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ON PERIODICITY IN TOPOLOGICAL SURGERY

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One of the distinguishing features of the topological category is the following periodicity in the set of homotopy TOP structures on X.

THEOREM (Siebenmann). *Let* X^m, $m \geqq 5$, *be a connected compact topological manifold with non-empty boundary. Then*

$$S_{\mathrm{TOP}}(X) \approx S_{\mathrm{TOP}}(I^4 \times X).$$

It was conjectured by Siebenmann (see [**3**], p. 283) that the analogous periodicity should also exist for noncompact manifolds.

The purpose of this paper is to prove that this is indeed the case, namely:

THEOREM 1. *Let* X^m, $m \geqq 6$, *be a connected noncompact topological manifold with non-empty boundary. Then*

$$S_{\mathrm{TOP}}(X) \approx S_{\mathrm{TOP}}(I^4 \times X).$$

The proof of Theorem 1 is an application of the proper surgery theory of S. Maumary [**4**], [**5**], L. Taylor [**13**] and F. Quinn's semi-simplicial formulation of Wall's surgery (see [**7**], [**6**]). Especially the algebraic determination of the proper surgery groups given in [**4**] is used essentially.

To make this note somewhat more self-contained we recall in the Preliminaries the basic notions from the proper surgery theory.

I am indebted to P. Vogel for useful conversations.

1. Preliminaries. In this section we recall the basic facts about proper surgery theory. For more information on this subject we refer to [**13**].

A continuous map $f:X \to Y$ between topological spaces X, Y is called *proper* if $f^{-1}(C)$ is compact whenever C is a compact subset of Y. By *proper category* we mean a category of topological spaces and proper maps. It is not difficult to observe that the ordinary homology and homotopy are not satisfactory tools in the proper category. The appropriate substitutions are so called Δ-homology and Δ-homotopy (see [**2**], [**13**]). As in the ordinary homology and homotopy there are functors from the proper category to the category of groups and homomorphisms.

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A lot of general properties and constructions which hold in the ordinary homology and homotopy (a long exact sequence, a cup product, Hurewicz Theorem, etc.) one can carry over to the Δ-case (see [**2**]).

Now a proper Poincarè complex of formal dimension m is defined to be a locally finite, finite dimensional CW complex X with an orientation class $w_1 \in H^1(X; Z_2)$ and a class $[X] \in H_m^{1.f}(X; Z^t)$ such that the cup product

$$\cap[X]:\Delta^{m-*}(X) \rightarrow \Delta_*(X)$$

is an isomorphism.

An obvious modification gives the notion of a proper Poincarè n-ad.

The basic example of a proper Poincarè complex of formal dimension m is an open (= noncompact without boundary) m-dimensional CAT manifold (CAT = TOP, PL or DIFF).

Suppose we have the following data (a proper normal map):

1. An m-dimensional open CAT manifold M.

2. A proper Poincarè complex X of formal dimension m with a CAT bundle ξ over X, where CAT bundle means a TOP-microbundle, a PL-microbundle, or a vector bundle respectively, and an orientation class $w \in H^1(X; Z_2)$.

3. A proper degree 1 map $f:M \rightarrow X$ i.e.,

$$f_*[M] = [X] \in H_m^{1.f}(X; Z^t).$$

4. A stable trivialization F of $\tau(M) \oplus f^*(\xi)$, where $\tau(M)$ is the CAT tangent bundle of M.

Now the task of a proper surgery theory is to decide whether there exists a CAT cobordism W with boundary $bW = M \cup N$, a proper map $g: W \rightarrow X$ with $g|W = f$, a stable trivialization F' of $\tau(W) \oplus g^*(\xi)$ naturally extending F, such that $g|N$ is a proper homotopy equivalence.

It turns out that there is a well-defined obstruction to solving this problem if the dimension of M is at least five. This obstruction lies in a group $L_m^h(X; w)$. The group $L_m^h(X; w)$ is defined as a certain bordism group strictly analogous to [**14**], chapter 9.

If one wants to obtain a proper simple homotopy equivalence $g|_N:N \rightarrow X$ (simple in the sense of [**1**], [**12**]) then the obstruction to solving this modified problem lies in a similarly defined group $L_m^s(X; w)$.

Now if K is an arbitrary locally finite, finite dimensional CW complex with an orientation class $w \in H^1(K; Z_2)$ then the groups $L_m^q(K; w)$, $q = h$ or s, are defined analogously to [**14**] by considering normal maps over $(K; w)$.

