i)}{m - i}\binom{\gamma_n}{m}^{-1}$$

$$= \left(\frac{\gamma_n^m}{m!} + \frac{(1 - m - 2c)\gamma_n^{m-1}}{2(m-1)!} + \cdots \right)\left(\frac{\gamma_n^m}{m!} + \frac{(1 - m)\gamma_n^{m-1}}{2(m-1)!} + \cdots \right)^{-1}$$

$$= 1 - \frac{1}{n} + O\left(\frac{1}{n^2} \right).$$

Hence, there exists a constant c' such that (4.13) holds for all n.

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let F be the distribution function of X. Our goal is to prove that F is absolutely continuous. By Propositions 4.1 and 4.2 and the discussion before Lemma 4.1, it is sufficient to show that the restriction of X to $\Omega_{\ell j} = \{\omega : Y_\ell(\omega) = j\}$ has a density for $cm \le j \le \gamma_{\ell-1}$.

Now, in view of Lemma 4.2, there exists a positive constant c' such that, for a fixed ℓ and for all $n \ge 0$,

$$\max_{0 \le k \le m(n+1)} P(Y_{\ell+n+1} = j + ck \mid Y_\ell = j)$$

$$\le \max_{0 \le k \le m(n+1)} \left\{ \sum_{i=0}^{m} P(Y_{\ell+n+1} = j + ck \mid Y_{\ell+n} = j + c(k-i)) \right.$$

$$\left. \times P(Y_{\ell+n} = j + c(k-i) \mid Y_\ell = j) \right\}$$

$$\le \max_{0 \le k \le m(n+1)} \left\{ \sum_{i=0}^{m} P(Y_{\ell+n+1} = j + ck \mid Y_{\ell+n} = j + c(k-i)) \right.$$

$$\left. \times \max_{0 \le i \le mn} P(Y_{\ell+n} = j + ci \mid Y_\ell = j) \right\}$$

$$= \max_{0 \le k \le m(n+1)} \sum_{i=0}^{m} P(Y_{\ell+n+1} = j + ck \mid Y_{\ell+n} = j + c(k-i))$$

$$\times \max_{0 \le i \le mn} P(Y_{\ell+n} = j + ci \mid Y_\ell = j)$$

$$\le \left[1 - \frac{1}{\ell+n} + \frac{c'}{(\ell+n)^2} \right] \max_{0 \le k \le mn} P(Y_{\ell+n} = j + ck \mid Y_\ell = j).$$

This implies that, for a fixed ℓ and for all $n \ge \ell + 1$,

$$\max_{0 \le k \le m(n-\ell)} P(Y_n = j + ck \mid Y_\ell = j)$$

$$\le \left[1 - \frac{1}{n-1} + \frac{c'}{(n-1)^2} \right] \max_{0 \le k \le m(n-1-\ell)} P(Y_{n-1} = j + ck \mid Y_\ell = j)$$

$$\le \left[1 - \frac{1}{n-1} + \frac{c'}{(n-1)^2} \right] \left[1 - \frac{1}{n-2} + \frac{c'}{(n-2)^2} \right]$$

$$\times \max_{0 \le k \le m(n-2-\ell)} P(Y_{n-2} = j + ck \mid Y_\ell = j)$$

$$\vdots \qquad \vdots$$

$$\le \prod_{i=\ell}^{n-1} \left(1 - \frac{1}{i} + \frac{c'}{i^2} \right)$$

$$\le c'' \exp\left(-\sum_{i=\ell}^{n-1} \frac{1}{i} \right),$$

for some positive constant c''. The last inequality can be derived from the fact that $1 - x \le e^{-x}$.

