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ON MAHLER'S COMPOUND BODIES

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Abstract

Let $1 \leq M \leq N-1$ be integers and K be a convex, symmetric set in Euclidean N-space. Associated with K and M, Mahler identified the M^{th} compound body of K, $\langle K \rangle_M$, in Euclidean $\binom{N}{M}$-space. The compound body $\langle K \rangle_M$ is describable as the convex hull of a certain subset of the Grassmann manifold in Euclidean $\binom{N}{M}$-space determined by K and M. The sets K and $\langle K \rangle_M$ are related by a number of well-known inequalities due to Mahler.

Here we generalize this theory to the geometry of numbers over the adèle ring of a number field and prove theorems which compare an adelic set with its adelic compound body. In addition, we include a comparison of the adelic compound body with the adelic polar body and prove an adelic general transfer principle which has implications to Diophantine approximation over number fields.

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1. Introduction

In 1955, Mahler [6] illustrated the relationship between compound matrices and geometry of numbers by developing the theory of compound convex bodies in Euclidean N-space. Specifically, Mahler compared a convex, symmetric set with its compound body by exhibiting inequalities involving their volumes and inequalities involving their successive minima. These results enabled Mahler to deduce a general transfer principle which has applications to Diophantine approximation. More recently, the theory of compound bodies was used in

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proving the very deep subspace theorem of Schmidt (see [9, 11, 8]).

In the present paper we address the issue of generalizing Mahler's original work to the setting of an arbitrary number field. We accomplish this by replacing the rôle of Euclidean space with the adèle ring of the number field.

The subject of geometry of numbers over the adèle ring of a number field was developed independently by McFeat [7] and Bombieri and Vaaler [2] who proved the analog of Minkowski's successive minima theorem. We state this theorem in Section 2. Recently in [3] we further expanded the subject by introducing the adelic polar body. We recall the basic results in Section 6.

In Section 2 we describe the relevant objects which will occur and define our notation. Briefly, let k be a number field and for each place v of k let k_v be the completion of k with respect to v. For each place v we write R_v for a nonempty subset of $(k_v)^N$ satisfying the following conditions. If v is an infinite place of k, then R_v is a bounded, convex, symmetric set with nonempty interior. If v is a finite place of k, then R_v is a compact, open $\mathscr{O}_v$-module, where $\mathscr{O}_v$ is the ring of v-adic integers. For almost all finite places v we require that $R_v = (\mathscr{O}_v)^N$. If we let $(k_{\mathbf{A}})^N$ be the N-fold product of the adèle ring of k, then we say a subset $\mathscr{R}$ of $(k_{\mathbf{A}})^N$ is admissible if it has the form

$$\mathscr{R} = \prod_v R_v.$$

The set $\mathscr{R}$ is the adelic analog of the convex, symmetric set in the classical geometry of numbers, and the rôle of the lattice $\mathbb{Z}^N$ in $\mathbb{R}^N$ is replaced by the discrete subgroup isomorphic to $(k)^N$ in $(k_{\mathbf{A}})^N$. For each place v we normalize a Haar measure β_v^N on $(k_v)^N$ and write V_N for the Haar measure on $(k_{\mathbf{A}})^N$ induced by the product measure $\prod_v \beta_v^N$. The Haar measure V_N on $(k_{\mathbf{A}})^N$ is the analog of volume in $\mathbb{R}^N$. Just as in the classical geometry of numbers, one can define the successive minima of an admissible adelic set with respect to the lattice $(k)^N$. This requires a notion of dilation. Let $\sigma > 0$ be a real number. Dilation of an admissible adelic set $\mathscr{R}$ by σ is defined by

$$\sigma\mathscr{R} = \prod_{v|\infty} \sigma R_v \times \prod_{v\nmid\infty} R_v.$$

For each integer n, $1 \le n \le N$, the n^{th} successive minimum of $\mathscr{R}$ with respect to $(k)^N$ is defined by

$$\lambda_n = \inf\{\sigma > 0 : (\sigma\mathscr{R}) \cap (k)^N \text{ contains } n \text{ linearly independent vectors over } k\}.$$

Given integers $1 \leq M \leq N-1$ and $\mathscr{R} \subseteq (k_{\mathbf{A}})^N$ an admissible set, we shall define in Section 4, the M^{th} adelic compound of $\mathscr{R}$, denoted by $\langle R\rangle_M \subseteq (k_{\mathbf{A}})^{\binom{N}{M}}$. We begin by proving

THEOREM 1.1. *Let $\mathscr{R} \subseteq (k_{\mathbf{A}})^N$ be an admissible set. Then*

$$\gamma\left\{N^{d(r+s)M\binom{N}{M}/2}\right\}^{-1} \leq V_{\binom{N}{M}}(\langle\mathscr{R}\rangle_M)V_N(\mathscr{R})^{-\binom{N-1}{M-1}} \leq \gamma\left\{N^{d(r+s)M\binom{N}{M}/2}\right\}.$$

Here d is the degree of k over $\mathbb{Q}$, r and s are the number of real and complex places of k, respectively, and $\gamma = \gamma(k, M, N)$ is a constant explicitly defined in Section 4.

Next let $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $\mu_1, \mu_2, \ldots, \mu_{\binom{N}{M}}$ be the successive minima of $\mathscr{R}$ and $\langle\mathscr{R}\rangle_M$, respectively. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_{\binom{N}{M}}$ be the $\binom{N}{M}$ products of M distinct λ_n's and ordered so that $\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_{\binom{N}{M}}$. We then show:

THEOREM 1.2. *Let $\mu_1, \mu_2, \ldots, \mu_{\binom{N}{M}}$ and $\Lambda_1, \Lambda_2, \ldots, \Lambda_{\binom{N}{M}}$ be as above. Then for all $l = 1, 2, \ldots, \binom{N}{M}$,*

$$\gamma_1\left\{N^{d(r+s)M\binom{N}{M}/2}\right\}^{-1}\Lambda_l^d \leq \mu_l^d \leq \Lambda_l^d,$$

where the constant $\gamma_1 = \gamma_1(k, M, N)$ is defined in Section 5.

We then compare the $(N-1)^{th}$ adelic compound of $\mathscr{R}$ with the adelic polar body of $\mathscr{R}$. This requires us to introduce the concept of idelic dilations of admissible sets. Finally we prove an adelic general transfer principle and as an application, prove a transference result in Diophantine approximation over number fields in the context of the ring of S-integers.

2. Notation and normalizations

Let k be an algebraic number field of degree d over $\mathbb{Q}$. We write V_k for the collection of all nontrivial places of k. Suppose $v \in V_k$. If v is an archimedean place, we say v lies over infinity, denoted by $v|\infty$. If v is a nonarchimedean place then there exists a finite rational prime p such that v extends the place of p to V_k. In this case we say v lies over the finite rational prime p, written as $v\nmid\infty$ or $v|p$.

For each $v \in V_k$ we write k_v for the completion of k with respect to the place v. We define the local degree as

$$d_v = [k_v : \mathbb{Q}_v].$$

We now normalize two absolute values. For each place v of k, we normalize the absolute value $\| \ \|_v$ as follows:

(i) if $v|p$ then $\|p\|_v = p^{-1}$,

(ii) if $v|\infty$ then for $x \in k_v$, $\|x\|_v = |x|$ where $| \ |$ is the usual Euclidean absolute value on $\mathbb{R}$ or $\mathbb{C}$.

Thus $\| \ \|_v$ extends the usual p-adic absolute value if $v|p$ and the Euclidean absolute value if $v|\infty$. Our second normalized absolute value $| \ |_v$ is defined by

$$|x|_v = \|x\|_v^{d_v/d}.$$

This normalization gives to the *product formula*:

$$\prod_{v \in V_k} |x|_v = 1$$

for all $x \in k$, $x \neq 0$.

We extend our absolute value to vectors as follows. Let

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

denote a column vector in $(k_v)^N$. We define

$$|\vec{x}|_v = \max_{1 \le n \le N} \{|x_n|_v\}.$$

We extend the absolute value $\| \ \|_v$ by declaring

$$\|\vec{x}\|_v = \begin{cases} \left(\sum_{n=1}^{N} \|x_n\|_v^2 \right)^{1/2} & \text{if } v|\infty \\ \max_{1 \le n \le N} \{\|x_n\|_v\} & \text{if } v\nmid\infty. \end{cases}$$

Assume now that v is a finite place of k. We write $\mathcal{O}_v$ for the maximal compact (open) subring of k_v,

$$\mathcal{O}_v = \{x \in k_v : |x|_v \le 1\}.$$

A subset R_v in $(k_v)^N$ is called a k_v*-lattice* if it is a compact open $\mathscr{O}_v$-module in $(k_v)^N$. Clearly $(\mathscr{O}_v)^N$ is a k_v-lattice in $(k_v)^N$.

Let $k_{\mathbf{A}}$ denote the adèle ring of k. Elements of $k_{\mathbf{A}}$ shall be written as $x = (x_v)$ where x_v is the v-component of x for all $v \in V_k$. We write $(k_{\mathbf{A}})^N$ for the N-fold product of the adèles.

The additive group $k_{\mathbf{A}}$ is locally compact and thus there exists a Haar measure on $k_{\mathbf{A}}$ which is unique up to a multiplicative constant. We normalize this as follows.

(i) If $v|\infty$ and $k_v \cong \mathbb{R}$ we let β_v denote ordinary Lebesgue measure on $\mathbb{R}$.

(ii) If $v|\infty$ and $k_v \cong \mathbb{C}$ we let β_v denote Lebesgue measure on the complex plane multiplied by 2.

(iii) If $v|p$ we let β_v denote Haar measure on k_v normalized so that

$$\beta_v(\mathscr{O}_v) = |\mathscr{D}_v|_v^{d/2},$$

where $\mathscr{D}_v$ is the local different of k at v.