Suppose X is a proper Poincarè complex of formal dimension m. The set of homotopy CAT structures on X, $S_{\mathrm{CAT}}^q(X)$, where CAT = TOP, PL or DIFF and $q = h$ or s is defined as follows.

An element of $S_{\mathrm{CAT}}^q(X)$ is represented by a proper (simple when $q = s$)

homotopy equivalence $f:M \to X$ defined on an m-dimensional CAT manifold M. Another such equivalence $f':M' \to X$ represents the same element $[f'] = [f]$ in $S^q_{\mathrm{CAT}}(X)$ if and only if there is a proper CAT h-cobordism (s-cobordism) (W, M, M') rel boundary and a proper (simple) homotopy equivalence $F:W \to X \times I$ that is product with I near $I \times bX$ while $F(x) = (f(x), 0)$ for $x \in M$, and $F(x) = (f'(x), 1)$ for $x \in M'$ after the identification

$$bW - F^{-1}(\mathring{I} \times bX) \approx M \cup M'$$

is made. When $m \geqq 5$, and $Wh(X) = 0$ then every proper h-cobordism is a product cobordism (see [**12**]), so $[f'] = [f]$ then simply means that there exists a CAT isomorphism $h:M \to M'$ with $f'h$ properly homotopic with f.

If we do not wish to distinguish between $S^h_{\mathrm{CAT}}(X)$ and $S^s_{\mathrm{CAT}}(X)$ then we write simply $S_{\mathrm{CAT}}(X)$.

A similar definition holds for X a proper Poincarè n-ad.

Analogously as in the compact case one gets the long Sullivan-Wall exact sequence (see [**13**])

$$[\Sigma X, G/\mathrm{CAT}] \to L^q_{t+1}(X; w) \to S^q_{\mathrm{CAT}}(X)$$
$$\to [X, G/\mathrm{CAT}] \overset{\Theta}{\to} L^q_t(X; w)$$

where [,] denotes the set of homotopy classes of maps and ΣX denotes the suspension of X.

It is well known that in the compact case the surgery groups of a space depend only on its fundamental group. In the proper case the groups $L^q_t(X; w)$ depend only on the system of fundamental groups of X (i.e., a proper 1-equivalence of spaces induces an isomorphism between their surgery groups)(see [**13**]).

Remark 1.1. We will follow the notational convention of Appendix C of [**3**]. Hence we will write simply $L^q_t(X)$, $S^q_{\mathrm{CAT}}(X)$ for $L^q_t(X, bX; w)$, $S^q_{\mathrm{CAT}}(X, bX)$ respectively.

The proper surgery groups were defined in a purely geometrical manner, which makes their computability extremely complicated. Fortunately there exists an exact sequence due to S. Maumary (see [**4**], [**5**]) which relates the proper surgery groups to the Wall-Novikov groups, namely:

Let X be a locally finite, finite dimensional connected CW complex and let $X_1 \supset X_2 \supset X_3 \supset \ldots$ be a sequence of neighborhoods of ∞, formed by a sequence of subcomplexes X_t with only noncompact components (in finite number). Denote by $\overline{X - X_t}$ any finite subcomplex of X such that $\overline{X - X_t} \cup X_t = X$ and let

$$\dot{X}_t = \overline{X - X_t} \cap X_t,$$

which is a finite subcomplex of X containing the frontier of X_t.

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Let

$$\prod_t^p(X) = \prod_{j=1}^{\infty} L_t^p(\pi_1(X_j))$$

and let

$$\prod_t(X) = \prod_{j=1}^{\infty} L_t(\pi_1 X_j))$$

where $L_t^p(\pi_1(X_j))$ are the U-groups and $L_t(\pi_1(X))$ are the V-groups from [**8**]. We define homomorphisms

$$(1 - s): \prod_t(X) \to L_t(\pi_1(X)) \oplus \prod_t(X)$$
$$(1 - s): \prod_t^p(X) \to L_t^p(\pi_1(X)) \oplus \prod_t^p(X)$$

by

$$(1 - s)(a_1, a_2, a_3, \ldots) \doteq (-i_*(a_1), a_1 - i_*(a_2), a_2 - i_*(a_3), \ldots),$$

where i_* denotes the map induced by the inclusions

$$i:X_t \to X_{t-1} \quad (t \geqq 1, X_0 = X).$$

THEOREM (Maumary). *The proper surgery groups $L_t^h(X)$ fit into an exact sequence*

$$(*) \quad \prod_t(X) \xrightarrow{1-s} L_t(\pi_1(X)) \oplus \prod_t(X) \to L_t^h(X)$$
$$\to \prod_{t-1}^p(X) \xrightarrow{1-s} L_{t-1}^p(\pi_1(X)) \oplus \prod_{t-1}^p(X).$$

