ON THE HASSE-MINKOWSKI INVARIANT OF THE KRONECKER PRODUCT OF MATRICES

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1. Introduction. Let $R = (r_{ij})$ be an $m \times n$ matrix and let $S = (s_{lk})$ be a $p \times q$ matrix defined over a field F. The Kronecker product $R \times S$ of R and S is defined as follows:

Definition 1.1. The Kronecker product $R \times S$ of the matrices R and S is given by

1.1 $$R \times S = \begin{bmatrix} r_{11}\, S & r_{12}\, S \ldots r_{1n}\, S \\ r_{21}\, S & r_{22}\, S \ldots r_{2n}\, S \\ \ldots & \ldots \\ r_{m1}\, S & r_{m2}\, S \ldots r_{mn}\, S \end{bmatrix}$$

where $r_{ij}\, S$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, is itself a $p \times q$ matrix **(1,** 69–70**)**.

We shall always use the symbol "$\times$" in a product of matrices to denote the Kronecker product. The ordinary product of R and S (whenever it exists) will be denoted by $R \cdot S$ or $R\, S$.

The Hasse-Minkowski invariant is a number-theoretic function occurring in the arithmetical theory of quadratic forms. With respect to the matrix $A = (a_{ij})$ of a quadratic form

$$Q = \sum_{i,\, j=1}^{n} a_{ij}\, x_i\, x_j,$$

it is defined as follows:

Definition 1.2. Let A be an $n \times n$ non-singular symmetric matrix with rational elements and let $D_i (i = 1, 2, \ldots, n)$ denote the leading principal minor determinant of order i in the matrix A. Suppose further that none of the D_i is zero. Then the integer

1.2 $$c_p = c_p(A) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p$$

is called the Hasse-Minkowski invariant of A where p is a prime and $(a, b)_p$ is the Hilbert norm residue symbol **(2)**.

From the properties of the Hilbert norm residue symbol we get the following expressions for $c_p(A)$ equivalent to 1.2:

1.3 $$c_p(A) = (-1, -1)_p \prod_{i=1}^{n} (D_i, -D_{i-1})_p$$

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where $D_0 = 1$, and

1.4 $$c_p(A) = (-1, -D_1)_p \prod_{i=1}^{n-1} (D_{i+1}, -D_i)_p.$$

The problem considered in this paper is that of obtaining the value of $c_p(A \times B)$ in terms of $c_p(A)$, $c_p(B)$ and the determinants $|A|$ and $|B|$ of A and B.

In the next section we prove a theorem giving the exact relation between $c_p(A \times B)$ on one side and $c_p(A)$, $c_p(B)$, $|A|$ and $|B|$ on the other. In §3 are considered some particular cases of this result.

2. The Hasse-Minkowski invariant $c_p(A \times B)$: We shall first prove the following Lemma:

LEMMA 2.1. *If A is a square matrix of order m and B is a square matrix of order n, then the leading principal minor determinant $|D_u|$ of order u in the Kronecker product $A \times B$ is given by*

2.1 $$|D_u| = |A_r|^{n-s}|A_{r+1}|^s|B|^r|B_s|$$

where $rn + s = u$; $0 \leqslant r < m$; $0 \leqslant s \leqslant n$; $|A_r|$ denotes the leading principal minor determinant of order r in A, $|B_s|$ denotes the leading principal minor determinant of order s in B, and none of the determinants $|A_r|$ is zero.

Proof. In the first place observe that for any given u, $1 \leqslant u \leqslant mn$, there is one and only one pair of integers r and s such that $rn + s = u$ with $0 \leqslant r < m$ and $0 < s \leqslant n$. We therefore have:

2.2 $$|D_u| = \begin{vmatrix} a_{11}B & a_{12}B & \dots a_{1r}B & a_{1,r+1}B^{(s)} \\ a_{21}B & a_{22}B & \dots a_{2r}B & a_{2,r+1}B^{(s)} \\ \dots & & \dots & \dots \\ a_{r1}B & a_{r2}B & \dots a_{rr}B & a_{r,r+1}B^{(s)} \\ a_{r+1,1}B_{(s)} & a_{r+1,2}B_{(s)} & \dots a_{r+1,r}B_{(s)} & a_{r+1,r+1}B_{(s)}^{(s)} \end{vmatrix}$$

where $B^{(s)}$ is the $n \times s$ matrix obtained from B by deleting the last $(n - s)$ columns, $B_{(s)}$ is the $s \times n$ matrix obtained from B by deleting the last $(n - s)$ rows, and $B_{(s)}^{(s)}$ is the $s \times s$ matrix obtained from B by deleting the last $(n - s)$ columns and $(n - s)$ rows. From (2.2) we have, since $|A_1| = a_{11} \neq 0$,

$$|D_u| = a_{11}^n \begin{vmatrix} B & 0_{n\times n} & 0_{n\times n} \dots & 0_{n\times n} & 0_{n\times n} \\ \frac{a_{21}}{a_{11}}B & a_{22}^{(1)}B & a_{23}^{(1)}B \dots & a_{2r}^{(1)}B & a_{2,r+1}^{(1)}B^{(s)} \\ \frac{a_{31}}{a_{11}}B & a_{32}^{(1)}B & a_{33}^{(1)}B \dots & a_{3r}^{(1)}B & a_{3,r+1}^{(1)}B^{(s)} \\ \dots & & \dots & & \dots \\ \frac{a_{r1}}{a_{11}}B & a_{r2}^{(1)}B & a_{r3}^{(1)}B \dots & a_{rr}^{(1)}B & a_{r,r+1}^{(1)}B^{(s)} \\ \frac{a_{r+1,1}}{a_{11}}B_{(s)} & a_{r+1,2}^{(1)}B_{(s)} & a_{r+1,3}^{(1)}B_{(s)} \dots & a_{r+1,r}^{(1)}B^{(s)} & a_{r+1,r+1}^{(1)}B_{(s)}^{(s)} \end{vmatrix}$$

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so that

$$|D_u| = a_{11}^n|B| \begin{vmatrix} a_{22}^{(1)}B & a_{23}^{(1)}B & \dots & a_{2r}^{(1)}B & a_{2,r+1}^{(1)}B^{(s)} \\ a_{32}^{(1)}B & a_{33}^{(1)}B & \dots & a_{3r}^{(1)}B & a_{3,r+1}^{(1)}B^{(s)} \\ \dots & & \dots & & \dots \\ a_{r,2}^{(1)}B & a_{r,3}^{(1)}B & \dots & a_{rr}^{(1)}B & a_{r,r+1}^{(1)}B^{(s)} \\ a_{r+1,2}^{(1)}B_{(s)} & a_{r+1,3}^{(1)}B_{(s)} & \dots & a_{r+1,r}^{(1)}B_{(s)} & a_{r+1,r+1}^{(1)}B_{(s)}^{(s)} \end{vmatrix}$$

where $0_{m\times n}$ is the zero matrix of order $m \times n$,

$$a_{ij}^{(1)} = \frac{a_{ij}\,a_{11} - a_{i1}\,a_{1j}}{a_{11}}, \qquad i, j = 1, 2, \dots, r+1;$$

and in particular

$$a_{22}^{(1)} = |A_2|/|A_1| \neq 0.$$

Proceeding in this way we finally get, since none of $|A_v|$ is zero,

$$|D_u| = |A_r|^n|B|^{r-1} \begin{vmatrix} B & 0_{n\times s} \\ \frac{a_{r+1,r}^{(r-1)}}{a_{rr}^{(r-1)}} B_{(s)} & a_{r+1,r+1}^{(r)} B_{(s)}^{(s)} \end{vmatrix}$$

where

$$a_{r+1,r+1}^{(r)} = \frac{a_{r+1,r+1}^{(r-1)}\,a_{rr}^{(r-1)} - a_{r+1,r}^{(r-1)}\,a_{r,r+1}^{(r-1)}}{a_{rr}^{(r-1)}} = \frac{|A_{r+1}|}{|A_r|} \neq 0.$$

Hence

$$|D_u| = |A_r|^{n-s}|A_{r+1}|^s|B|^r|B_s|,$$

which proves the Lemma.