We now define a Haar measure β on $k_{\mathbf{A}}$ to be the product measure of the previously normalized local Haar measures:

$$\beta = \prod_{v \in V_k} \beta_v.$$

Technically, β determines a Haar measure on all open subgroups of the form

$$\prod_{v \in S} k_v \times \prod_{v \notin S} \mathscr{O}_v$$

where S is a finite collection of places of k containing all infinite places. Therefore the Haar measure on $k_{\mathbf{A}}$ is the unique measure which agrees with the product measure on these open subgroups. For each place v of k we let β_v^N denote the product measure on $(k_v)^N$. Similarly we define V_N to be the product measure β^N on $(k_{\mathbf{A}})^N$ (see [13]).

We may view k as a subset $k_{\mathbf{A}}$ by the natural diagonal map. The set $k \subseteq k_{\mathbf{A}}$ is referred to as the set of *principal adèles* and is a discrete subgroup of $k_{\mathbf{A}}$ with $k_{\mathbf{A}}/k$ compact.

Let $x = (x_v)$ be an element of $k_{\mathbf{A}}$ and α be a positive real number. We define scalar multiplication, αx, to be the point $y = (y_v)$ in $k_{\mathbf{A}}$ determined by

$$y_v = \begin{cases} \alpha x_v & \text{if } v|\infty \\ x_v & \text{if } v\nmid\infty. \end{cases}$$

We shall view elements of $(k_{\mathbf{A}})^N$ as column vectors $\vec{x}$ and extend our notion of scalar multiplication to vectors $\vec{x} \in (k_{\mathbf{A}})^N$ by

$$\alpha\vec{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_N \end{pmatrix}.$$

If $X \subseteq (k_{\mathbf{A}})^N$ then $\alpha X \subseteq (k_{\mathbf{A}})^N$ is obtained by applying scalar multiplication by α to each $\vec{x} \in X$.

For $v|\infty$ we say a subset $R_v \subseteq (k_v)^N$ is *symmetric* if $R_v = \alpha R_v$ for all $\alpha \in k_v$ with $\|\alpha\|_v = 1$. Let v be any place of k. We call a nonempty subset $R_v \subseteq (k_v)^N$ a *regular set* if it has the following form.

(i) If $v|\infty$ then R_v is a bounded, convex, symmetric subset with non- empty interior.

(ii) If $v\nmid\infty$ then R_v is a k_v-lattice in $(k_v)^N$.

For each $v \in V_k$ let R_v be a regular set in $(k_v)^N$. Assume that for almost all places v,

$$R_v = (\mathcal{O}_v)^N.$$

We now define

$$\mathscr{R} = \prod_{v \in V_k} R_v.$$

From our above assumption it is clear that $\mathscr{R} \subseteq (k_{\mathbf{A}})^N$. We call a subset $\mathscr{R}$ of $(k_{\mathbf{A}})^N$ *admissible* if it has the form described above. The set $\mathscr{R}$ is the adelic analog of the convex, symmetric set K in the classical geometry of numbers, and the rôle of the lattice $\mathbb{Z}^N$ in $\mathbb{R}^N$ is replaced by the discrete subgroup $(k)^N$ in $(k_{\mathbf{A}})^N$.

Let $\mathscr{R}$ be an admissible set in $(k_{\mathbf{A}})^N$. For each integer n, $1 \leq n \leq N$, we define the n^{th} successive minimum λ_n of $\mathscr{R}$ with respect to $(k)^N$ by

$$\lambda_n = \inf\{\sigma > 0 : (\sigma\mathscr{R}) \cap (k)^N \text{ contains } n \text{ linearly independent vectors over } k\}.$$

By our assumptions on $\mathscr{R}$,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N < \infty$$

(see [2]). We now recall the adelic successive minima theorem of Bombieri and Vaaler ([2, Theorem 3]).

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THEOREM 2.1. *Let $\mathscr{R}$ be an admissible subset of $(k_{\mathbf{A}})^N$ and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the successive minima of $\mathscr{R}$ with respect to $(k)^N$. Then*

$$\frac{2^{dN}\pi^{sN}}{(N!)^r(2N!)^s|\Delta_k|^{N/2}} \le (\lambda_1\lambda_2\ldots\lambda_N)^d V_N(\mathscr{R}) \le 2^{dN},$$

where Δ_k is the discriminant of k and r and s are the number of real and complex places of k, respectively.

3. Grassmann co-ordinates and local compound bodies

We begin this section by defining some notation which will facilitate our computations. Let N and M be integers such that $1 \le M \le N-1$. Define

$$\mathscr{J} = \big\{J \subseteq \{1, 2, \ldots, N\} : J \text{ contains } |J| = M \text{ elements}\big\}.$$

Clearly $\mathscr{J}$ has $|\mathscr{J}| = \binom{N}{M}$ elements. For each $J \in \mathscr{J}$, we write $J = \{j_1, j_2, \ldots, j_M\}$ where

$$1 \le j_1 < j_2 < \cdots < j_M \le N.$$

We order the elements of $\mathscr{J}$ using the lexicographical ordering:

$$\mathscr{J} = \{J_1, J_2, \ldots, J_{\binom{N}{M}}\}.$$

Next, suppose $A = (a_{nm})$ is an $N \times M$ matrix over k_v. For $J \in \mathscr{J}$ we define the $M \times M$ matrix ${}_JA$ by:

$${}_JA = (a_{nm}), \qquad n \in J, \quad 1 \le m \le M.$$

For an $N \times N$ matrix $B = (b_{nm})$ and for $J_l \in \mathscr{J}$, $J_h \in \mathscr{J}$ we define the $M \times M$ matrix ${}_{J_l}B_{J_h}$ by

$${}_{J_l}B_{J_h} = (b_{nm}), \qquad n \in J_l, \quad m \in J_h.$$

For $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_M \in (k_v)^N$, we write X for the $N \times M$ matrix given by:

$$X = (\vec{x}_1\vec{x}_2\cdots\vec{x}_M).$$

We define $\vec{\mathfrak{X}} = \vec{\mathfrak{X}}(X) \in (k_v)^{\binom{N}{M}}$ by

$$\vec{\mathfrak{X}}(X) = \begin{pmatrix} \mathfrak{X}_1(X) \\ \mathfrak{X}_2(X) \\ \vdots \\ \mathfrak{X}_{\binom{N}{M}}(X) \end{pmatrix},$$

where $\mathfrak{X}_l(X) = \det({}_{J_l}X)$ for $l = 1, 2, \ldots, \binom{N}{M}$.

Let B be an $N \times N$ nonsingular matrix over k_v. We define the M^{th} *compound of* B, $\langle B\rangle_M$ to be the $\binom{N}{M} \times \binom{N}{M}$ matrix given by:

$$\langle B\rangle_M = (\mathfrak{B}_{lh}(B)),$$

where $\mathfrak{B}_{lh}(B) = \det({}_{J_l}B_{J_h})$. It is well-known that (see [1]):

$$\det(\langle B\rangle_M) = \{\det(B)\}^{\binom{N-1}{M-1}}. \tag{3.1}$$

Let R_v be a regular subset of $(k_v)^N$. Below we define the M^{th} *local compound body of* R_v, $\langle R_v\rangle_M$: Let

$$(R_v)_M = \Big\{\vec{\mathfrak{X}}(X) \in (k_v)^{\binom{N}{M}} : X = (\vec{x}_1\vec{x}_2\cdots\vec{x}_M) \\ \text{with } \vec{x}_m \in R_v \text{ for } m = 1, 2, \ldots, M\Big\}.$$

For $v|\infty$ define $\langle R_v\rangle_M$ to be the convex hull of $(R_v)_M$ in $(k_v)^{\binom{N}{M}}$.

For $v\nmid\infty$ define $\langle R_v\rangle_M$ to be the $\mathcal{O}_v$-module in $(k_v)^{\binom{N}{M}}$ generated by $(R_v)_M$. It is clear that for all v, $\langle R_v\rangle_M$ is a regular subset of $(k_v)^{\binom{N}{M}}$. We remark that rather than introducing additional notation, we write $\langle\ \rangle_M$ to indicate both the compound of a matrix and the compound of a set. Of course the meaning of $\langle\ \rangle_M$ will be clear from the context in which it occurs. We now demonstrate the relationship between the M^{th} compound of a matrix and the M^{th} compound of a subset.