2. Periodicity. In this section we give a proof of Theorem 1. First we prove the S^h_{TOP}-version and we next show how to obtain the S^s_{TOP}-case.

The analysis of L. Siebenmann's proof of

$$S_{\mathrm{TOP}}(X) \approx S_{\mathrm{TOP}}(I^4 \times X)$$

in the compact case clearly shows that the basis for this periodicity is the periodicity in the surgery groups.

The geometric approach to proper surgery groups gives no sign that these groups are periodic. Fortunately it turns out that they are periodic with the periodicity (like in the compact case) given by the multiplication by CP^2. A proof of this is in fact our main task. To obtain the periodicity in the set of homotopy TOP structures from the surgerical periodicity we proceed analogously to [**3**], [**6**].

We start from the following

THEOREM 2.1. *Let X be a locally finite, finite dimensional connected* CW *complex. Then*

$$L_t^p(X) \overset{\approx}{\to} L_{t+4}^p(X), \quad t \geqq 6,$$

and this isomorphism is given by multiplying by CP^2.

Proof. Consider the exact sequence (*) of Maumary

$$\prod_t(X) \xrightarrow{1-s} L_t(\pi_1(X)) \oplus \prod_t(X) \xrightarrow{\tau} L_t^h(X)$$

$$\xrightarrow{\sigma} \prod_{t-1}^p(X) \xrightarrow{1-s} L_{t-1}^p(\pi_1(X)) \oplus \prod_{t-1}^p(X).$$

All groups in this sequence, except $L_t^h(X)$, are periodic with the period equal to [4] (see **[8]**) so we can write

$$\begin{array}{ccccccc}
\prod_t(X) & \xrightarrow{1-s} & L_t(\pi_1(X)) & \oplus & \prod_t(X) & \xrightarrow{\tau} & L_t^h(X) \\
\approx\downarrow & (1) & \approx\downarrow & \oplus & \downarrow\approx & & \\
\prod_{t+4}(X) & \xrightarrow{1-s} & L_{t+4}(\pi_1(X)) & \oplus & \prod_{t+4} & &
\end{array}$$

$$\begin{array}{cccccc}
\sigma & \prod_{t-1}^p(X) & \xrightarrow{1-s} & L_{t-1}^p(\pi_1(X)) & \oplus & \prod_{t-1}^p(X) \\
& \approx\downarrow & (2) & \approx\downarrow & \oplus & \downarrow\approx \\
& \prod_{t+3}^p(X) & \xrightarrow{1-s} & L_{t+3}^p(\pi_1(X)) & \oplus & \prod_{t+3}^p(X)
\end{array}$$

where the diagrams (1), (2) are commutative by definition.

Now we define a homomorphism

$$\times CP^2 : L_t^p(X) \to L_{t+4}^p(X \times CP^2)$$

as follows:

Let $[f] \in L_t^h(X)$ be represented by a proper normal map (f, F, g) over X, i.e., let $f:M \to Y$ be a proper degree 1 map between a connected t-dimensional CAT manifold M and a proper Poincarè complex Y of formal dimension t, where F is a stable trivialization of $\pi(M) \oplus f^*(\xi)$, and $g:Y \to X$ satisfies the following condition: if $w \in H^1(X; Z_2)$ is the orientation class of X then

$$g^*(w) \in H^1(Y; Z_2)$$

is the orientation class of Y. Then

$$\times CP^2(f) \in L_{t+4}(X \times CP^2)$$

is represented by the proper normal map $(f \times \mathrm{id}, F \times \mathrm{id}, g \times \mathrm{id})$, where

$$f \times \mathrm{id}: M \times \mathrm{CP}^2 \to Y \times \mathrm{CP}^2$$

and $F \times \mathrm{id}$, $g \times \mathrm{id}$ are the obvious terms.

