THE GIBBS PHENOMENON AND LEBESGUE CONSTANTS FOR REGULAR SONNENSCHEIN MATRICES

W. T. SLEDD

1. Introduction. If $\psi(x)$ is a real-valued function which has a jump discontinuity at $x = \xi$ and otherwise satisfies the Dirichlet conditions in a neighbourhood of $x = \xi$ then $\{s_n(x)\}$, the sequence of partial sums of the Fourier series for $\psi(x)$, cannot converge uniformly at $x = \xi$. Moreover, it can be shown that given τ in $[-\pi, \pi]$ then there is a sequence $\{t_n\}$ such that $t_n \to \xi$ and

$$\lim_{n\to\infty} s_n(t_n) = \frac{\psi(\xi+0)+\psi(\xi-0)}{2} + \frac{\psi(\xi+0)-\psi(\xi-0)}{\pi}\int_0^\tau \frac{\sin y}{y}\,dy.$$

This behaviour of $\{s_n(x)\}$ is called the Gibbs phenomenon. If $\{\sigma_n(x)\}$ is the transform of $\{s_n(x)\}$ by a summability method T, and if $\{\sigma_n(x)\}$ also has the property described then we say that T preserves the Gibbs phenomenon.

Miracle **(10)** has proved that in order to show that a regular summability method T preserves the Gibbs phenomenon it suffices to show that if τ is in $[-\pi, \pi]$ there is a sequence $\{t_n\}$ such that $t_n \to 0$ and

$$\lim_{n\to\infty} \sigma_n(t_n) = \int_0^\tau \frac{\sin y}{y}\,dy.$$

Here $\{\sigma_n(x)\}$ is the T-transform of the sequence of partial sums of the Fourier series for the particular function

$$\phi(x) = \begin{cases} -\pi/2 & -\pi < x < 0 \\ \pi/2 & 0 < x < \pi \end{cases}$$

$$\phi(-\pi) = \phi(0) = \phi(\pi) = 0 \qquad \phi(x) = \phi(x+2\pi).$$

An extensive bibliography for the Gibbs phenomenon may be found in Miracle's paper **(10)**.

The Lebesgue constants for a summability method determined by an infinite matrix $T = (t_{nk})$ are defined by

$$L_n(T) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\left|\sum_{k=0}^{\infty} t_{nk}\sin(2k+1)t\right|}{\sin t}\,dt.$$

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The sequence $\{L_n(T)\}$ is of considerable interest in the theory of summability of Fourier series. For if T satisfies

$$(1.1) \qquad \sum_{k=0}^{\infty} |t_{nk}|\, k < \infty \qquad n = 0, 1, 2, \ldots$$

if $\sup_n|L_n(T)| < \infty$ and if $\psi(x)$ is continuous in $[a, b]$ then the Fourier series for $\psi(x)$ is uniformly T-summable to $\psi(x)$ in $[a, b]$. However, if $\{L_n(T)\}$ is not a bounded sequence then there are continuous functions whose Fourier series are not always T-summable **(5**, pp. 58–60**)**.

Lorch **(7**; **8)** has studied the Lebesgue constants for the $(E, 1)$ and Borel methods, Lorch and Newman **(9)** have studied them for the Hausdorff methods, and Livingston **(6)** has studied them for the (E, p) methods.

Now if $f(z)$ is a function that is analytic in a neighbourhood of the origin, we may form an infinite matrix $F = (f_{nk})$ in the following manner. Let

$$[f(z)]^n = \sum_{k=0}^{\infty} f_{nk}\, z^k \qquad n = 1, 2, \ldots$$

$$f_{00} = 1 \qquad f_{0k} = 0 \qquad k = 1, 2, \ldots .$$

Then F is said to be generated by $f(z)$ and F determines a sequence-to-sequence transformation.

Here we shall study the Gibbs phenomenon and the Lebesgue constants for summability methods of the type just described. We shall see that if suitable restrictions are imposed on $f(z)$ then the summability method determined by F preserves the Gibbs phenomenon and that an asymptotic expansion can be obtained for $L_n(F)$, showing that $\{L_n(F)\}$ is unbounded.

2. The main theorems. The restrictions on $f(z)$ are that

$$(2.1) \qquad f(z) \text{ be analytic for } |z| < R \text{ where } R > 1,$$

$$(2.2) \qquad f(1) = 1,$$

and

$$(2.3) \qquad |f(z)| < 1 \text{ when } |z| \leqslant 1, \qquad z \neq 1.$$

The matrix F generated by such a function $f(z)$ is called a Sonnenschein matrix **(12)**. If we also assume that

(2.4) $\operatorname{Re} A \neq 0$ where A is defined by

$$f(z) - z^{\zeta} = Ai^p(z-1)^p + O(1)(z-1)^{p+1}, \qquad z \to 1$$

and $\zeta = f'(1)$, then F is a regular matrix **(1)**. In fact, if $f(z)$ satisfies the hypotheses (2.1), (2.2), (2.3) and if $f(z)$ is not of the form $f(z) = z^k$ ($k = 1, 2, \ldots$) then (2.4) is a necessary condition that F be regular **(3)**.