LEMMA 3.1. *Let B_v be an $N \times N$ nonsingular matrix over k_v. Let R_v be a regular subset of $(k_v)^N$. Then*

$$\langle B_vR_v\rangle_M = \langle B_v\rangle_M\langle R_v\rangle_M.$$

PROOF. Let $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_M$ be elements of R_v. Set

$$\vec{x}_m = B_v\vec{y}_m \quad \text{for } m = 1, 2, \ldots, M.$$

Write $X = (\vec{x}_1\vec{x}_2 \cdots \vec{x}_M)$ and $Y = (\vec{y}_1\vec{y}_2 \cdots \vec{y}_M)$ for the corresponding $N \times M$ matrices over k_v. As before, let

$$\vec{\mathfrak{X}}(X) = \begin{pmatrix} \mathfrak{X}_1(X) \\ \mathfrak{X}_2(X) \\ \vdots \\ \mathfrak{X}_{\binom{N}{M}}(X) \end{pmatrix} \quad \text{and} \quad \vec{\mathfrak{Y}}(Y) = \begin{pmatrix} \mathfrak{Y}_1(Y) \\ \mathfrak{Y}_2(Y) \\ \vdots \\ \mathfrak{Y}_{\binom{N}{M}}(Y) \end{pmatrix},$$

where $\mathfrak{X}_l(X) = \det({}_{J_l}X)$ for $l = 1, 2, \ldots, \binom{N}{M}$. We now compute:

$$\begin{aligned} \mathfrak{X}_l(X) = \det({}_{J_l}X) &= \sum_{n=1}^{\binom{N}{M}} \det\{({}_{J_l}B_{v\,J_n})({}_{J_n}Y)\} \\ &= \sum_{n=1}^{\binom{N}{M}} \det({}_{J_l}B_{vJ_n}) \det({}_{J_n}Y) \\ &= \sum_{n=1}^{\binom{N}{M}} \mathfrak{B}_{ln}(B_n)\mathfrak{Y}_n(Y). \end{aligned}$$

Therefore we have just shown that

$$\vec{\mathfrak{X}}(X) = \langle B_v \rangle_M \vec{\mathfrak{Y}}(Y) \tag{3.2}$$

and thus $(B_vR_v)_M = \langle B_v \rangle_M (R_v)_M$. It now follows in both the archimedean and nonarchimedean cases that

$$\langle B_vR_v \rangle_M = \langle B_v \rangle_M \langle R_v \rangle_M.$$

Identity (3.2) is useful and immediately implies the following

COROLLARY 3.2. *Let X_v be an $N \times M$ matrix over k_v and B_v be an $N \times N$ nonsingular matrix over k_v. Then*

$$\vec{\mathfrak{X}}(B_vX_v) = \langle B_v \rangle_M \vec{\mathfrak{X}}(X_v).$$

REMARK. Suppose $v \nmid \infty$. If R_v is any k_v-lattice in $(k_v)^N$ then there exists an $N \times N$ nonsingular matrix B_v over k_v such that

$$R_v = B_v(\mathcal{O}_v)^N$$

(see [13, Chapter II, Section 2]). Hence by Lemma 3.1,

$$\langle R_v\rangle_M = \langle B_v\rangle_M \langle (\mathcal{O}_v)^N\rangle_M.$$

It is a straightforward calculation to verify that

$$\langle (\mathcal{O}_v)^N\rangle_M = (\mathcal{O}_v)^{\binom{N}{M}}.$$

Therefore we may conclude that

$$\langle R_v\rangle_M = \langle B_v\rangle_M (\mathcal{O}_v)^{\binom{N}{M}}. \tag{3.3}$$

4. The adelic compound body

Given an admissible subset $\mathscr{R} = \prod_v R_v$ of $(k_{\mathbf{A}})^N$, we define the M^{th} *adelic compound body of* $\mathscr{R}$, $\langle\mathscr{R}\rangle_M$, by

$$\langle\mathscr{R}\rangle_M = \prod_v \langle R_v\rangle_M.$$

From our previous remarks in Section 3 we conclude that $\langle\mathscr{R}\rangle_M$ is an admissible subset of $(k_{\mathbf{A}})^{\binom{N}{M}}$.

For $v|\infty$ we define $S_v \subseteq (k_v)^N$ to be the v-adic unit L^2-ball

$$S_v = \{\vec{x} \in (k_v)^N : \|\vec{x}\|_v \le 1\}.$$

We define the positive constant $\gamma = \gamma(k, M, N)$ by

$$\gamma = |\Delta_k|^{(M-1)\binom{N}{M}/2} \prod_{v|\infty} \left\{ \beta_v^{\binom{N}{M}}(\langle S_v\rangle_M) \left(\beta_v^N(S_v)^{-\binom{N-1}{M-1}}\right)\right\},$$

where Δ_k is the discriminant of k. We remark that

$$\beta_v^N(S_v) = \begin{cases} \pi^{N/2}\Gamma(\frac{1}{2}N+1)^{-1} & \text{for } v \text{ real} \\ (2\pi)^N\Gamma(N+1)^{-1} & \text{for } v \text{ complex.} \end{cases}$$

The following theorem provides a relationship between the volume of $\mathscr{R}$ and its M^{th} compound body.

THEOREM 4.1. *Let $\mathscr{R} \subseteq (k_{\mathbf{A}})^N$ be an admissible set. Then*

$$\gamma\left\{N^{d(r+s)M\binom{N}{M}/2}\right\}^{-1} \le V_{\binom{N}{M}}(\langle\mathscr{R}\rangle_M) V_N(\mathscr{R})^{-\binom{N-1}{M-1}} \le \gamma\left\{N^{d(r+s)M\binom{N}{M}/2}\right\},$$

where r and s are the number of real and complex places of k, respectively.

In the classical situation, Mahler [6] proved this result by appealing to a theorem of Jordan (see John [5] or Schmidt [10]) which, in essence, states that every convex symmetric set in $\mathbb{R}^N$ can be approximated by an ellipsoid. We prove Theorem 4.1 by utilizing this approximation technique at each archimedean place. Thus we need a version of Jordan's theorem over $\mathbb{C}^N$. By a complex ellipsoid we mean a nonsingular linear transformation of the unit L^2-ball in $\mathbb{C}^N$. The proof of a generalized Jordan theorem in an N-dimensional vector space over any archimedean field is very similar to the classical one and thus we merely outline the argument below.

LEMMA 4.2. *Let v be an archimedean place of k and $R_v \subseteq (k_v)^N$ a regular subset. Then there exists an ellipsoid E_v centered about the origin satisfying*

$$E_v \subseteq R_v \subseteq \sqrt{N}\, E_v.$$

SKETCH OF PROOF. If $k_v \cong \mathbb{R}$ then this is Jordan's result, thus we need only prove the lemma for $k_v \cong \mathbb{C}$. Since R_v is compact in k_v^N, there exists an ellipsoid E_v with maximal volume satisfying

$$E_v \subset R_v.$$

Without loss of generality we may assume that $E_v = S_v$, the unit L^2-ball in k_v^N. We claim that E_v is the ellipsoid which the lemma asserts exists. If not, then there must exist a vector $\vec{w} \in R_v$ with $\vec{w} \notin \sqrt{N} E_v$, that is

$$\|\vec{w}\|_v > \sqrt{N}.$$

Let $\mathfrak{A}$ be the subspace of k_v^N spanned by $\vec{w}$ and $\mathfrak{A}^\perp$ be the orthogonal complement with respect to the Hermitian inner product. We define the orthogonal projection matrices $P_{\vec{w}}$ and $P_{\vec{w}}^\perp$ onto $\mathfrak{A}$ and $\mathfrak{A}^\perp$, respectively, by:

$$P_{\vec{w}} = \vec{w}(\vec{w}^*\vec{w})^{-1}\vec{w}^*$$

and

$$P_{\vec{w}}^\perp = \mathbf{1}_N - P_{\vec{w}},$$

where $\vec{w}^*$ is the complex conjugate transpose of $\vec{w}$ and $\mathbf{1}_N$ is the $N \times N$ identity matrix. Let

$$r = \sqrt{\frac{\|\vec{w}\|_v^2}{\|\vec{w}\|_v^2 - 1}}, \tag{4.1}$$

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and define the cone

$$\mathscr{C} = \left\{ \vec{z} \in k_v^N : \frac{\| P_{\vec{w}} \vec{z} \|_v}{\| \vec{w} \|_v} + \frac{\| P_{\vec{w}}^{\perp} \vec{z} \|_v}{r} \leq 1 \right\}.$$

For positive real numbers a and b consider the ellipsoid

$$E(a, b) = \left\{ \vec{z} \in k_v^N : a^2 \| P_{\vec{w}} \vec{z} \|_v^2 + b^2 \| P_{\vec{w}}^{\perp} \vec{z} \|_v^2 \leq 1 \right\}.$$

We note that

$$\begin{aligned} \frac{\| P_{\vec{w}} \vec{z} \|_v}{\| \vec{w} \|_v} + \frac{\| P_{\vec{w}}^{\perp} \vec{z} \|_v}{r} &= \frac{a}{\| \vec{w} \|_v a} \| P_{\vec{w}} \vec{z} \|_v + \frac{b}{rb} \| P_{\vec{w}}^{\perp} \vec{z} \|_v \\ &\leq \left(\frac{1}{\| \vec{w} \|_v^2 a^2} + \frac{1}{r^2 b^2} \right)^{1/2} \left(a^2 \| P_{\vec{w}} \vec{z} \|_v^2 + b^2 \| P_{\vec{w}}^{\perp} \vec{z} \|_v^2 \right)^{1/2}. \end{aligned}$$

Hence if

$$\frac{1}{\| \vec{w} \|_v^2 a^2} + \frac{1}{r^2 b^2} = 1$$

then

$$E(a, b) \subseteq \mathscr{C}.$$

Furthermore, if $b > a$ then the previous identity implies that

$$\frac{1}{b^2} < 1$$

and thus $P_{\vec{w}}^{\perp}(E(a, b)) \subseteq E_v$. As $E(a, b)$ is an ellipsoid contained in $\mathscr{C}$ with the property that

$$P_{\vec{w}}^{\perp}(E(a, b)) \subseteq E_v,$$

it follows that $E(a, b)$ is contained in the convex hull of E_v and $\alpha \vec{w}$ with $\|\alpha\|_v = 1$. That is, $E(a, b)$ is contained in the smallest convex, symmetric set containing E_v and $\vec{w}$, and therefore we conclude

$$E(a, b) \subseteq R_v.$$

Thus we wish to maximize the volume of $E(a, b)$ given the constraints

$$\frac{1}{\| \vec{w} \|_v^2 a^2} + \frac{1}{r^2 b^2} = 1 \quad \text{and} \quad b > a. \tag{4.2}$$

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A short calculation reveals

$$\beta_v^N(E(a,b)) = (ab^{(N-1)})^{-d_v}\beta_v^N(S_v). \tag{4.3}$$

So we wish to minimize $a^2b^{2(N-1)}$ given the constraints of (4.2). This minimum occurs when

$$a = \frac{\sqrt{N}}{\|\vec{w}\|_v} \quad \text{and} \quad b = \frac{\sqrt{N}}{r\sqrt{N-1}}.$$

By (4.1) and (4.3) we have

$$\begin{aligned}\beta_v^N(E(a,b))^2/\beta_v^N(E_v)^2 &= (a^{-2}b^{-2(N-1)})^{d_v}\\ &= \left\{\left(\frac{\|\vec{w}\|_v^{2N}}{(\|\vec{w}\|_v^2-1)^{(N-1)}}\right)\left(\frac{N^N}{(N-1)^{(N-1)}}\right)^{-1}\right\}^{d_v}.\end{aligned}$$

The function $f(x) = x^N(x-1)^{1-N}$ is increasing for real $x \geq N$. Since $\|\vec{w}\|_v^2 > N$, $f(\|\vec{w}\|_v^2) > f(N)$ and thus

$$\beta_v^N(E(a,b)) > \beta_v^N(E_v).$$

This contradicts the maximality of E_v.

