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ASYMPTOTICS OF A GAUSS HYPERGEOMETRIC FUNCTION WITH TWO LARGE PARAMETERS: A NEW CASE

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Abstract

Asymptotic expansions of the Gauss hypergeometric function with large parameters, $F(\alpha + \epsilon_1\tau, \beta + \epsilon_2\tau; \gamma + \epsilon_3\tau; z)$ as $|\tau| \to \infty$, are known for many special cases, but not for one that the author encountered in recent work on fluid mechanics: $\epsilon_2 = 0$ and $\epsilon_3 = \epsilon_1 z$. This paper gives the leading term for that case if β is not a negative integer and z is not on the branch cut $[1, \infty)$, and it shows how subsequent terms can be found.

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1. Introduction

Asymptotic expansions as $|\tau| \to \infty$ of the hypergeometric function

$$F(\alpha + \epsilon_1\tau, \beta + \epsilon_2\tau; \gamma + \epsilon_3\tau; z), \tag{1.1}$$

where $\alpha, \beta, \gamma, \epsilon_i$ and z are finite, have been studied for over 100 years, but the theory is still not complete [2, 5–8]. Most recently Paris [6] and then Cvitković et al. [2] considered $\varepsilon_2 = 0$, but their methods did not apply if $\varepsilon_3 = \varepsilon_1 z$. This paper deals with that case by putting $a = \alpha - \gamma/z$, $b = \beta$, $\lambda = \gamma + \epsilon_3\tau$, so that unless $z = 0$, $|\lambda| \to \infty$ and

$$F(\alpha + \epsilon_1\tau, \beta; \gamma + \epsilon_3\tau; z) = F(a + \lambda/z, b; \lambda; z). \tag{1.2}$$

We refer to this function simply as F if no ambiguity would arise. If $z = 0$, then (1.1) gives $F = 1$; its asymptotic expansion is just 1.

Our theory will assume that b is not a negative integer, so that there is a branch cut in the complex z-plane, and that $z \notin [1, \infty)$ in order to avoid it. If b were a negative

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integer, F would be a polynomial in z and a rational function of λ, no branch cut would exist and the asymptotic series would converge if $|\lambda| > |b| - 1$.

The motivation for studying the case $\varepsilon_2 = 0, \varepsilon_3 = \varepsilon_1 z$ was the author's recent theory [3] for a gas bubble rising in a solution of a substance (for example, common salt in water) that raises the surface tension. Finding the drag on the bubble led to a series $\sum f_n \sigma_n$, in which $|f_n| \to 0$ as $n \to \infty$, and the values of σ_n were

$$\sigma_n = \frac{F(3c_n + 2, 1; 2c_n + 2; 2/3)}{1 + 2c_n} - \frac{F(3c_n + 3, 1; 2c_n + 3; 2/3)}{1 + c_n}, \tag{1.3}$$

where $c_n \to \infty$ as $n \to \infty$ and every $c_n > 0$. Equation (1.3) involves two special cases of (1.2): $a = -d/2$, $b = 1$, $z = 2/3$, $\lambda = 2c_n + d$, where $d = 2$ or 3. Section 3 will show that $|\sigma_n| = O(c_n^{-1/2})$; the series $\sum f_n \sigma_n$ converges.

2. Integral representation of F

Let ph : $\mathbb{C}\backslash\{0\} \to (-\pi, \pi]$ be the single-valued phase function [5, Section 1.9(i)], let $\theta_\lambda = \mathrm{ph}(\lambda)$, $\theta_z = \mathrm{ph}(z)$, let a, b and z be finite real or complex constants and let $G(\lambda, b)$, $I(\lambda, a, b, z)$, $p(t, z)$, $q(t, a, b, z)$, $r(t, z, \theta_\lambda)$ be the functions

$$\begin{aligned} G(\lambda, b) &= \frac{\Gamma(\lambda)}{\Gamma(b)\Gamma(\lambda - b)}, \\ I(\lambda, a, b, z) &= \int_0^1 t^{b-1}(1-t)^{\lambda-b-1}(1-zt)^{-(\lambda/z)-a}\,dt \\ &= \int_0^1 \exp\,(-\lambda p(t,z))q(t,a,b,z)\,dt, \qquad (2.1) \\ p(t, z) &= z^{-1}\ln(1-zt) - \ln(1-t), \\ q(t, a, b, z) &= t^{b-1}(1-t)^{-b-1}(1-zt)^{-a}, \\ r(t, z, \theta_\lambda) &= \mathrm{Re}(e^{i\theta_\lambda}p(t,z)). \qquad (2.2) \end{aligned}$$