It is easy to see that $\times \mathrm{CP}^2$ is a homomorphism and if we are able to show that the following diagram is commutative, then by the five-lemma we will conclude that

$$\times \mathrm{CP}^2 : L_t^h(X) \to L_{t+4}^h(X \times \mathrm{CP}^2)$$

is an isomorphism.

$$(**)\quad \begin{array}{ccccccc}
\prod_t(X) & \xrightarrow{1-s} & L_t(\pi_1(X)) & \oplus & \prod_t(X) & \xrightarrow{\tau} & L_t^h(X) \\
\approx\big\downarrow & (1) & \approx\big\downarrow & \oplus & \big\downarrow\approx & (3) & \big\downarrow \\
\prod_{t+4}(X) & \xrightarrow{1-s} & L_{t+4}(\pi_1(X)) & \oplus & \prod_{t+4}(X) & \xrightarrow{\bar{\tau}} & L_{t+4}^h(X \times \mathrm{CP}^2)
\end{array}$$

$$\begin{array}{cccccc}
\xrightarrow{\sigma} & \prod_{t-1}^p(X) & \xrightarrow{1-s} & L_{t-1}^p(\pi_1(X)) & \oplus & \prod_{t-1}^p(X) \\
(4) & \approx\big\downarrow & (2) & \approx\big\downarrow & \oplus & \big\downarrow\approx \\
\xrightarrow{\bar{\sigma}} & \prod_{t+3}^p(X) & \xrightarrow{1-s} & L_{t+3}^p(\pi_1(X)) & \oplus & \prod_{t+3}^p(X)
\end{array}.$$

All new terms in (**) will be defined later in the proof. At the beginning we describe the homomorphism

$$\sigma: L_t^h(X) \to \prod\nolimits_{t-1}^p(X) = \prod_{j \geqq 1} L_{t-1}(\pi_1(X_j)).$$

Roughly speaking, σ is defined as follows (see [**4**]). Let $[f] \in L_t^h(X)$ be given by a proper normal map over X say (f, F, g) with $f: M \to Y$ of degree 1. Assume that t is an odd number of the form $2q + 1$. Let us take a fundamental system of neighborhoods of ∞ in Y, $Y_l \supset Y_2 \supset Y_3 \supset \dots$ such that

$$\forall_{j \geqq 1} \ \overline{Y - Y_j}$$

is compact. By transversality we can assume that $f^{-1}(Y_j) = M_j$ is a submanifold in M and

$$bM_j = f^{-1}(\dot{Y}_j).$$

Doing a sequence of surgeries we get a sequence of finitely generated projective $Z(\pi_1(Y_j))$-modules $\{K_q(bM_j)\}$ (see [**4**]).

Every such module $K_q(bM_j)$ is equipped with a special Hermitian form coming from the intersection numbers on bM_j. Hence we get a sequence of elements c_j with

$$c_j \in L^p_{t-1}(\pi_1(X_j)), \quad j = 1, 2, \dots,$$

and σ is defined as

$$\sigma([f]) = \{c_j\} \in \prod_{j \geqq 1} L^p_{t-1}(\pi_1(X_j)).$$

Now let us choose the fundamental system of neighborhoods of ∞ in $X \times \mathrm{CP}^2$ as

$$X_1 \times \mathrm{CP}^2 \supset X_2 \times \mathrm{CP}^2 \supset \dots$$

and define $\bar{\sigma}$ in the same manner as σ was defined. Then the periodicity for the surgery groups (see Proposition 8.2 in [**9**], p. 259) shows that the diagram (4) in (**) is commutative.

Now we show the commutativity of (3). As before t is assumed to be an odd number of the form $2q + 1$. Let $\{a_j\}_{j=0,1,2,\dots}$ be an element in

$$L_t(\pi_1(X)) \oplus \prod{}_t(X) = \prod_{j \geqq 0} L_t(\pi_1(X_j)),$$

where $X_0 = X$.