PROOF OF THEOREM 4.1. We write

$$\mathscr{R} = \prod_v R_v,$$

where R_v is a regular subset of $(k_v)^N$ for each v. For $v\nmid\infty$ select an $N \times N$ nonsingular matrix B_v over k_v such that

$$R_v = B_v(\mathscr{O}_v)^N.$$

For each place $v|\infty$, let $E_v \subseteq (k_v)^N$ be the v-adic ellipsoid of Lemma 4.2. That is,

$$E_v \subseteq R_v \subseteq \sqrt{N}E_v.$$

For $v|\infty$ let B_v be an $N \times N$ nonsingular matrix over k_v so that

$$E_v = B_vS_v,$$

where $S_v \subseteq (k_v)^N$ is the v-adic unit L^2-ball. Next define

$$\mathscr{E} = \prod_v T_v \subseteq (k_{\mathbf{A}})^N$$

by

$$T_v = \begin{cases} E_v = B_v S_v & \text{for } v|\infty \\ B_v(\mathscr{O}_v)^N & \text{for } v\nmid\infty. \end{cases}$$

Clearly $\mathscr{E}$ is an admissible set and

(4.4) $$\mathscr{E} \subseteq \mathscr{R} \subseteq \sqrt{N}\mathscr{E}.$$

By Lemma 3.1 and the remarks which follow it we have

$$\langle T_v\rangle_M = \begin{cases} \langle B_v\rangle_M \langle S_v\rangle_M & \text{for } v|\infty \\ \langle B_v\rangle_M (\mathscr{O}_v)^{\binom{N}{M}} & \text{for } v\nmid\infty. \end{cases}$$

We now compute the volume of $\langle T_v\rangle_M$.

$$\beta_v^{\binom{N}{M}}(\langle T_v\rangle_M) = \begin{cases} |\det\langle B_v\rangle_M|_v^d \beta_v^{\binom{N}{M}}(\langle S_v\rangle_M) & \text{for } v \mid \infty \\ |\det\langle B_v\rangle_M|_v^d |\mathscr{D}_v|_v^{\frac{d}{2}\binom{N}{M}} & \text{for } v\nmid\infty. \end{cases}$$

Thus,

$$V_N(\mathscr{E}) = |\Delta_k|^{-N/2}\left(\prod_v |\det B_v|_v^d\right)\left(\prod_{v|\infty}\beta_v^N(S_v)\right)$$

and

$$V_{\binom{N}{M}}(\langle\mathscr{E}\rangle_M) = |\Delta_k|^{-\binom{N}{M}/2}\left(\prod_v |\det\langle B_v\rangle_M|_v^d\right)\left(\prod_{v|\infty}\beta_v^{\binom{N}{M}}(\langle S_v\rangle_M)\right).$$

By (3.1) we may write

$$V_{\binom{N}{M}}(\langle\mathscr{E}\rangle_M) = |\Delta_k|^{-\binom{N}{M}/2}\left(\prod_v |\det B_v|_v^d\right)^{\binom{N-1}{M-1}}\left(\prod_{v|\infty}\beta_v^{\binom{N}{M}}(\langle S_v\rangle_M)\right).$$

Hence

(4.5) $$V_{\binom{N}{M}}(\langle\mathscr{E}\rangle_M)\cdot V_N(\mathscr{E})^{-\binom{N-1}{M-1}} = \gamma.$$

We may also report the compound body analog of (4.4):

(4.6) $$\langle\mathscr{E}\rangle_M \subseteq \langle\mathscr{R}\rangle_M \subseteq \sqrt{N}^M\langle\mathscr{E}\rangle_M.$$

From (4.4) and (4.6) we conclude

$$\begin{aligned} V_{\binom{N}{M}}(\langle\mathscr{E}\rangle_M)\cdot V_N(\sqrt{N}\mathscr{E})^{-\binom{N-1}{M-1}} &\le V_{\binom{N}{M}}(\langle\mathscr{R}\rangle_M)\cdot V_N(\mathscr{R})^{-\binom{N-1}{M-1}} \\ &\le V_{\binom{N}{M}}(\sqrt{N}^M\langle\mathscr{E}\rangle_M)\cdot V_N(\mathscr{E})^{-\binom{N-1}{M-1}}. \end{aligned}$$

Clearly

$$V_N(\sqrt{N}\mathscr{E}) = \left(\prod_{v|\infty} N^{dN/2}\right) V_N(\mathscr{E})$$

and

$$V_{\binom{N}{M}}(\sqrt{N}^M) = \left(\prod_{v|\infty} N^{dM\binom{N}{M}/2}\right) V_{\binom{N}{M}}(\langle\mathscr{E}\rangle_M).$$

The theorem now follows from (4.5).

5. Successive minima

Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be the successive minima of $\mathscr{R}$ and let $\mu_1, \mu_2, \dots, \mu_{\binom{N}{M}}$ be the successive minima associated with $\langle\mathscr{R}\rangle_M$ in $(k_{\mathbf{A}})^{\binom{N}{M}}$. For $J \in \mathscr{J}$,

$$J = \{1 \le i_1 < i_2 < \cdots < i_M \le N\},$$

define

$$P_J = \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_M}.$$

Let

$$\mathscr{M} = \left\{P_{J_l} : l = 1, 2, \dots \binom{N}{M}\right\}.$$

We now select a permutation $\sigma : \{1, 2, \dots, \binom{N}{M}\} \to \{1, 2, \dots, \binom{N}{M}\}$ such that if we write

$$\Lambda_l = P_{J_{\sigma(l)}}$$

for $l = 1, 2, \dots, \binom{N}{M}$, then

$$0 < \Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_{\binom{N}{M}} < \infty.$$

We remark that a simple counting argument shows

(5.1) $$(\lambda_1\lambda_2\dots\lambda_N)^{\binom{N-1}{M-1}} = \Lambda_1\Lambda_2\dots\Lambda_{\binom{N}{M}}.$$

Next we define the positive constant $\gamma_1 = \gamma_1(k, M, N)$ by

$$\gamma_1 = \frac{2^{d(1-M)\binom{N}{M}}\pi^{s\binom{N}{M}}\gamma^{-1}}{\left(\binom{N}{M}!\right)^r (2\binom{N}{M}!)^s|\Delta_k|^{\binom{N}{M}/2}},$$

where r and s are the number of real and complex places of k, respectively and γ is the constant from Section 4.

We now show that the μ_l's and Λ_l's are compatible.

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THEOREM 5.1. *Let* $\mu_1, \mu_2, \ldots, \mu_{\binom{N}{M}}$ *and* $\Lambda_1, \Lambda_2, \ldots, \Lambda_{\binom{N}{M}}$ *be as above. Then for all* $l = 1, 2, \ldots, \binom{N}{M}$,

$$\gamma_1 \left\{ N^{d(r+s)M\binom{N}{M}/2} \right\}^{-1} \Lambda_l^d \leq \mu_l^d \leq \Lambda_l^d.$$

PROOF. Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N$ be linearly independent vectors in $(k)^N$ associated with the successive minima $\lambda_1, \lambda_2, \ldots, \lambda_N$ of $\mathscr{R}$. That is, for each $n = 1, 2, \ldots, N$ and $\lambda > \lambda_n$,

$$\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\} \subseteq \lambda \mathscr{R}.$$

Write U for the $N \times N$ nonsingular matrix over k given by

$$U = (\vec{u}_1 \vec{u}_2 \cdots \vec{u}_N).$$

For each $l = 1, 2, \ldots, \binom{N}{M}$, define $\vec{\mathfrak{U}}_l \in (k)^{\binom{N}{M}}$ by

$$\vec{\mathfrak{U}}_l = \begin{pmatrix} \mathfrak{U}_1(U_{J_l}) \\ \mathfrak{U}_2(U_{J_l}) \\ \vdots \\ \mathfrak{U}_{\binom{N}{M}}(U_{J_l}) \end{pmatrix},$$

where $\mathfrak{U}_h(U_{J_l}) = \det({}_{J_h}U_{J_l})$ for $h = 1, 2, \ldots, \binom{N}{M}$. We remark that since $\lambda_{i_1}^{-1}\vec{u}_{i_1}, \lambda_{i_2}^{-1}\vec{u}_{i_2}, \ldots, \lambda_{i_M}^{-1}\vec{u}_{i_M}$ are all in $\mathscr{R}$, it follows that

$$\vec{\mathfrak{U}}_l \in P_{J_l}\langle \mathscr{R} \rangle_M.$$

Also, (3.1) reveals that $\vec{\mathfrak{U}}_1, \vec{\mathfrak{U}}_2, \ldots, \vec{\mathfrak{U}}_{\binom{N}{M}}$ are linearly independent vectors in $(k)^{\binom{N}{M}}$. Next we define real numbers $\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_{\binom{N}{M}}$ by:

$$\Gamma'_l = \inf\{\sigma > 0 : \vec{\mathfrak{U}}_l \in \sigma\langle \mathscr{R} \rangle_M\}.$$

Trivially, for all $l = 1, 2, \ldots, \binom{N}{M}$,

$$\Gamma'_l \leq P_{J_l}. \tag{5.2}$$

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_{\binom{N}{M}}$ be a permutation of the numbers $\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_{\binom{N}{M}}$ such that

$$0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_{\binom{N}{M}} < \infty.$$

Fix an l, $1 \leq l \leq \binom{N}{M}$, then for any collection of integers $1 \leq j_1 < j_2 < \cdots < j_l \leq \binom{N}{M}$ we have

$$\Gamma_l \leq \max\{\Gamma'_{j_1}, \Gamma'_{j_2}, \ldots, \Gamma'_{j_l}\}.$$

Select integers $j_1, j_2, \ldots, j_l$ so that

$$\max\{P_{J_{j_1}}, P_{J_{j_2}}, \ldots, P_{J_{j_l}}\} = \Lambda_l.$$

Thus by (5.2) we have

$$\Gamma_l \leq \Lambda_l.$$

Since $\vec{\mathfrak{U}}_1, \vec{\mathfrak{U}}_2, \ldots, \vec{\mathfrak{U}}_l$ are linearly independent and contained in $\Gamma_l\langle\mathscr{R}\rangle_M$, we must have

$$\mu_l \leq \Lambda_l \tag{5.3}$$

which is our upper bound.