If $z \neq 0$ or 1, $|\mathrm{ph}(1-z)| < \pi$ and $\mathrm{Re}(\lambda) > \mathrm{Re}(b) > 0$, then [5, Section 15.6]

$$F(a + \lambda/z, b; \lambda; z) = G(\lambda, b)I(\lambda, a, b, z). \tag{2.3}$$

The asymptotic form of $G(\lambda, b)$ is well known [5, Section 5.11(iii)]. If $|\lambda| \to \infty$ and $|\theta_\lambda| \leqslant \pi - \delta < \pi$, then

$$G(\lambda, b) \sim \frac{\lambda^b}{\Gamma(b)} \sum_{s=0}^{\infty} \binom{b}{s} \frac{B_s^{(b+1)}}{\lambda^s} = \frac{\lambda^b}{\Gamma(b)} \sum_{s=0}^{\infty} \frac{G_s}{\lambda^s}, \text{ say,}$$

where the $B_s^{(b+1)}$ are Bernoulli numbers of order $b + 1$ [5, Section 24.16(i)]; then

$$\begin{aligned} G_0 &= 1, \\ G_1 &= -b(b+1)/2, \\ G_2 &= b(b-1)(b+1)(3b+2)/24. \end{aligned}$$

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3. Real $z < 1, z \neq 0$

Because we use both (2.1) and (2.3), we assume henceforth that $|\theta_\lambda| < \pi/2$, $z \neq 0$ and $z \notin [1, \infty)$. We also assume for the time being that $\mathrm{Re}(\lambda) > \mathrm{Re}(b) > 0$ and $z < 1$, but the restriction on $\mathrm{Re}(b)$ will be relaxed in Section 3.2 and the restriction on z will be relaxed in Section 4.

3.1. Re(b) > 0 Let S be the open sector $\{t \mid 0 < |t| < 1, |\mathrm{ph}(t)| < \pi/2\}$. Let D be the open disc $\{t \mid |t| < \min(1, 1/|z|)\}$. Let A be the closed annular sector

$$A = \{\lambda \mid |\lambda| \geqslant R_{\min} > \max(|b| + 1, |z| + |a|),\ |\theta_\lambda| \leqslant \pi/2 - \delta < \pi/2\}.$$

We may use Laplace's method [5, Section 2.4(iii)] if $z \in (-\infty, 0) \cup (0, 1)$ to find the asymptotic series of $I(\lambda, a, b, z)$ by integrating along $P_1 = [0, 1]$, because the following conditions are all satisfied.

(1) Both $p(t, z)$ and $q(t, a, b, z)$ are analytic for $t \in S$ and the path of integration with t real, from 0 to 1, is in S except for its ends.

(2) If $t \in D$, then $p(t, z)$ and $q(t, a, b, z)$ have these convergent series in powers of t, with $p_0 \neq 0$ and $q_0 \neq 0$:

$$p(t, z) = \sum_{n=0}^{\infty} p_n t^{n+2}, \qquad p_n = \frac{1 - z^{n+1}}{n + 2},$$
$$q(t, a, b, z) = \sum_{n=0}^{\infty} q_n t^{n+b-1}, \quad q_n = (-1)^n \sum_{m=0}^{n} \binom{-b-1}{n-m} \binom{-a}{m} z^m. \tag{3.1}$$

If b is not an integer, we use the principal value of t^b in (3.1).

(3) $I(\lambda, a, b, z)$ converges at $t = 1$ absolutely and uniformly with respect to $\lambda \in A$.

(4) If $0 < t < 1$, then $r(t, z, \theta_\lambda) > r(0, z, \theta_\lambda) = 0$.

(5) If $\lambda \in A$, $r(t, z, \theta_\lambda)$ is bounded away from zero uniformly with respect to θ_λ as $t \to 1$ along P_1.