We choose an open connected $2q$-dimensional CAT manifold N such that the inverse system $\{\pi_1(N_j)\}_{j=0,1,2,\dots}$ is conjugate equivalent to $\{\pi_1(X_j)\}_{j=0,1,2,\dots}$. Let us recall that the conjugate equivalence of inverse systems means that the commutativity of a diagram which is required in the ordinary equivalence is replaced by the commutativity up to action of $\pi_1(X_j)$ on itself (comp. [**5**]). Next we do (see [**4**]) a series of surgeries on $\mathrm{id}:N \to N$ (these surgeries depend on $\{a_j\}_{j=0,1,2,\dots}$) to get a new manifold N_1 properly homotopy equivalent to N under a proper homotopy equivalence $f_1:N_1 \to N$. Because $f_1:N_1 \to N$ was obtained by a sequence of surgeries we then have a cobordism say

$$f:M^{2q+1} \to N \times I$$

between $\mathrm{id}:N \to N$ and $f_1:N_1 \to N$. It turns out that

$$f:(M, bM) \to (N, bN)$$

provides the surgery data we are looking for in $L^h_{2q+1}(X)$ (see [**4**]).

Now we define

$$\bar{\tau}(\{\bar{a}_j\}_{j=0,1,2,\dots}) = [f \times \mathrm{id}] \in L^h_{t+4}(X \times \mathrm{CP}^2),$$

where

$$(f \times \mathrm{id}):(M, bM) \times \mathrm{CP}^2 \to (N, bN) \times \mathrm{CP}^2$$

and $\bar{a}_j$ comes from a_j by periodicity. As a consequence we obtain the commutativity of (3) in (**).

The exactness of

$$\xrightarrow{1-s} \cdot \xrightarrow{\bar{\tau}} \cdot \xrightarrow{\bar{\sigma}} \cdot \xrightarrow{1-s} \cdot$$

is also evident so we get the isomorphism

$$L_t^h(X) \underset{\approx}{\xrightarrow{\times \mathrm{CP}^2}} L_{t+4}^h(X \times \mathrm{CP}^2)$$

in the case t odd.

When t is an even number then in fact nothing is changed. The homomorphisms σ, τ are defined in the same manner (see [**4**]) and we can proceed as in the case t odd. Therefore we can take $\times \mathrm{CP}^2$ to be the isomorphism for all $t \geqq 6$.

Remark. The condition $t \geqq 6$ which occurs in Theorem 2.1 is a consequence of assumptions in [**4**].

Now to obtain the isomorphism

$$\times \mathrm{CP}^2 : L_t^h(X) \to L_{t+4}^h(X)$$

postulated in Theorem 2.1 it is enough to prove the following:

PROPOSITION 2.2 *Choose* $x_0 \in \mathrm{CP}^2$. *Then the inclusion*

$$i:X \to X \times \mathrm{CP}^2$$

given by $i(x) = (x, x_0)$ *induces the isomorphism*

$$i_*:L_t^h(X) \to L_t^h(X \times \mathrm{CP}^2).$$

Proof. As mentioned in the Preliminaries, to prove Proposition 2.2 it suffices to show that the map

$$i:X \to X \times \mathrm{CP}^2$$

is a proper 1-equivalence. To see this let $\mathscr{K}$ denote the family of all compact subsets of X. It is clear that the family

$$\bar{\mathscr{K}} = \{K \times \mathrm{CP}^2\}_{K \in \mathscr{K}}$$

of compact subsets in $X \times \mathrm{CP}^2$ is a cofinal collection of compacta in $X \times \mathrm{CP}^2$ (i.e., given a compact set $A \subset X \times \mathrm{CP}^2$, then there is some $A' \in \bar{\mathscr{K}}$ containing A).

It is not difficult to prove that this implies that

$$i_*:\Delta(X, \{p\}; \pi_1, \text{no cov}) \to \Delta(X \times \mathrm{CP}^2, \{ (p, x_0) \}, \pi_1, \text{no cov})$$

is an isomorphism (the Δ-objects above are defined in [**2**]). Hence by Theorem 2.19 in [**2**]

$$i:X \to X \times \mathrm{CP}^2$$

is a proper 1-equivalence and consequently we get the isomorphism

$$i_*: L_t^h(X) \to L_t^h(X \times \mathrm{CP}^2).$$

This finishes the proof of Theorem 2.1.

The periodicity

$$L_t^h(X) \xrightarrow[\approx]{\times \mathrm{CP}^2} L_{t+4}^h(X)$$

is the crucial point in the proof of Theorem 1. To deduce Theorem 1 from Theorem 2.1 we can proceed analogously as L. Siebenmann in [**3**], but some care is needed. This is because of the following:

Remark 2.3. As was observed by A. Nicas (see [**6**]) Theorem C.5 of [**3**], p. 283 is incorrectly stated. Namely, in the case when X is a closed manifold $S_{\mathrm{TOP}}(X) \not\approx S_{\mathrm{TOP}}(I^4 \times X)$ in general, contradicting Siebenmann's claim for the periodicity in this case also.