From the hypotheses (2.1)–(2.4) it follows that $\zeta = f'(1) > 0$, that $\xi = -\operatorname{Re} A > 0$, and that p is an even positive integer, where A and p are given by (2.4). Further, with $f(e^{2it}) = Re^{i\theta}$, we may write

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(2.5) $$R = 1 - \xi(2t)^p + O(t^{p+1}) \qquad t \to 0,$$

and

(2.6) $$\theta = 2\zeta t + O(t^2) \qquad t \to 0.$$

We may now prove the following theorems about the matrix F generated by a function $f(z)$ satisfying (2.1)–(2.4).

THEOREM 2.1. *The summability method determined by F preserves the Gibbs phenomenon.*

THEOREM 2.2. *An asymptotic expansion for $L_n(F)$ is given by*

$$L_n(F) = \frac{4}{p\pi^2}\log\frac{n^{p-1}\zeta^p}{\xi} - \frac{4\gamma}{p\pi^2} - \frac{4}{\pi^2}\log\frac{\pi}{2} + 2\int_0^1 \log\Gamma(t)\cos\pi t\,dt + o(1) \qquad n \to \infty$$

where γ is Euler's constant, and the numbers p, ζ, and ξ are as defined above.

Proof of Theorem 2.1. The sequence of partial sums of the Fourier series for the function $\phi(x)$ defined in the Introduction is known (**2**, p. 296) to be given by

$$s_n(x) = \int_0^x \frac{\sin 2nt}{\sin t}\,dt \qquad n = 0, 1, 2, \ldots .$$

Assume that $|x| \leqslant \pi/2$. The F-transform of $\{s_n(x)\}$ is then given by

$$\sigma_n(x) = \sum_{k=0}^{\infty} f_{nk}\int_0^x \frac{\sin 2kt}{\sin t}\,dt.$$

But (2.1) implies (1.1), and using (1.1) and the fact that

$$|\sin 2kt| \leqslant k\pi|\sin t| \qquad |t| \leqslant \pi/2$$

we see that the order of summation and integration may be reversed and we may write

$$\sigma_n(x) = \int_0^x \frac{\psi_n}{\sin t}\,dt \qquad n = 0, 1, 2, \ldots$$

where

$$\psi_n = \frac{R^n e^{in\theta} - R_1^n e^{in\theta_1}}{2i}$$

and

$$R_1 e^{i\theta_1} = f(e^{-2it}).$$

Using (2.6) we have that

$$\sin n\theta = \sin 2n\zeta t + O(nt^2) \qquad nt^2 \to 0.$$

And by (2.5) and (2.6) it follows that

(2.7) $$\theta_1 = -\theta + O(t^2) \qquad t \to 0$$

and

(2.8) $$R_1 = R + O(t^{p+1}) \qquad t \to 0.$$

Combining these results and using (2.5)

$$\psi_n = \sin 2n\zeta t + O(nt^2) \qquad nt^2 \to 0.$$

Hence

$$\sigma_n(x) = \int_0^x \frac{\sin 2n\,\zeta\,t}{t}\,dt + O(nx^2) \qquad nx^2 \to 0.$$

Thus, given τ such that $0 \leqslant \tau \leqslant \pi$, let $t_n = \tau/(2n\zeta)$. Then

$$\sigma_n(t_n) = \int_0^\tau \frac{\sin y}{y}\,dy + O(n^{-1}) \qquad n \to \infty.$$

And if $-\pi \leqslant \tau \leqslant 0$, then $0 \leqslant -\tau \leqslant \pi$, so that since $\sigma_n(x) = -\sigma_n(-x)$ we may choose $t_n = \tau/(2n\zeta)$ and still have

$$\sigma_n(t_n) = \int_0^\tau \frac{\sin y}{y}\,dy + O(n^{-1}) \qquad n \to \infty.$$

This together with an application of Miracle's results mentioned in the Introduction completes the proof of Theorem 2.1.

Proof of Theorem 2.2. The Lebesgue constants for F are given by

$$L_n(F) = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\lambda_n|}{\sin t}\,dt \qquad n = 0, 1, 2, \ldots$$

where

$$\lambda_n = \frac{R^n e^{i(n\theta+t)} - R_1^n e^{i(n\theta_1-t)}}{2i}.$$

Using (2.5) and (2.6)

$$\lambda_n = R^n \sin[n(\theta-\theta_1)/2 + t]e^{in(\theta+\theta_1)/2} + O(nt^{p+1}) \qquad nt^{p+1} \to 0.$$