For the lower bound, we recall the adelic successive minima theorem:

$$\frac{2^{dN}\pi^{sN}}{(N!)^r(2N!)^s|\Delta_k|^{N/2}} \leq (\lambda_1\lambda_2\ldots\lambda_N)^d V_N(\mathscr{R}) \leq 2^{dN}.$$

This together with (5.1) yields

$$(\mu_1\mu_2\ldots\mu_{\binom{N}{M}})^d V_{\binom{N}{M}}(\langle\mathscr{R}\rangle_M)((\lambda_1\lambda_2\ldots\lambda_N)^d V_N(\mathscr{R}))^{-\binom{N-1}{M-1}}$$
$$[100pt]\prod_{h=1}^{\binom{N}{M}}(\mu_h/\Lambda_h)^d V_{\binom{N}{M}}(\langle\mathscr{R}\rangle_M)\cdot V_N(\mathscr{R})^{-\binom{N-1}{M-1}} \geq \gamma\gamma_1.$$

Theorem 4.1 along with (5.3) and the previous inequality show

$$\gamma_1(N^{d(r+s)M\binom{N}{M}/2})^{-1}\Lambda_l^d \leq \mu_l^d$$

which is the required lower bound.

6. The compound body $\langle\mathscr{R}\rangle_{(n-1)}$ and the polar body $\mathscr{R}^*$

The set $\langle\mathscr{R}\rangle_{(N-1)}$ is readily seen to be a subset of $(k_{\mathbf{A}})^N$, and thus has the same dimension as $\mathscr{R}$. In [6], Mahler demonstrated a relationship between $\langle\mathscr{R}\rangle_{(N-1)}$ and $\mathscr{R}$. In fact, just as in the classical setting, the adelic compound body $\langle\mathscr{R}\rangle_{(N-1)}$ is compatible with the adelic polar body. We begin by briefly

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recalling the adelic polar body as described in [3] and then considering the local situation.

Let $\mathscr{R} = \prod_v R_v$ be an admissible subset of $(k_{\mathbf{A}})^N$. For each place v of k, define the *local polar body* R_v^*, of R_v by:

$$R_v^* = \left\{ \vec{x} \in (k_v)^N : \left| \sum_{n=1}^{N} x_n y_n \right|_v \leq 1 \text{ for all } \vec{y} \in R_v \right\}.$$

If R_v is a regular subset of $(k_v)^N$ then the same is true for R_v^* and

$$(R_v^*)^* = R_v.$$

Also if A_v is an $N \times N$ nonsingular matrix over k_v then

$$(A_v R_v)^* = (A_v^T)^{-1} R_v^*,$$

where A_v^T is the transpose of A_v. We define the *adelic polar body* $\mathscr{R}^*$ by:

$$\mathscr{R}^* = \prod_v R_v^* \subseteq (k_{\mathbf{A}})^N.$$

The sets $\mathscr{R}$ and $\mathscr{R}^*$ possess two fundamental reciprocal properties. The first is that

$$1 \ll V_N(\mathscr{R}) V_N(\mathscr{R}^*) \ll 1,$$

and the second is if $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*$ are the successive minima of $\mathscr{R}$ and $\mathscr{R}^*$, respectively, then for each $n = 1, 2, \ldots, N$,

$$1 \leq (\lambda_n \lambda_{N+1-n}^*)^d \ll 1.$$

Here the constants implied the Vinogradov symbol depend only upon the number field k and N, and are explicitly given in [3].

In what follows, v is an archimedean place of k. Again, we write $S_v \subseteq (k_v)^N$ for the unit L^2-ball:

$$S_v = \{\vec{x} \in (k_v)^N : \|\vec{x}\|_v \leq 1\}.$$

Below we prove that $S_v^* = \langle S_v \rangle_{(N-1)} = S_v$. It is a well-known fact that $S_v^* = S_v$, thus we need only prove the second equality.

It will be useful to define the $N \times N$ matrix $(\pm\mathbf{1})_N = (e_{mn})$ where

$$e_{mn} = \begin{cases} (-1)^{n+1} & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

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Alternatively, $(\pm\mathbf{1})_N$ has the following shape:

$$(\pm\mathbf{1})_N = \begin{pmatrix} 1 & & & & \\ & -1 & & \bigcirc & \\ & & 1 & & \\ & \bigcirc & & -1 & \\ & & & & \ddots \end{pmatrix}.$$

Finally, recall that for $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_M \in (k_v)^N$ we write Y for the associated $N \times M$ matrix defined by:

$$Y = (\vec{y}_1 \vec{y}_2 \ldots \vec{y}_M)$$

and write

$$\vec{\mathfrak{Y}}(Y) = \begin{pmatrix} \mathfrak{Y}_1(Y) \\ \mathfrak{Y}_2(Y) \\ \vdots \\ \mathfrak{Y}_{\binom{N}{M}}(Y) \end{pmatrix} \in (k_v)^{\binom{N}{M}}$$

where $\mathfrak{Y}_l(Y) = \det({}_{J_l}Y)$ for $l = 1, 2, \ldots, \binom{N}{M}$. We note that by the Cauchy-Binet formula (see [2]) we have

$$\|\vec{\mathfrak{Y}}_l(Y)\|_v = \|\det(Y^*Y)\|_v^{1/2}, \tag{6.1}$$

where Y^* is the complex conjugate transpose of Y. We now prove the following:

LEMMA 6.1. *Given $S_v \subseteq (k_v)^N$ as above,*

$$\langle S_v \rangle_{(N-1)} = S_v.$$

PROOF. Mahler proved this in the case when $k_v \cong \mathbb{R}$ (see [6, Section 16]), so we need only consider the case $k_v \cong \mathbb{C}$. First suppose $\vec{x} \in S_v$. Let $c \in k_v$ with $\|c\|_v = 1$ be a constant to be chosen later. Select orthogonal (with respect to the Hermitian inner product) vectors $c\vec{y}_1, \vec{y}_2, \ldots \vec{y}_{N-1}$ in $(k_v)^N$ such that the following hold:

(i) $\vec{x}$ is orthogonal to $(\pm\mathbf{1})_N \vec{y}_n$ for $n = 1, 2, \ldots, N-1$;
(ii) $\|\vec{y}_1\|_v = \|\vec{x}\|_v$;
(iii) $\|\vec{y}_n\|_v = 1$ for $n = 2, 3, \ldots, N-1$.

Let Y be the $N \times (N-1)$ matrix over k_v defined by

$$Y = (c\vec{y}_1 \vec{y}_2 \ldots \vec{y}_{N-1}).$$

It is simple to verify that $\vec{\mathfrak{Y}}(Y) \in (k_v)^N$ is orthogonal to $(\pm\mathbf{1})_N \vec{y}_n$ for each $n = 1, 2, \ldots, N-1$. Also, by orthogonality and (6.1) we have

$$\|\vec{\mathfrak{Y}}(Y)\|_v = \| \det(Y^*Y)\|_v^{1/2} = \prod_{n=1}^{N-1} \|\vec{y}_n\|_v.$$

From (ii) and (iii) this implies

$$\|\vec{\mathfrak{Y}}(Y)\|_v = \|\vec{x}\|_v.$$

Therefore we see that $\vec{\mathfrak{Y}}(Y)$ and $\vec{x}$ are dependent vectors with the same L^2-norm. Select $c \in k_v$ with $\|c\|_v = 1$ so that $\vec{\mathfrak{Y}}(Y) = \vec{x}$. Since $\vec{x} \in S_v$, $\{c\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_{N-1}\} \subseteq S_v$. Thus

$$\vec{x} = \vec{\mathfrak{Y}}(Y) \in (S_v)_{(N-1)} \subseteq \langle S_v \rangle_{(N-1)},$$

and $S_v \subseteq \langle S_v \rangle_{(N-1)}$.