Then, as $\lambda \to \infty$ in S,

$$I(\lambda, a, b, z) \sim \sum_{s=0}^{\infty} \Gamma\Big(\frac{s + b}{2}\Big) \frac{a_s}{\lambda^{(s+b)/2}},$$

where the coefficient a_s is the residue at $t = 0$ of

$$\frac{q(t, a, b, z)}{2p(t, z)^{(s+b)/2}},$$

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which has a pole of order s there. Expressions for a_s are also known in terms of partial ordinary Bell polynomials [9, 10] and of ordinary potential polynomials [4]. We have

$$a_0 = \frac{q_0}{2p_0^{b/2}} = \frac{2^{b/2-1}}{(1-z)^{b/2}},$$
$$a_1 = \left[\frac{q_1}{2} - \frac{(b+1)p_1q_0}{4p_0}\right]\frac{1}{p_0^{(b+1)/2}},$$
$$a_2 = \left[\frac{q_2}{2} - \frac{(b+2)p_1q_1}{4p_0} + \{(b+4)p_1^2 - 4p_0p_2\}\frac{(b+2)q_0}{16p_0^2}\right]\frac{1}{p_0^{(b+2)/2}}.$$

By the duplication formula for gamma functions [5, (5.5.5)], the leading term when z is real is

$$F\Big(a + \frac{\lambda}{z}, b; \lambda; z\Big) \sim \frac{2^{-b/2}\pi^{1/2}\,\lambda^{b/2}}{(1-z)^{b/2}\Gamma(1/2+b/2)}. \tag{3.2}$$

3.2. Re(b) ⩽ 0 Equation (3.2) has been proved for $\mathrm{Re}(b) > 0$. It still holds if $b = 0$, because then $F = 1$. It also holds for any $b \in \mathbb{C}$, except a negative integer. That is because the value for $\mathrm{Re}(b) > -1$ follows from those for $b+1$ and $b+2$ [1, (15.2.11)]. It agrees with (3.2) and shows that each term after the first in the asymptotic series is $O(\lambda^{-1/2})$ times the previous one. In the same way we find the same results for $\mathrm{Re}(b) > -2, -3, -4, \ldots$. If on the other hand b is a negative integer, the right-hand side of (3.2) is zero but the left-hand side is not. The asymptotic expansion then begins further along the series, which now converges and contains only integer powers of λ.

4. Complex z

We still require $\theta_\lambda \in (-\pi/2, \pi/2)$, $z \neq 0$ and $z \notin [1, \infty)$. Checking where $r(t, z, \theta_\lambda) > 0$ when $z \in \mathbb{C}$ need not require dealing with $\mathrm{Im}(z) < 0$, because $r(t, z, \theta_\lambda) = r(t, \bar{z}, -\theta_\lambda)$. If the least value of $r(t, z, \theta_\lambda)$ for $t \in P_1 = [0, 1]$ is still at $t = 0$ even when $\mathrm{Im}(z) \neq 0$, then the result of Section 3 still holds.

We now show that a sufficient condition is $\ddot{r}(t, z, \theta_\lambda)|_{t=0} > 0$, where dots indicate t-derivatives. With θ_λ and z fixed, $r(t, z, \theta_\lambda)$ defined in (2.2) is a real function of t infinitely differentiable on P_1; its least value on P_1 is thus either at $t = 0$ or at another point $t = t'$, where $\dot{r}(t, z, \theta_\lambda) = 0$. That gives a quadratic equation for t; its two roots are 0 and t'. Because $r(t, z, \theta_\lambda) \to +\infty$ as $t \to 1$ along P_1, $t' \in (0, 1)$ would require $r(t', z, \theta_\lambda) < 0$ and $\dot{r}(t', z, \theta_\lambda) = 0$. Now

$$\dot{r}(t, z, \theta_\lambda) = \mathrm{Re}\Big(\frac{e^{i\theta_\lambda}}{1-t} - \frac{e^{i\theta_\lambda}}{1-zt}\Big),$$
$$\therefore t' = \frac{\ddot{r}(0, z, \theta_\lambda)}{|z|\cos(\theta_\lambda - \theta_z) - |z|^2\cos\theta_\lambda},$$
$$\ddot{r}(t, z, \theta_\lambda) = \mathrm{Re}\Big(\frac{e^{i\theta_\lambda}(1-z)(1-zt^2)}{(1-t)^2(1-zt)^2}\Big).$$