It is interesting to note that if we let $r = m - 1$ and $s = n$, then 2.1 reduces to a well-known result **(3)**

2.3 $$|A \times B| = |A|^n|B|^m.$$

We are now in a position to prove the following:

THEOREM 2.1. *Let A be a symmetric matrix of order m and B be a symmetric matrix of order n. Let the elements of A and B be rational numbers and assume that all the leading principal minor determinants of A and B are different from zero. Then*

2.4 $$c_p(A \times B) = (-1, -1)_p^{m+n-1}\,\{c_p(A)\}^n\,\{c_p(B)\}^m\,(|A|, -1)_p^{\frac{1}{2}n(n-1)} \times (|B|, -1)_p^{\frac{1}{2}m(m-1)}\,(|A|, |B|)_p^{mn-1}$$

Proof. In the first place observe that $A \times B$ is symmetric and none of the leading principal minor determinants of $A \times$ B is zero since the same properties hold for A and B. Thus $c_p(A \times B)$ has a meaning. Further from 1.3 and 2.1 we have

2.5 $$c_p(A \times B) = (-1, -1)_p \prod_{r=0}^{m-1} \prod_{s=1}^{n} (|A_r|^{n-s}|A_{r+1}|^s|B|^r|B_s|, - |A_r|^{n-s+1}|A_{r+1}|^{s-1}|B|^r|B_{s-1}|)_p.$$

Since the Hilbert norm residue symbol $(a, b)_p$ satisfies the property

2.6 $$(a_1 a_2, b)_p = (a_1, b)_p (a_2, b)_p$$

it follows that the factor written after the product signs on the right hand side of 2.5 breaks into twenty factors which are simplified as follows, where we have dropped, for convenience, the subscript p:

(i) $(|A_r|^{n-s}, -1) = (|A_r|, -1)^{n-s}$,
(ii) $(|A_r|^{n-s}, |A_r|^{n-s+1}) = (|A_r|, |A_r|)^{(n-s)(n-s+1)} = 1$,
(iii) $(|A_r|^{n-s}, |A_{r+1}|^{s-1}) = (|A_r|, |A_{r+1}|)^{(n-s)(s-1)} = (|A_r|, |A_{r+1}|)^{n(s-1)}$,
(iv) $(|A_r|^{n-s}, |B|^r) = (|A_r|, |B|)^{r(n-s)}$,
(v) $(|A_r|^{n-s}, |B_{s-1}|) = (|A_r|, |B_{s-1}|)^{n-s}$,
(vi) $(|A_{r+1}|^s, -1) = (|A_{r+1}|, -1)^s$,
(vii) $(|A_{r+1}|^s, |A_r|^{n-s+1}) = (|A_r|, |A_{r+1}|)^{s(n-s+1)} = (|A_r|, |A_{r+1}|)^{ns}$,
(viii) $(|A_{r+1}|^s, |A_{r+1}|^{s-1}) = (|A_{r+1}|, -1)^{s(s-1)} = 1$,
(ix) $(|A_{r+1}|^s, |B|^r) = (|A_{r+1}|, |B|)^{rs}$.
(x) $(|A_{r+1}|^s, |B_{s-1}|) = (|A_{r+1}|, |B_{s-1}|)^s$,
(xi) $(|B|^r, -1) = (|B|, -1)^r$,
(xii) $(|B|^r, |A_r|^{n-s+1}) = (|A_r|, |B|)^{r(n-s+1)}$,
(xiii) $(|B|^r, |A_{r+1}|^{s-1}) = (|A_{r+1}|, |B|)^{r(s-1)}$,
(xiv) $(|B|^r, |B|^r) = (|B|, -1)^r$,
(xv) $(|B|^r, |B_{s-1}|) = (|B|, |B_{s-1}|)^r$,
(xvi) $(|B_s|, -1)$,
(xvii) $(|B_s|, |A_r|^{n-s+1}) = (|A_r|, |B_s|)^{n-s+1}$,
(xviii) $(|B_s|, |A_{r+1}|^{s-1}) = (|A_{r+1}|, |B_s|)^{s-1}$,
(xix) $(|B_s|, |B|^r) = (|B|, |B_s|)^r$,
(xx) $(|B_s|, |B_{s-1}|) = (|B_{s-1}|, |B_s|)$.