Next, let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{N-1}\} \subseteq S_v$. We claim that $\vec{\mathfrak{X}}(X) \in S_v$. It follows from the Cauchy-Binet formula (6.1), and an application of Hadamard's inequality (or directly from an inequality of Fisher [4]), that

$$\|\vec{\mathfrak{X}}(X)\|_v = \| \det(X^*X)\|_v^{1/2} \leq \prod_{n=1}^{N-1} \|\vec{x}_n\|_v.$$

Since each L^2-norm in the product is bounded above by 1, we have $\vec{\mathfrak{X}}(X) \in S_v$. Thus

$$(S_v)_{(N-1)} \subseteq S_v.$$

Since $\langle S_v \rangle_{(N-1)}$ is the convex hull of $(S_v)_{(N-1)}$ and being that S_v is convex we conclude $\langle S_v \rangle_{(N-1)} \subseteq S_v$. Hence

$$\langle S_v \rangle_{(N-1)} = S_v.$$

We are now in a position to analyze arbitrary regular subset $R_v \subseteq (k_v)^N$ for v archimedean. We begin by observing that the $(N-1)$ compound of a nonsingular matrix is similar to its adjugate matrix. Specifically,

$$(6.2) \qquad (A_v^T)^{-1} = (\pm\mathbf{1})_N \det(A_v)^{-1} \langle A_v \rangle_{(N-1)} (\pm\mathbf{1})_N$$

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where A_v is an $N \times N$ nonsingular matrix over k_v, for any place v of k. Next we define the constant $\tau_v(N)$ by:

$$\tau_v(N) = \begin{cases} \pi^{-N/2}\Gamma(\frac{1}{2}N+1) & \text{for } v \text{ real} \\ (2\pi)^{-N}\Gamma(N+1) & \text{for } v \text{ complex,} \end{cases}$$

where Γ is the gamma function. We note that for all $v|\infty$,

$$\beta_v^N(S_v) = \tau_v(N)^{-1}.$$

THEOREM 6.2. *Let v be an archimedean place of k and $R_v \subseteq (k_v)^N$ a regular subset. Then*

$$N^{-\frac{1}{2}N}(\tau_v(N)\beta_v^N(R_v))^{1/d_v} R_v^* \subseteq \langle R_v\rangle_{(N-1)} \subseteq N^{\frac{1}{2}N}(\tau_v(N)\beta_v^N(R_v))^{1/d_v} R_v^*.$$

PROOF. By Lemma 4.2, there exists an ellipsoid E_v centered about the origin satisfying

$$E_v \subseteq R_v \subseteq \sqrt{N} E_v. \tag{6.3}$$

Clearly there exists an orthogonal (unitary, in the case when v is complex) $N \times N$ matrix U over k_v such that

$$(\pm\mathbf{1})_N(UE_v) = UE_v.$$

If we were to multiply each set of (6.3) by U we would merely rotate the sets in space. Thus without loss of generality, we may assume that E_v is already invariant under the action of $(\pm\mathbf{1})_N$. That is,

$$(\pm\mathbf{1})_N E_v = E_v. \tag{6.4}$$

Next we write the ellipsoid as

$$E_v = A_v S_v,$$

where A_v is an $N \times N$ nonsingular matrix over k_v. Without loss of generality, A_v may be chosen so that

$$\det(A_v) = \|\det(A_v)\|_v.$$

From Lemma 3.1 and Lemma 6.1 we have

$$\langle E_v\rangle_{(N-1)} = \langle A_v S_v\rangle_{(N-1)} = \langle A_v\rangle_{(N-1)}\langle S_v\rangle_{(N-1)} = \langle A_v\rangle_{(N-1)} S_v.$$

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By the containments of (6.3) the above implies

$$\langle A_v\rangle_{(N-1)} S_v \subseteq \langle R_v\rangle_{(N-1)} \subseteq \sqrt{N}^{(N-1)} \langle A_v\rangle_{(N-1)} S_v. \tag{6.5}$$

Next we observe that

$$E_v^* = (A_v S_v)^* = (A_v^T)^{-1} S_v^* = (A_v^T)^{-1} S_v.$$

Alternatively, by (6.2), this may be expressed as

$$E_v^* = (\pm\mathbf{1})_N \det(A_v)^{-1} \langle A_v\rangle_{(N-1)} (\pm\mathbf{1})_N S_v. \tag{6.6}$$

Clearly,

$$(\pm\mathbf{1})_N S_v = S_v$$

and by (6.4)

$$E_v^* = ((\pm\mathbf{1})_N E_v)^* = (\pm\mathbf{1})_N E_v^*.$$

In view of these remarks, (6.6) becomes

$$E_v^* = \det(A_v)^{-1} \langle A_v\rangle_{(N-1)} S_v. \tag{6.7}$$

We remark that by the definition of the polar body, if $R_v \subseteq T_v$ then $T_v^* \subseteq R_v^*$. Thus from (6.3) we have

$$\sqrt{N}^{-1} E_v^* \subseteq R_v^* \subseteq E_v^*,$$

and by (6.7) this yields

$$\sqrt{N}^{-1} \det(A_v)^{-1} \langle A_v\rangle_{(N-1)} S_v \subseteq R_v^* \subseteq \det(A_v)^{-1} \langle A_v\rangle_{(N-1)} S_v.$$

It now follows from (6.5) that

$$\det(A_v) R_v^* \subseteq \langle R_v\rangle_{(N-1)} \subseteq \sqrt{N}^{N} \det(A_v) R_v^*. \tag{6.8}$$

We note that

$$\beta_v^N(E_v)/\beta_v^N(S_v) = \|\det(A_v)\|_v^{d_v} = (\det(A_v))^{d_v}.$$

Also from (6.3),

$$\beta_v^N(E_v) \le \beta_v^N(R_v) \le \sqrt{N}^{N d_v} \beta_v^N(E_v).$$

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Therefore (6.8) implies:

$$N^{-\frac{1}{2}N}(\beta_v^N(S_v)^{-1}\beta_v^N(R_v))^{1/d_v}R_v^* \subseteq \langle R_v\rangle_{(N-1)}$$
$$\subseteq N^{\frac{1}{2}N}(\beta_v^N(S_v)^{-1}\beta_v^N(R_v))^{1/d_v}R_v^*,$$

which is the conclusion of the theorem.

We now turn our attention to the nonarchimedean places of k. It is a straightforward calculation to show that

$$((\mathcal{O}_v)^N)^* = (\mathcal{O}_v)^N \quad \text{and} \quad \langle(\mathcal{O}_v)^N\rangle_{(N-1)} = (\mathcal{O}_v)^N.$$

Next we wish to consider arbitrary k_v-lattices in $(k_v)^N$. We pause momentarily to give an outline of our plan of attack. We wish to prove a result similar to Theorem 6.2. Here in the nonarchimedean case, the set $(\mathcal{O}_v)^N$ will play the rôle of the L^2-ball, S_v, in the archimedean setting. Recall that any k_v-lattice, R_v may be expressed as

$$R_v = B_v(\mathcal{O}_v)^N,$$

where B_v is an $N \times N$ nonsingular matrix over k_v. Thus at the finite places, very regular set is an "ellipsoid." Hence there is no need for a nonarchimedean form of Lemma 4.2. This suggests that the sets $\langle R_v\rangle_{(N-1)}$ and R_v^* differ only by a constant multiple.

Clearly $(\mathcal{O}_v)^N$ is invariant under the action of multiplication of $(\pm\mathbf{1})_N$. However the k_v-lattice R_v might not have this strong symmetry property which would then prevent us from utilizing the identity of (6.2). This issue was quickly dispensed with in the archimedean case by the basic fact that we may always find an orthogonal (unitary) matrix which rotates the ellipsoid into the appropriate position. Thus we need to insure that for any given $R_v \subseteq (k_v)^N$, there exists an $N \times N$ nonsingular matrix U over k_v such that:

(i) the transformation of $(k_v)^N$ by U is, in some sense, a "rotation"

and

(ii) $(\pm\mathbf{1})_N U R_v = U R_v$.

Issues of orthogonality in a general nonarchimedean setting are discussed in [13, Chapter II, Section 1] and in this particular situation in [12].

Below we review the basics of orthogonality in $(k_v)^N$, for v a nonarchimedean place. Let $\vec{x}$ and $\vec{y}$ be vectors in $(k_v)^N$. We say that $\vec{x}$ is *orthogonal* to $\vec{y}$ if

$$\|\vec{x} + \vec{y}\|_v = \max\{\|\vec{x}\|_v,\ \|\vec{y}\|_v\}.$$

We say that an $N \times N$ matrix U is *orthogonal* if $\|U\vec{w}\|_v = \|\vec{w}\|_v$ for all $\vec{w} \in (k_v)^N$

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LEMMA 6.3. *Let U be an $N \times N$ nonsingular matrix over k_v, $v \nmid \infty$. If U is an orthogonal matrix and $\vec{x}$ is orthogonal to $\vec{y}$ then $U\vec{x}$ is orthogonal to $U\vec{y}$.*

PROOF. From the hypothesis we have

$$\|U\vec{w}\|_v = \|\vec{w}\|_v \quad \text{for all } \vec{w} \in (k_v)^N.$$

Assume that $\vec{x}$ is orthogonal to $\vec{y}$, that is,

$$\|\vec{x} + \vec{y}\|_v = \max\{\|\vec{x}\|_v, \ \|\vec{y}\|_v\}.$$

Thus we see

$$\begin{aligned} \|U\vec{x} + U\vec{y}\|_v &= \|U(\vec{x} + \vec{y})\|_v = \|\vec{x} + \vec{y}\|_v \\ &= \max\{\|\vec{x}\|_v, \|\vec{y}\|_v\} \\ &= \max\{\|U\vec{x}\|_v, \|U\vec{y}\|_v\}. \end{aligned}$$

Therefore $U\vec{x}$ is orthogonal to $U\vec{y}$.