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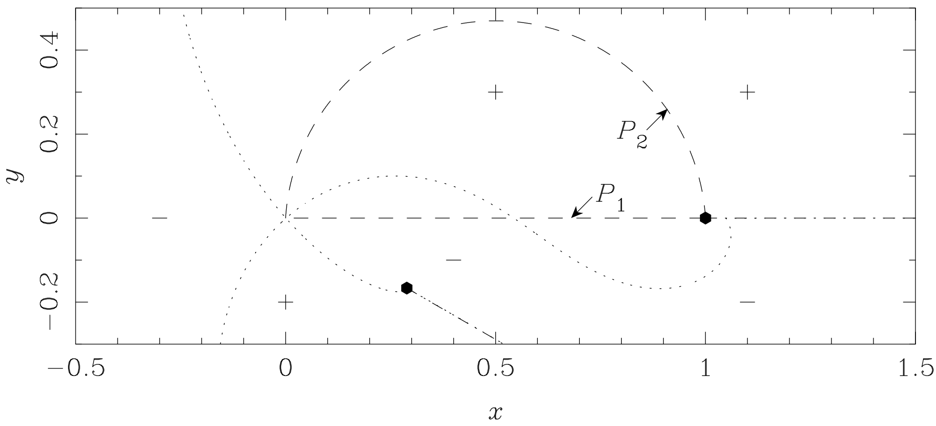

FIGURE 1. The complex t-plane ($t = x + iy$), in a case where P_2 is useful but P_1 is not: $|z| = 3$, $\theta_z = 30°$, $\theta_\lambda = -36°$. Dotted lines on which $r(t, z, \theta_\lambda) = 0$ separate regions where it is of constant sign, marked + and −. The lines spiral in towards the branch points on other Riemann sheets; only the principal sheet is shown. Dashed lines: the straight line P_1 and arc P_2. Dot-dash-dot-dash lines: the branch cuts for $\ln(1 - zt)/z$ and $\ln(1 - t)$. Black circles: the branch points $t = 1$ and $1/z$.

If $\ddot{r}(0, z, \theta_\lambda) > 0$, then $r(t, z, \theta_\lambda) > 0$ when $0 < t \ll 1$ and there would have to be a maximum of r at $t'' \in (0, t')$ before r decreased to its minimum at t'. The quadratic equation would then have three roots, which is impossible. Therefore, if $\ddot{r}(0, z, \theta_\lambda) > 0$, then $t' \notin (0, 1)$ and the leading contribution to the integral (2.1) is from t near 0, with P_1 still the path of integration.

Experiments with Waterloo Maple computer algebra revealed (to the author's initial surprise) that the results of Section 3 still held even when $\ddot{r}(0, z, \theta_\lambda) < 0$. That is because there are other paths through complex values from $t = 0$ to $t = 1$ along which $r(t, z, \theta_\lambda)$ has its least value at $t = 0$. Figures 1 and 2 illustrate the possibilities. In Figure 1 $|z| = 3$, $\theta_z = 30°$, $\theta_\lambda = -36°$, $\omega \approx 86°$, and in Figure 2 $|z| = 8$, $\theta_z = 3°$, $\theta_\lambda = 83°$, $\omega \approx 47°$, where $\omega = -\{\mathrm{ph}(1 - z) + \theta_\lambda\}/2$.

Let $\theta_t = \mathrm{ph}(t)$. Then

$$\begin{aligned} r(t, z, \theta_\lambda) &\sim \mathrm{Re}\big(\tfrac{1}{2}(1 - z)t^2 e^{i\theta_\lambda}\big) \quad \text{as } t \to 0 \\ &= |\tfrac{1}{2}(1 - z)t^2| \cos(2\theta_t - 2\omega). \end{aligned}$$

Therefore, $r(t, z, \theta_\lambda)$ begins increasing as t leaves 0 if $|\theta_t - \omega| < \pi/4$, and it does so as fast as possible if $\theta_t = \omega$. The other direction of steepest ascent of r (or steepest descent of $e^{-\lambda p}$) from $t = 0$ is $\theta_t = \omega \pm \pi$, but it will not concern us. Let P_2 be the circular arc passing through $t = 0$ and $t = 1$ with its tangent at 0 in the direction $\theta_t = \omega$, so that if $\sin\omega \neq 0$, then

$$t = \frac{(\sin\phi + \sin\omega) + i(\cos\phi - \cos\omega)}{2\sin\omega}, \quad \phi \in [-\omega, \omega].$$

Numerical work with $|z|$ from 0.5 to 10, θ_z from 3° to 177° and θ_λ from −87° to +87° showed that $r(t, z, \theta_\lambda) > 0$ everywhere except $t = 0$ on at least one of P_1 and P_2, except