Therefore, for any x and x', $0 \leq x < x' \leq 1$,

$$P(x \leq X_n \leq x' \mid Y_\ell = j) = P(x\gamma_n \leq Y_n \leq x'\gamma_n \mid Y_\ell = j)$$

$$\leq [(x' - x)\gamma_n + 1]c'' \exp\left(-\sum_{i=\ell}^{n-1} \frac{1}{i}\right).$$

We obtain

$$\limsup_n P(x \leq X_n \leq x' \mid Y_\ell = j) \leq \lim_{n\to\infty} [(x' - x)\gamma_n + 1]c'' \exp\left(-\sum_{i=\ell}^{n-1} \frac{1}{i}\right)$$

$$= (x' - x)c''',$$

where c''' is a positive constant depending only on ℓ. Now fix ℓ. Let $cm \leq j \leq \gamma_{\ell-1}$. For any given $\varepsilon > 0$, let $\delta = \varepsilon/c'''$. Let $x_1 < x'_1 \leq x_2 < x'_2 \leq \cdots \leq x_s < x'_s$ and $\sum_{i=1}^{s} |x'_i - x_i| < \delta$. We have, by Fatou's lemma (see [16]),

$$\sum_{i=1}^{s} P(\{x_i < X < x'_i\} \cap \Omega_{\ell j}) \leq \sum_{i=1}^{s} \liminf_{n\to\infty} E[\mathbf{1}_{\{x_i \leq X_n \leq x'_i\}} \mid Y_\ell = j] P(\Omega_{\ell j})$$

$$\leq \sum_{i=1}^{s} \liminf_{n\to\infty} P(x_i \leq X_n \leq x'_i \mid Y_\ell = j) P(\Omega_{\ell j})$$

$$\leq \sum_{i=1}^{s} (x'_i - x_i)c'''$$

$$< \varepsilon.$$

By [2, Theorem 31.7], the restriction of X to $\Omega_{\ell j}$ thus has a density. Hence, the proof is complete.

Remark 4.3. For the urn model with r different colors, as mentioned in Remark 4.2, our method can also be used to prove that $X = (X_1, \ldots, X_r)$ is absolutely continuous. This requires a little modification and rewriting, i.e.

$$\Omega_\ell = \{\omega : Y_{\ell i} \geq cm \text{ for all } i = 1, \ldots, r\}, \qquad \Omega_{\ell, j} = \{\omega : Y_\ell = (Y_{\ell 1}, \ldots, Y_{\ell r}) = j\}$$

and

$$\sum_t P(Y_{n+1} = j + ck \mid Y_n = j + c(k - t)) \leq 1 - \frac{r-1}{n} + \frac{c'}{n^2},$$

where $j = (j_1, \ldots, j_r)$, $k = (k_1, \ldots, k_r)$, and $t = (t_1, \ldots, t_r)$.

Appendix A.