Clearly if D is an $N \times N$ nonsingular diagonal matrix over k_v and $R_v = D(\mathcal{O}_v)^N$ then

$$(\pm\mathbf{1})_N R_v = R_v.$$

So given an arbitrary k_v-lattice

$$R_v = B_v(\mathcal{O}_v)^N,$$

we wish to find an orthogonal matrix U which, in some sense, diagonalizes B_v. This is accomplished via a proposition of Weil [13, Chapter II, Proposition 4]. We state it here in our present notation:

LEMMA 6.4. *Let A_1 and A_2 be two $N \times N$ nonsingular matrices over k_v, $v \nmid \infty$. Then there exists an $N \times N$ nonsingular matrix, $W = (\vec{w}_1\vec{w}_2 \cdots \vec{w}_N)$, over k_v with columns $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_N$ such that:*

(i) $\|\vec{w}_n\|_v = 1$ *for all* $n = 1, 2, \ldots, N$,

(ii) *for all* $\vec{x} \in (k_v)^N$,

$$\|A_1 W\vec{x}\|_v = \max_{1 \le n \le N}\{\|A_1\vec{w}_n\|_v \|x_n\|_v\}$$

and

$$\|A_2 W\vec{x}\|_v = \max_{1 \le n \le N}\{\|A_2\vec{w}_n\|_v \|x_n\|_v\}.$$

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We now show that every k_v-lattice may be rotated in order to achieve certain symmetry properties.

LEMMA 6.5. *Let v be a finite place of k, and let R_v be a k_v-lattice in $(k_v)^N$. Let Φ be any diagonal $N \times N$ matrix whose diagonal entries are units in k_v. Then there exists an $N \times N$ orthogonal matrix U over k_v such that*

$$\Phi(UR_v) = UR_v.$$

PROOF. Let B_v be an $N \times N$ nonsingular matrix over k_v such that

$$R_v = B_v(\mathcal{O}_v)^N.$$

We now apply Lemma 6.4 with $A_1 = \mathbf{1}_N$ ($N \times N$ identity matrix) and $A_2 = B_v^{-1}$. Thus there exists a matrix $W = (\vec{w}_1 \vec{w}_2 \dots \vec{w}_N)$ satisfying:

$$\|\vec{w}_n\|_v = 1 \quad \text{for } n = 1, 2, \dots, N,$$

and for each $\vec{x} \in (k_v)^N$,

$$\|W\vec{x}\|_v = \max_{1 \le n \le N} \{\|\vec{w}_n\|_v \|x_n\|_v\} = \max_{1 \le n \le N} \{\|x_n\|_v\} = \|\vec{x}\|_v \tag{6.9}$$

and

$$\|B_v^{-1} W\vec{x}\|_v = \max_{1 \le n \le N} \{\|B_v^{-1}\vec{w}_n\|_v \|x_n\|_v\}. \tag{6.10}$$

Lemma 6.3 and equality (6.9) show that W is an orthogonal matrix. By making the change of variables in (6.9), $\vec{x} \to W^{-1}\vec{y}$, we immediately conclude that W^{-1} is also an orthogonal matrix. We claim that $U = W^{-1}$. To see this, select $\{\delta_1, \delta_2, \dots, \delta_N\} \subseteq k_v \backslash \{0\}$ so that $\|\delta_n\|_v = \|B_v^{-1}\vec{w}_n\|_v$ for each $n = 1, 2, \dots, N$. Define the $N \times N$ diagonal matrix D by:

$$D = \begin{pmatrix} \delta_1^{-1} & & & \\ & \delta_2^{-1} & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & \\ & & & \delta_N^{-1} \end{pmatrix}.$$

Next we recall

$$R_v = \{\vec{x} \in (k_v)^N : \|B_v^{-1}\vec{x}\|_v \le 1\}.$$

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Therefore by (6.10) we see that

$$\begin{aligned} UR_v &= \{\vec{x} \in (k_v)^N : \|B_v^{-1}U^{-1}\vec{x}\|_v \leq 1\} \\ &= \{\vec{x} \in (k_v)^N : \|B_v^{-1}W\vec{x}\|_v \leq 1\} \\ &= \{\vec{x} \in (k_v)^N : \max_{1\leq n\leq N}\{\|\delta_n\|_v\|x_n\|_v\} \leq 1\} \\ &= \{\vec{x} \in (k_v)^N : \|D^{-1}\vec{x}\|_v \leq 1\} \\ &= D(\mathcal{O}_v)^N. \end{aligned}$$

However, since D is diagonal, from our previous remarks we have

$$\Phi(D(\mathcal{O}_v)^N) = D(\mathcal{O}_v)^N$$

and hence

$$\Phi(UR_v) = UR_v.$$

We are finally prepared to prove the nonarchimedean version of Theorem 6.2.

THEOREM 6.6. *Let v be a nonarchimedean place of k and $R_v \subseteq (k_v)^N$ a regular subset. Let B_v be the $N \times N$ nonsingular matrix over k_v such that $R_v = B_v(\mathcal{O}_v)^N$. Then*

$$\langle R_v\rangle_{(N-1)} = \det(B_v)R_v^*.$$

PROOF. By Lemma 6.5 we may find an $N \times N$ orthogonal matrix U so that

$$(\pm\mathbf{1})_N(UR_v) = UR_v.$$

Thus without loss of generality, we may assume that

$$(\pm\mathbf{1})_N R_v = R_v.$$

It now follows from (6.2) and Lemma 3.1 that

$$\begin{aligned} R_v^* &= ((\pm\mathbf{1})_N R_v)^* \\ &= (\pm\mathbf{1})_N (B_v^T)^{-1}((\mathcal{O}_v)^N)^* \\ &= \det(B_v^{-1})\langle B_v\rangle_{(N-1)}(\pm\mathbf{1})_N(\mathcal{O}_v)^N \\ &= \det(B_v)^{-1}\langle B_v\rangle_{(N-1)}\langle(\mathcal{O}_v)^N\rangle_{(N-1)} \\ &= \det(B_v)^{-1}\langle B_v(\mathcal{O}_v)^N\rangle_{(N-1)} \\ &= \det(B_v)^{-1}\langle R_v\rangle_{(N-1)}. \end{aligned}$$

Let (α_v) be an idèle in $k_{\mathbf{A}}$. The *volume of the idèle*, $V((\alpha_v))$, is defined to be:

$$V((\alpha_v)) = \prod_v |\alpha_v|_v.$$

Let $\mathscr{R} = \prod_v R_v$ be an admissible subset of $(k_{\mathbf{A}})^N$. We define the *idelic dilation of* $\mathscr{R}$ *by* (α_v), $(\alpha_v)\mathscr{R}$, by:

$$(\alpha_v)\mathscr{R} = \prod_v \alpha_v R_v.$$

This is clearly a generalization of the usual real dilation at the infinite places which we recall here. If σ is a real number then $\sigma\mathscr{R} = \prod_{v|\infty} \sigma R_v \times \prod_{v\nmid\infty} R_v$. Of course, for ease of notation, one could dilate in both manners simultaneously:

$$\sigma(\alpha_v)\mathscr{R} = \prod_{v|\infty} \sigma\alpha_v R_v \times \prod_{v\nmid\infty} \alpha_v R_v.$$

At last we compare $\langle\mathscr{R}\rangle_{(N-1)}$ with $\mathscr{R}^*$. Again write $\mathscr{R} = \prod_v R_v$. For each $v\nmid\infty$ write $R_v = B_v(\mathscr{O}_v)^N$, where B_v is an $N \times N$ nonsingular matrix over k_v. Define the idèle (α_v) by:

$$\alpha_v = \begin{cases} (\tau_v(N)\beta_v^N(R_v))^{1/d_v} & \text{for } v|\infty \\ \det(B_v) & \text{for } v\nmid\infty. \end{cases}$$

The following is now immediate from Theorem 6.2 and Theorem 6.6.

THEOREM 6.7. *Let* $\mathscr{R}$ *and* (α_v) *be as above. Then*

$$N^{-\frac{1}{2}N}(\alpha_v)\mathscr{R}^* \subseteq \langle\mathscr{R}\rangle_{(N-1)} \subseteq N^{\frac{1}{2}N}(\alpha_v)\mathscr{R}^*.$$

Moreover,

$$V((\alpha_v)) = 2^{-sN/d}\pi^{-N/2}\Gamma(\tfrac{1}{2}N+1)^{r/d}\Gamma(N+1)^{s/d}|\Delta_k|^{N/2}V_N(\mathscr{R})^{1/d}.$$

REMARK. One can prove theorems in geometry of numbers over the adèle space using the idelic dilation outlined here, and it is of some independent interest.

7. A general transfer principle over number fields

Below we present an application of Theorem 5.1 in Diophantine approximation over number fields. For each place v of k, let $C_v(L)$ be the v-adic cube in $(k_v)^L$:

$$C_v(L) = \{\vec{x} \in (k_v)^L : |\vec{x}|_v^{d/d_v} \leq 1\}.$$

We remark that given our normalizations on the absolute values $|\ |_v$ and $\|\ \|_v$ we may also write $C_v(L)$ as

$$C_v(L) = \left\{\vec{x} \in (k_v)^L : \max_{1\leq l\leq L}\{\|x_l\|_v\} \leq 1\right\}.$$

We begin by demonstrating that the sets $\langle C_v(N)\rangle_M$ and $C_v(\binom{N}{M})$ are similar.

LEMMA 7.1. *Let N and M be integers such that $1 \leq M \leq N-1$. Then*
(i) *if v is an archimedean place of k then*

$$m^{-\frac{1}{2}M}\langle C_v(N)\rangle_M \subseteq C_v\left(\binom{N}{M}\right) \subseteq 2\binom{N}{M}\langle C_v(N)\rangle_M$$

and
(ii) *if v is a nonarchimedean place of k then*

$$\langle C_v(N)\rangle_M = C_v\left(\binom{N}{M}\right).$$

PROOF. We first consider the case when v is archimedean. Let $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_M\} \subseteq C_v$ and write $Y = (\vec{y}_1\vec{y}_2 \ldots \vec{y}_M)$ for the associated $N \times M$ matrix over k_v. We write

$$\vec{\mathfrak{Y}}(Y) \in (C_v(N))_m \subseteq (k_v)^{\binom{N}{M}}$$

for the M^{th} compound of Y. By Hadamard's inequality we have

$$|\vec{\mathfrak{Y}}(Y)|_v^{d/d_v} \leq M^{\frac{1}{2}M},$$

so $\vec{\mathfrak{Y}}(Y) \in M^{\frac{1}{2}M}C_v(\binom{N}{M})$. Thus

$$(C_v(N))_M \subseteq M^{\frac{1}{2}M}C_v\left(\binom{N}{M}\right)$$

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and since $C_v(\binom{N}{M})$ is convex, it follows that

$$M^{-\frac{1}{2}M}\langle C_v(N)\rangle_M \subseteq C_v\left(\binom{N}{M}\right),$$

which is the required first containment.