Also we have that

$$\frac{1}{\sin t} = \frac{1}{t} + O(t) \qquad t \to 0.$$

Thus if $1/p > a > 1/(p+1)$ we have that

$$\int_0^{n^{-a}} \frac{|\lambda_n|}{\sin t}\,dt = \int_0^{n^{-a}} \frac{R^n|\sin[n(\theta-\theta_1)/2 + t]|}{\sin t}\,dt + O(n^{1-(p+1)a}) \qquad n \to \infty$$

$$= \int_0^{n^{-a}} \frac{R^n|\sin[n(\theta-\theta_1)/2]|}{t}\,dt + O(n^{1-(p+1)a}) \qquad n \to \infty.$$

By (2.5)

$$R^n = e^{-n\xi(2t)^p}(1 + O(nt^{p+1})) \qquad nt^{p+1} \to 0.$$

So that

$$\int_{n^{-a}}^{\pi/2} \frac{R^n|\sin[n(\theta - \theta_1)/2 + t]|}{\sin t}\, dt \leqslant \frac{2}{\pi}\, e^{-n\xi(2n^{-a})^p} \log \frac{\pi n^a}{2} + O(n^{1-(p+1)a}) \qquad n \to \infty.$$

Hence

$$L_n(F) = \int_0^{n^{-a}} \frac{R^n|\sin[n(\theta - \theta_1)/2]|}{t}\, dt + o(1) \qquad n \to \infty.$$

Again using (2.6) with $\omega = (\theta - \theta_1)/2\zeta$

$$R^n = e^{-n\xi\omega^p}(1 + O(nt^{p+1})) \qquad nt^{p+1} \to 0.$$

Thus

$$\begin{aligned} L_n(F) &= \frac{2}{\pi} \int_0^{n^{-a}} \frac{e^{-n\xi\omega^p}|\sin n\zeta\omega|}{t}\, dt + o(1) \\ &= \frac{2}{\pi} \int_0^{\omega(n^{-a})/2} \frac{e^{-n\xi\omega^p}|\sin n\zeta\omega|}{\omega}\, d\omega + o(1) \\ &= \frac{2}{\pi} \int_0^{\pi/2} \frac{e^{-n\xi\omega^p}|\sin(n\zeta + 1)\omega|}{\omega}\, d\omega + o(1) \qquad n \to \infty. \end{aligned}$$

Then, using results due to Lorch **(7)**,

$$\begin{aligned} L_n(F) &= \frac{2}{\pi} \int_0^{\pi/2} \frac{|\sin(n\zeta + 1)\omega|}{\omega}\, d\omega - \frac{4}{\pi^2} \int_0^{\pi/2} \frac{1 - e^{-n\xi\omega^p}}{\omega}\, d\omega + o(1) \\ &= \frac{2}{\pi} \int_0^{\pi/2} \frac{|\sin(n\zeta + 1)\omega|}{\omega}\, d\omega - \frac{4}{p\pi^2} \log n\xi(\pi/2)^p - \frac{4\gamma}{p\pi^2} + o(1) \qquad n \to \infty. \end{aligned}$$

However, the first term in this last expression for $L_n(F)$ is just L_n, the nth Lebesgue constant for ordinary convergence. Using the known asymptotic expansion for L_n **(4)** we may then write

$$\begin{aligned} L_n(F) = {} & \frac{4}{p\pi^2} \log \frac{n^{p-1}\zeta^p}{\xi} - \frac{4\gamma}{p\pi^2} \\ & + 2\int_0^1 \log \Gamma(t) \cos \pi t\, dt - \frac{4}{\pi^2} \log \frac{\pi}{2} + o(1) \qquad n \to \infty. \end{aligned}$$

This completes the proof of Theorem 2.2.

3. An example. Let

$$f(z) = \frac{\alpha + (1 - \alpha - \beta)z}{1 - \beta z}$$

where α and β are complex numbers. Then the matrix F generated by this $f(z)$ is called a Karamata matrix **(1)**. Conditions that F be regular **(11)** are that

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$$\alpha = \beta = 0 \quad \text{or} \quad 1 - |\alpha|^2 > (1 - \bar{\alpha})(1 - \beta) > 0. \tag{3.1}$$

Here

$$R^2 = 1 - \frac{4(\beta + \bar{\alpha}(1 - \alpha - \beta))\sin^2 t}{|1 - \beta e^{2it}|^2}$$

$$= 1 - \frac{4(\beta + \bar{\alpha}(1 - \alpha - \beta))t^2}{|1 - \beta|^2} + O(t^3) \qquad t \to 0.$$

and

$$\theta = 2t\left(\frac{1 - \alpha}{1 - \beta}\right) + O(t^2) \qquad t \to 0.$$

Thus under conditions (3.1) F preserves the Gibbs phenomenon and

$$L_n(F) = \frac{2}{\pi^2} \log \frac{n(1 - \alpha)^2(1 - \bar{\beta})}{2(1 - \beta)(\beta + \bar{\alpha}(1 - \alpha - \beta))}$$

$$- \frac{2\gamma}{\pi^2} + 2\int_0^1 \log \Gamma(t) \cos \pi t \, dt - \frac{4}{\pi^2} \log \frac{\pi}{2} + o(1) \qquad n \to \infty.$$

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University of Kentucky
Michigan State University