In this appendix, more examples are illustrated, in Figure 3, to show the effects of different initial numbers of black balls b and white balls w, and numbers, m, of balls drawn each time. As in Section 3, $M(b, w, m)$ denotes our new model with initial data (b, w, m) and beta(α, β) denotes a beta distribution with parameters α and β.
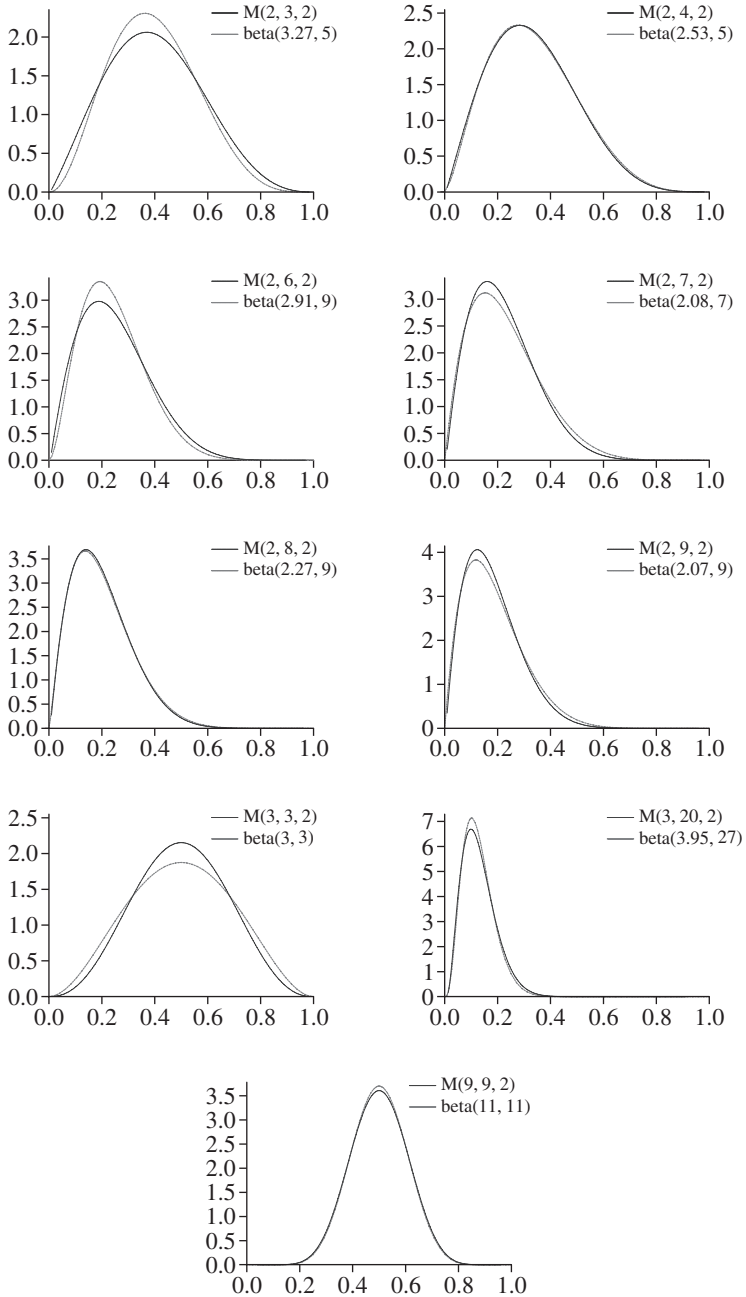
FIGURE 3: Nine examples with $n = 100$.

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References

[1] Bagchi, A. and Pal, A. K. (1985). Asymptoic normality in the generalized Pólya–Eggenberger urn model, with an application to computer data structures. *SIAM J. Algebraic Discrete Meth.* **6,** 394–405.

[2] Billingsley, P. (1995). *Probability and Measure,* 3rd edn. John Wiley, New York.

[3] Eggenberger, F. and Pólya, G. (1923). Über die statistik verketteter vorgänge. *Z. Angewandte Math. Mech.* **1,** 279–289.

[4] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd edn. John Wiley, New York.

[5] Gouet, R. (1989). A martingale approach to strong convergence in a generalized Pólya–Eggenberger urn model. *Statist. Prob. Lett.* **8,** 225–228.

[6] Gouet, R. (1993). Martingale functional central limit theorems for a generalized Pólya urn. *Ann. Prob.* **21,** 1624–1639.

[7] Hall, P. and Heyde, C. C. (1980). *Martingale Limit Theory and Its Application*. Academic Press, New York.

[8] Hill, B., Lane, D. and Sudderth, W. (1980). A strong law for some generalized urn processes. *Ann. Prob.* **8,** 214–226.

[9] Johnson, N. L. and Kotz, S. (1977). *Urn Models and Their Application*. John Wiley, New York.

[10] Kotz, S. and Balakrishnan, N. (1997). Advances in urn models during the past two decades. In *Advances in Combinatorial Methods and Applications to Probability and Statistics*, Birkhäuser, Boston, MA, pp. 203–257.

[11] Maistrov, L. E. (1974). *Probability Theory: a Historical Sketch*. Academic Press, New York.

[12] Pemantle, R. (1990). A time-dependent version of Pólya's urn. *J. Theoret. Prob.* **3,** 627–637.

[13] Schreiber, S. J. (2001). Urn models, replicator processes, and random genetic drift. *SIAM J. Appl. Math.* **61,** 2148–2167.

[14] Stewart, I. (1989). *Galois Theory*, 2nd edn. Chapman and Hall, New York.

[15] Wei, C. Z. (1993). Martingale transforms with non-atomic limits and stochastic approximation. *Prob. Theory Relat. Fields* **95,** 103–114.

[16] Wheeden, R. L. and Zygmund, A. (1977). *Measure and Integral*. Marcel Dekker, New York.