For the second containment, we first assume that v is a real place of k. By considering all possible permutations of rows and permutations of columns of the $N \times M$ matrix

$$\begin{pmatrix} \mathbf{1}_M \\ \cdots \\ \bigcirc \end{pmatrix},$$

one may quickly show that $\vec{e}_l \in \langle C_v(N)\rangle_M$ for $l = 1, 2, \ldots, \binom{N}{M}$, where $\vec{e}_l$ is the l^{th} column of the $\binom{N}{M} \times \binom{N}{M}$ identity matrix $\mathbf{1}_{\binom{N}{M}}$. Therefore by convexity, $\langle C_v(N)\rangle_M$ contains the unit L^1-ball in $(k_v)^{\binom{N}{M}}$:

$$L_v = \left\{\vec{x} \in (k_v)^{\binom{N}{M}} : \sum_{l=1}^{\binom{N}{M}} \|x_l\|_v \le 1\right\}.$$

It now follows that

$$C_v\left(\binom{N}{M}\right) \subseteq \binom{N}{M} L_v \subseteq \binom{N}{M}\langle C_v(N)\rangle_M,$$

which is even stronger than required.

For v complex, one may use a similar argument to show that every vector in $(k_v)^{\binom{N}{M}}$ having $\{\binom{N}{M} - 1\}$ components zero and one component a unit is contained in $\langle C_v(N)\rangle_M$. By the complex convexity of $\langle C_v(N)\rangle_M$, it follows that $\langle C_v(N)\rangle_M$ contains the unit L^1-ball in $\mathbb{R}^{2\binom{N}{M}} \cong (k_v)^{\binom{N}{M}}$. Therefore,

$$C_v\left(\binom{N}{M}\right) \subseteq 2\binom{N}{M}\langle C_v(N)\rangle_M,$$

which is the required containment of (i). Part (ii) is immediate from our remarks following Corollary 3.2.

We now fix some further notation. For each place v of k, let A_v be an $N \times N$ nonsingular matrix over k_v. Define sets R_v and T_v in $(k_v)^N$ and $(k_v)^{\binom{N}{M}}$, respectively, by:

$$R_v = \{\vec{x} \in (k_v)^N : |A_v\vec{x}|_v^{d/d_v} \le 1\}$$

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and

$$T_v = \{\vec{X} \in (k_v)^{\binom{N}{M}} : |\langle A_v\rangle_M \vec{X}|_v^{d/d_v} \le 1\}.$$

We assume that for almost all v, $R_v = (\mathcal{O}_v)^N$. Let $\mathcal{R} = \prod_v R_v$ and $\mathcal{T} = \prod_v T_v$. From our above assumption we have that $\mathcal{R}$ and $\mathcal{T}$ are admissible subsets of $(k_{\mathbf{A}})^N$ and $(k_{\mathbf{A}})^{\binom{N}{M}}$, respectively.

COROLLARY 7.2. *Let $\mathcal{R}$ and $\mathcal{T}$ be as above. Then*

$$M^{-\frac{1}{2}M}\langle\mathcal{R}\rangle_M \subseteq \mathcal{T} \subseteq 2\binom{N}{M}\langle\mathcal{R}\rangle_M.$$

PROOF. Clearly for each place v, $A_v R_v = C_v(N)$ and $\langle A_v\rangle_M T_v = C_v(\binom{N}{M})$. By Lemma 3.1,

$$\langle A_v R_v\rangle_M = \langle A_v\rangle_M\langle R_v\rangle_M.$$

The corollary now follows from Lemma 7.1.

We now state and prove Mahler's general transfer principle in this setting. We define the constants $\gamma_2 = \gamma_2(k, M, N)$ and $\gamma_3 = \gamma_3(k, M, N)$ by

$$\gamma_2 = \left\{2\binom{N}{M}\gamma_1^{-1/d} N^{(r+s)M\binom{N}{M}/2}\right\}^{1/M}$$

and

$$\gamma_3 = 2^{sN(M-1)/(d(N-1))} M^{\frac{1}{2}M} (\pi^{-s}|\Delta_k|^{1/2})^{N/d}\{(N!)^r(2N!)^s\}^{(N-M)/(d(N-1))},$$

where the constant γ_1 is defined in Section 5.

THEOREM 7.3. *Let $\mathcal{R}$ and $\mathcal{T}$ be as described above. Let λ_1 and σ_1 be the first successive minima of $\mathcal{R}$ and $\mathcal{T}$, respectively. Then*

$$\lambda_1 \le \gamma_2\sigma_1^{1/M} \quad \textit{and} \quad \sigma_1 \le \gamma_3\left(\prod_v |\det(A_v)|_v\right)^{\frac{M-1}{N-1}} \lambda_1^{\frac{N-M}{N-1}}.$$

PROOF. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $\mu_1, \mu_2, \ldots, \mu_{\binom{N}{M}}$ be the successive minima of $\mathcal{R}$ and $\langle\mathcal{R}\rangle_M$, respectively. Write $\Lambda_1, \Lambda_2, \ldots, \Lambda_{\binom{N}{M}}$ for the corresponding M-products as in Section 5. From Lemma 7.2 we have

$$\left\{2\binom{N}{M}\right\}^{-1}\mu_1 \le \sigma_1 \le M^{\frac{1}{2}M}\mu_1. \tag{7.1}$$

Trivially we have

$$\lambda_1^{dM} \leq (\lambda_1\lambda_2 \ldots \lambda_M)^d = \Lambda_1^d.$$

Therefore by Theorem 5.1 and (7.1) we conclude

$$\lambda_1 \leq \gamma_2\sigma_1^{1/M}.$$

For the second inequality, we begin by noting

$$(\lambda_2\lambda_3 \ldots \lambda_N)^{1/(N-1)} \leq (\lambda_{M+1}\lambda_{M+2} \ldots \lambda_N)^{1/(N-M)}. \tag{7.2}$$

From the upper bound in the adelic successive minima theorem and (7.2) we conclude

$$\begin{aligned}\Lambda_1^d = (\lambda_1\lambda_2 \ldots \lambda_M)^d &\leq (\lambda_{M+1}\lambda_{M+2} \ldots \lambda_N)^{-d} 2^{dN} V_N(\mathscr{R})^{-1} \\ &\leq (\lambda_2\lambda_3 \ldots \lambda_N)^{-d(N-M)/(N-1)} 2^{dN} V_N(\mathscr{R})^{-1}.\end{aligned}$$

By the lower bound in the successive minima theorem, the previous inequality yields:

$$\Lambda_1^d \leq \left\{\frac{(N!)^r(2N!)^s|\Delta_k|^{N/2}}{\pi^{sN}}\right\}^{\frac{N-M}{N-1}} (2^{dN}V_N(\mathscr{R}))^{\frac{M-1}{N-1}} \lambda^{\frac{d(N-M)}{N-1}}.$$

The theorem now follows from Theorem 5.1, (7.1) and the identity

$$V_N(\mathscr{R}) = 2^{dN}\left(\frac{\pi}{2}\right)^{sN} |\Delta_k|^{-N/2} \prod_v |\det(A_v)|_v^{-d}.$$

Let S be a finite set of places of k containing all the archimedean places. Write $\mathscr{O}_S$ for the ring of S-integers in k. That is,

$$\mathscr{O}_S = \{x \in k : \|x\|_v \leq 1 \quad \text{for all } v \notin S\}.$$

Define the function

$$\delta_v = \begin{cases} d_v/d & \text{if } v|\infty \\ 0 & \text{if } v\nmid\infty. \end{cases}$$

Then as an immediate consequence of Theorem 7.3 we have the following result:

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COROLLARY 7.4. *Let A_v be an $N \times N$ nonsingular matrix over k_v for each $v \in S$. For each $v \in S$, select $\varepsilon_v \in k_v \backslash \{0\}$ so that*

$$\prod_{v \in S} |\varepsilon_v^{-N} \det(A_v)|_v = 1.$$

If there exists an $\vec{x} \in (\mathcal{O}_S)^N$, $\vec{x} \neq \vec{0}$, such that

$$|A_v \vec{x}|_v \leq |\varepsilon_v|_v \quad \textit{for each } v \in S,$$

then there exists an $\vec{X} \in (\mathcal{O}_S)^{\binom{N}{M}}$, $\vec{X} \neq \vec{0}$, such that

$$|\langle A_v \rangle_M \vec{X}|_v \leq \gamma_3^{\delta_v} |\varepsilon_v|_v^M \quad \textit{for each } v \in S.$$

Similarly, if there exists an $\vec{X} \in (\mathcal{O}_S)^{\binom{N}{M}}$, $\vec{X} \neq \vec{0}$, satisfying

$$|\langle A_v \rangle_M \vec{X}|_v \leq |\varepsilon_v|_v^M \quad \textit{for each } v \in S,$$

then there exists an $\vec{x} \in (\mathcal{O}_S)^N$, $\vec{x} \neq \vec{0}$, satisfying

$$|A_v \vec{x}|_v \leq \gamma_2^{\delta_v} |\varepsilon_v|_v \quad \textit{for each } v \in S.$$

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