For example for $n \geqq 5$

$$S_{\mathrm{TOP}}(S^n) = 0$$

by the generalized Poincarè Conjecture. On the other hand

$$S_{\mathrm{TOP}}(I^4 \times S^n) = L_4(0) \approx Z$$

by the surgery exact sequence.

It was also observed by A. Nicas (see [**6**]) that the corresponding periodicity for a closed manifold X is the following exact sequence:

$$0 \to S_{\mathrm{TOP}}(X) \to S_{\mathrm{TOP}}(I^4 \times X) \to L_0(0).$$

Because of this discrepancy we in fact should follow [**6**], where the rigorous proof of the periodicity is given. We will not give all the details here, mainly because we ought to repeat (after a small modification) about forty pages of consideration from [**6**]. Therefore we assume familiarity with [**7**], [**6**] and give only a short sketch how to obtain Theorem 1 from Theorem 2.1.

For information concerning Δ-sets we refer to [**10**].

First a few words about notation. By $\mathbf{L}_m(G)$ we denote the surgery space for the group G (see [**7**], [**6**]). $\mathbf{L}_m(G)_0$ stands for the 0-component of $\mathbf{L}_m(G)$ and $S(X)$ denotes the singular complex of a space X.

If K is a locally finite, finite dimensional CW complex, then the proper surgery space for K is denoted by $\bar{\mathbf{L}}_m^q(K)$, $q = h$ or s. The definition of $\bar{\mathbf{L}}_m^q(K)$ is the obvious modification of the definition of $\mathbf{L}_m(G)$. Now let X be a connected noncompact CAT manifold. Then analogously as in the compact case we have the pointed Δ-sets $\bar{\mathbf{N}}_{\mathrm{CAT}}^q(X)$ and $\bar{\mathbf{S}}_{\mathrm{CAT}}^q(X)$, $q = h$ or s, of proper normal maps and proper homotopy CAT structures respectively. By the obvious modification of the definition from [**6**] one also gets the notion of a proper surgery mock bundle.

Now we go back to the proof of Theorem 1. Let

$$a(0, r): S(G/\mathrm{TOP}) \to \mathbf{L}_{4r}(0)_0 \quad r \geqq 2$$

$$a(k, r): S(\Omega(G/\mathrm{TOP})) \to \mathbf{L}_{4r+k}(0) \quad 4r + k \geqq 5, k \geqq 1$$

be the homotopy equivalences defined in [**3**] p. 279.

Suppose X is an m-dimensional connected noncompact topological manifold with a non-empty boundary bM. Then the following diagram, corresponding to the diagram (7) in [**3**], p. 280 is homotopy (not proper homotopy) commutative.

$$\begin{array}{ccccc}
\Delta(X_+;\ S(G/\mathrm{TOP})) & \xrightarrow{\Theta_{0,2}} & \Delta(X_+;\ \mathbf{L}_8(0)_0) & \xrightarrow{\sigma} & \overline{\mathbf{L}}^h_{m+8}(X) \\
\Pi \downarrow \simeq & & \simeq \downarrow \times \mathrm{CP}^2 & & \simeq \downarrow \times \mathrm{CP}^2 \\
\Delta(X_+;\ S(\Omega^4(G/\mathrm{TOP}))) & \xrightarrow{\Theta_{4,2}} & \Delta(X_+;\ \mathbf{L}_{12}(0)) & \xrightarrow{\sigma} & \overline{\mathbf{L}}^h_{m+12}(X)
\end{array}$$

Two points here are worth noting. First, observe that the map σ really goes to $\overline{\mathbf{L}}^h_{m+8}(X)$ and not to $\mathbf{L}^h_{m+8}(\pi_1(X))$. The reason for this is the fact that X is noncompact. Namely, the map σ is obtained as a composition of the glue map (see [**6**], p. 42) and the assembly map. As our manifold X is noncompact we glue infinitely many compact manifolds, hence the manifolds M, X in Theorem 3.3.2 in [**6**], p. 47 are now noncompact. It is also not difficult to see that the assembly procedure gives us the proper surgery problem, i.e., we get a proper normal map, therefore an element in $\overline{\mathbf{L}}^h_{m+8}(X)$.