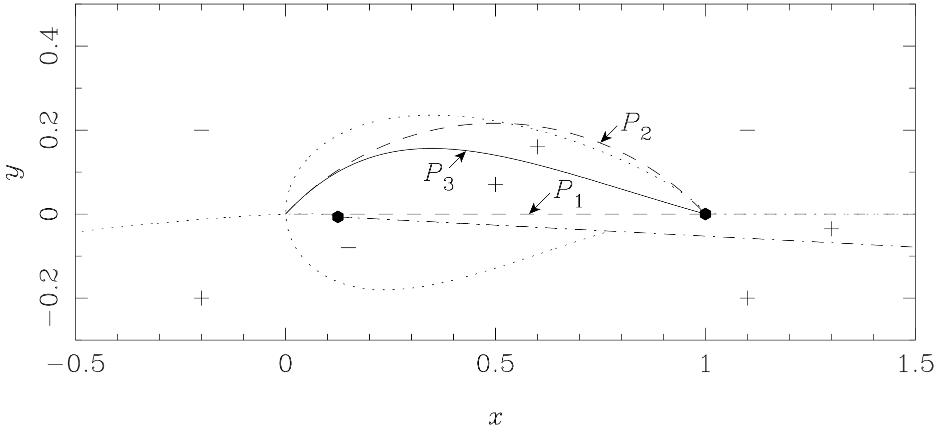

FIGURE 2. The complex t-plane in a case where P_3 is useful but P_1 and P_2 are not: $|z| = 8$, $\theta_z = 3°$, $\theta_\lambda = 83°$. Solid line: P_3. Other symbols as in Figure 1.

for a few cases where $r(t, z, \theta_\lambda) < 0$ at some points on each of P_1 and P_2. In those cases there is a path P_3 on which $r(t, z, \theta_\lambda) > 0$ everywhere except $t = 0$; we obtain P_3 by modifying P_2 to bring $y = \mathrm{Im}(t)$ nearer to 0 by an amount gradually increasing as ϕ increases:

$$y = \frac{\cos\phi - \cos\omega}{2\cosh(1 + \phi/\omega)\sin\omega}, \quad \phi \in [-\omega, \omega].$$

Because $t = 0$ when $\phi = -\omega$ and $t = 1$ when $\phi = \omega$, P_3 leaves $t = 0$ in the same direction as P_2, but arrives at $t = 1$ from a direction much closer to the real axis $y = 0$ (see Figure 2).

The path P_3 was needed if $|z| \gg 1$, $|\theta_z| \ll \pi/2$, so that z was near the positive real axis but far from 1, and θ_λ was near $\pm\pi/2$. Then $r < 0$ both on the straight line P_1 very near $t = 0$ and on much of the arc P_2; hence, both paths were unsuitable for Laplace's method. Because $\omega \approx 47°$ in Figure 2, the lower right negative region is between $\theta_t \approx +2°$ and $\theta_t \approx -88°$ near $t = 0$, and the path P_1 is initially inside it. The circular-arc path P_2 is in a negative region if $x > 0.53$, but P_3 is in a positive region for its whole length. Note that P_2 and P_3 are not paths of steepest descent, though they do leave $t = 0$ in the steepest-descent direction. As the NIST Handbook says [5, Section 2.4(iv)], "for the purpose of simply deriving the asymptotic expansions the use of steepest descent paths is not essential".

The remaining task was to choose the correct branch of $p_0^{b/2}$ in (3.2). That was the one on which $\mathrm{ph}(p_0) = \mathrm{ph}(1 - z)$ [5, equation (2.4.13)], whether the path of integration was P_1, P_2 or P_3.

5. Conclusion

The asymptotic form (3.2) for $F(a + \lambda/z, b; \lambda; z)$ as $|\lambda| \to \infty$ if $|\mathrm{ph}\lambda| < \pi/2$ was found in Section 3.1 for $\mathrm{Re}(b) > 0$, z real, $z \neq 0$ and $z < 1$. Section 3.2 shows that it still

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holds for any real or complex value of b except a negative integer if $z \in \mathbb{R} \setminus [1, \infty)$, and Section 4 extends that region to $z \in \mathbb{C} \setminus [1, \infty)$. In a recent paper on bubbles rising in a liquid [3], $b = 1$ and $z = 2/3$, a special case which revealed that the present investigation was needed. The present results appear not to be in previous work on hypergeometric functions.

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