All the well-known properties of the Hilbert norm residue symbol, in addition to 2.6, have been made use of in the above simplifications.

The factors (ii) and (viii) drop out automatically. The factors (iii) and (vii) give

$$(|A_r|, |A_{r+1}|)^{n(s-1)} (|A_r|, |A_{r+1}|)^{ns} = (|A_r|, |A_{r+1}|)^n.$$

The factors (iv) and (xii) give

$$(|A_r|, |B|)^{r(n-s+1)} (|A_r|, |B|)^{r(n-s)} = (|A_r|, |B|)^r.$$

The factors (ix) and (xiii) give

$$(|A_{r+1}|, |B|)^{rs} (|A_{r+1}|, |B|)^{r(s-1)} = (|A_{r+1}|, |B|)^r.$$

The factors (xi) and (xiv) give

$$(|B|, -1)^r (|B|, -1)^r = 1.$$

Hence the factor on the right hand side of 2.5 reduces to

2.7 $$\mathbf{f} = (|A_r|, |A_{r+1}|)^n (|A_r|, |B|)^r (|A_{r+1}|, |B|)^r (|A_r|, -1)^{n-s} (|A_{r+1}|, -1)^s \times \{(|A_r|, |B_s|)^{n-s+1}(|A_r|, |B_{s-1}|)^{n-s}\}\{(|A_{r+1}|, |B_s|)^{s-1} (|A_{r+1}|, |B_{s-1}|)^s\} \times \{(|B|, |B_s|)^r (|B|, |B_{s-1}|)^r\} \{(|B_s|, -1) (|B_s|, |B_{s-1}|)\}.$$

Now

$$\prod_{s=1}^{n} \{(|A_r|, |B_s|)^{n-s+1} (|A_r|, |B_{s-1}|)^{n-s}\} = (|A_r|, 1)^{n-1}\left\{\prod_{s=1}^{n-1} (|A_r|, |B_s|)^{2(n-s)}\right\} (|A_r|, |B|) = (|A_r|, |B|)$$

since $(|A_r|, 1) = (|A_r|, 1)_p = 1$ for any prime p.

Similarly

$$\prod_{s=1}^{n} \{(|A_{r+1}|, |B_s|)^{s-1} (|A_{r+1}|, |B_{s-1}|)^s\} = (|A_{r+1}|, |B|)^{n-1},$$

$$\prod_{s=1}^{n} \{(|B|, |B_s|)^r (|B|, |B_{s-1}|)^r\} = (|B|, -1)^r$$

and

$$\prod_{s=1}^{n} \{(|B_s|, -1) (|B_s|, |B_{s-1}|)\} = (-1, -1) \{c(B)\}$$

from 1.3 and 2.6.

Hence we get

$$\mathbf{F} = \prod_{s=1}^{n} \{\mathbf{f}\} = (|A_r|, |A_{r+1}|)^{n^2} (|A_r|, |B|)^{nr} (|A_{r+1}|, |B|)^{nr} \times (|A_r|, -1)^{\frac{1}{2}n(n-1)} (|A_{r+1}|, -1)^{\frac{1}{2}n(n+1)} (|A_r|, |B|) \times (|A_{r+1}|, |B|)^{n-1} (|B|, -1)^r (-1, -1) \{c(B)\}.$$

But

$$(|A_r|, |A_{r+1}|)^{n^2} = (|A_r|, |A_{r+1}|)^{n(n-1)+n} = (|A_{r+1}|, |A_r|)^n.$$