Second, in the case where X is not triangulable we take X_+ to be the triangulated stable normal disc bundle to X in a euclidean space $\mathbf{R}^n$ (see [**3**], p. 281). We should note that the considerations in step 7, p. 281 in [**3**] remain valid in the case of noncompact manifolds (i.e., the TOP transversality for proper maps).

Now we assume $\dim X \geqq 6$, $bM \neq 0$. Then we proceed in a way strictly analogous to [**6**] (with the obvious modifications of course).

As a final result we will obtain the homotopy commutative diagram.

$$\begin{array}{ccc}
\overline{\mathbf{S}}^h_{\mathrm{TOP}}(X) & \longrightarrow & \overline{\mathbf{L}}^h_m(X) \\
\Pi \downarrow \simeq & & \simeq \downarrow \times \mathrm{CP}^2 \\
\overline{\mathbf{S}}_{\mathrm{TOP}}(I^4 \times X) & \longrightarrow & \overline{\mathbf{L}}^h_{m+4}(X)
\end{array}$$

Hence on the π_0 level we get the periodicity

$$S^h_{\mathrm{TOP}}(X) \approx S^h_{\mathrm{TOP}}(I^4 \times X).$$

It is also clear that in the case $bM = 0$ we get the exact sequence

$$0 \to S^h_{\mathrm{TOP}}(X) \to S^h_{\mathrm{TOP}}(I^4 \times X) \to L_0(0).$$

To obtain a somewhat stronger result, namely the periodicity

$$S^s_{\mathrm{TOP}}(X) \approx S^s_{\mathrm{TOP}}(I^4 \times X),$$

and thus completing the proof of Theorem 1, we clearly only need to prove that there exists the periodicity

$$L^s_t(X) \xrightarrow[\approx]{\times \mathrm{CP}^2} L^s_{t+4}(X).$$

Because of the lack of a Maumary-type exact sequence for $L^s_t(X)$ we proceed as follows:

Let us consider the Rothenberg's exact sequence for proper surgery groups (see [**13**], compare also [**11**])

$$\ldots \to A_{t+1}(X) \to L^s_t(X) \to L^h_t(X) \to A_t(X) \to \ldots$$

where

$$A_t(X) = \{\sigma \in Wh(X) \mid \sigma = (-1)^t \sigma^*\} / \{\tau + (-1)^t \tau^* \mid \tau \in Wh(X)\},$$

$Wh(X)$ is the proper Whitehead group of X (see [**1**], [**12**]) and

$$*: Wh(X) \to Wh(X)$$

is the canonical involution on $Wh(X)$.

It is not difficult to observe that the required periodicity

$$L^s_t(X) \xrightarrow[\approx]{\mathrm{CP}^2} L^s_{t+4}(X), \quad t \geqq 6,$$

can be deduced from Rothenberg's exact sequence by the five-lemma argument. This implies the periodicity

$$S^s_{\mathrm{TOP}}(X) \approx S^s_{\mathrm{TOP}}(I^4 \times X)$$

in the case when X is a connected noncompact manifold with $\dim X \geqq 6$ and $bM \neq 0$. When $bM = 0$ then analogously to our previous argument we get the exact sequence

$$0 \to S^s_{\mathrm{TOP}}(X) \to S^s_{\mathrm{TOP}}(I^4 \times X) \to L_0(0).$$

This completes the proof of Theorem 1.

Remark 2.4. The results in [**6**] are formulated and proved only for orientable manifolds. But all ideas in [**6**] which we need work in the nonorientable case as well. Hence there is no restriction concerning the orientability of X in Theorem 1.

Remark 2.5. To see that in the open case there in no periodicity

$$S_{\mathrm{TOP}}(X) \approx S_{\mathrm{TOP}}(I^4 \times X)$$

in general let us take $X = S^n \times R$, $n \geqq 3$. Then (Siebenmann)

$$S_{\mathrm{TOP}}(S^n \times R) = 0$$

compare [M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. *17*(1982), 357-453]. On the other hand one can use the projective surgery theory of Pedersen and Ranicki (see Topology *19*(1980), 239-254) to show that

$$S_{\mathrm{TOP}}(S^n \times R \times I^4) \approx Z \quad \text{for } n \geqq 3.$$

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