Hence

2.8 $$\mathbf{F} = (-1, -1) \{c(B)\} (|B|, -1)^r (|A_r|, -1)^{\frac{1}{2}n(n-1)} \times (|A_{r+1}|, -1)^{\frac{1}{2}n(n-1)} (|A_{r+1}|, -|A_r|)^n \times \{(|A_r|, |B|)^{nr+1} (|A_{r+1}|, |B|)^{nr+n-1}\}.$$

Now

$$\prod_{r=0}^{m-1} (|B|, -1)^r = (|B|, -1)^{\frac{1}{2}m(m-1)},$$

$$\prod_{r=0}^{m-1} \{(|A_r|, -1)^{\frac{1}{2}n(n-1)} (|A_{r+1}|, -1)^{\frac{1}{2}n(n-1)}\} = (|A|, -1)^{\frac{1}{2}n(n-1)}$$

and

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$$\prod_{r=0}^{m-1} \{(|A_{r+1}|, -|A_r|)^n\} = \{(-1, -1)\, c(A)\}^n$$

from 1.3 and 2.6.

Finally

$$\prod_{r=0}^{m-1} \{(|A_r|, |B|)^{nr+1} (|A_{r+1}|, |B|)^{nr+n-1}\} = (|A|, |B|)^{mn-1}.$$

Making use of all these simplifications, we get

$$\prod_{r=0}^{m-1} \{\mathbf{F}\} = (-1, -1)^m \{c(B)\}^m \times (|A|, -1)^{\frac{1}{2}n(n-1)} \times \{(-1, -1)\, c(A)\}^n (|B|, -1)^{\frac{1}{2}m(m-1)} (|A|, |B|)^{mn-1}.$$

This, after a little rearrangement and restoring the prime p, reduces to 2.4. This completes the proof of the theorem.

3. Some particular cases. We shall show that two well-known formulae are particular cases of the result 2.4.

Jones **(2)** has shown that if a is a non-zero rational number and B is an $n \times n$ matrix whose Hasse-Minkowski invariant is defined, then

3.1 $$c_p(a\,B) = c_p(B)\ (a, -1)_p^{\frac{1}{2}n(n+1)}\ (a, |B|)_p^{n-1}.$$

This can be easily deduced from 2.4 by observing that $a\ B = a \times B$, a being a scalar.

MacDuffee **(3)** has defined the direct sum of matrices A and B by the relation

$$A \dotplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where 0 is a null matrix of appropriate order. Let B be an $n \times n$ matrix whose Hasse-Minkowski invariant is defined and let

$$\Delta_m = B \dotplus B \dotplus \ldots \dotplus B$$

there being m B's in the direct sum. Bose and Connor **(4)** have shown that

3.2 $$c_p(\Delta_m) = (-1, -1)_p^{m-1} \{c_p(B)\}^m (|B|, -1)_p^{\frac{1}{2}m(m-1)}.$$

This can also be deduced as a particular case of 2.4 by observing that $\Delta_m = B \dotplus B \dotplus B \dotplus \ldots \dotplus B = I_m \times B$ where I_m is the identity matrix of order m.

Applications of the result 2.4 to some combinatorial problems connected with statistical designs are being investigated **(5)**.

4. Summary and acknowledgment. In this paper the Hasse-Minkowski invariant $c_p(A \times B)$ of the Kronecker product of matrices A and B is obtained in terms of $c_p(A)$, $c_p(B)$, $|A|$ and $|B|$; and two well known results regarding the Hasse-Minkowski invariant are shown to be particular cases of the result.

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References

1. F. D. Murnaghan, *The Theory of Group Representation* (Baltimore, 1938).
2. B. W. Jones, *The Arithmetic Theory of Quadratic Forms* (New York, 1950).
3. C. C. MacDuffee *Theory of Matrices* (New York, 1946).
4. R. C. Bose and W. S. Connor, *Combinatorial properties of group-divisible incomplete block designs,* Ann. Math. Stat., *23* (1952), 367–383.
5. M. N. Vartak, *On an application of the Kronecker product of matrices to statistical designs,* Ann. Math. Stat., *26* (1955), 420–438.

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