A CHOQUET-DENY THEOREM FOR AFFINE FUNCTIONS ON A CHOQUET SIMPLEX

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1. Introduction

The closed wedges in $C(X)$ (the space of real continuous functions on a compact Hausdorff space X) which are also inf-lattices have been characterized by Choquet and Deny (**2**); see also (**5**). The present note extends their result to certain wedges of affine continuous functions on a Choquet simplex, the generalization being in the same spirit as the generalization of the Kakutani-Stone theorem obtained by Edwards in (**4**).

I should like to thank my supervisor, Dr D. A. Edwards, for suggesting this problem and for his subsequent help. I am also grateful to the referee for correcting several slips.

2. Preliminaries

Throughout X will denote a Choquet simplex and X_e the set of extreme points of X. $A(X)$ denotes the space of real-valued affine continuous functions on X with the uniform norm. We define, for each $x \in X_e$,

$$N_x = \{f \in A(X): f \leqq 0, f(x) = 0\}.$$

The proof of the theorem will depend on the following criterion for a continuous functional on $A(X)$ to take non-negative values on N_x.

Lemma. *Let ϕ be a continuous functional on $A(X)$ and let $x \in X_e$. Then $\phi(N_x) \subseteq \mathbf{R}^+$ if and only if there exists a constant $\alpha \in \mathbf{R}$ such that $\phi \leqq \alpha\varepsilon_x$, ε_x being the evaluation functional $f \mapsto f(x)$ $(f \in A(X))$.*

Since ε_x is zero on N_x, the condition is clearly sufficient. To prove necessity, we suppose ϕ is non-negative on N_x and define $\alpha \in (-\infty, \infty]$ by

$$\alpha = \sup\{\phi(f): f \in A(X), f \geqq 0, f(x) = 1\}.$$

We show first that $\alpha < \infty$. Take any $f \in A(X)$ such that $f \geqq 0$ and $f(x) = 1$. Choose $h \in A(X)$ such that $0 \leqq h \leqq f$, 1 and $h(x) = 1$. That such an h exists is a consequence of Edwards's separation theorem ((**3**), corollary to theorem 3). We have $\phi(f) = \phi(h) + \phi(f-h)$. Since $h-f \in N_x$, $\phi(f-h) \leqq 0$. The continuity of ϕ implies that ϕ is bounded above, by M say, on the set $\{g \in A(X): 0 \leqq g \leqq 1\}$. Therefore $\phi(f) \leqq M$ for every $f \in A(X)$ such that $f \geqq 0, f(x) = 1$. Consequently $\alpha < \infty$.

A routine calculation shows that, with this choice of α, $\phi \leqq \alpha\varepsilon_x$ and the proof of the Lemma is complete.

3. The main theorem

We suppose K to be a non-empty wedge in $A(X)$ and replace the inf-lattice condition of Choquet and Deny's paper by the following filtering condition on K:

(F) for each $h \in A(X)$, the family $V = \{k \in K: k > h\}$ is either empty, or downward filtering in the sense that, for each pair of functions $k_1, k_2 \in K$, there exists $k \in V$ with $k_1, k_2 > k$.

Theorem. *Let K be a wedge in $A(X)$ satisfying (F) and let $f \in A(X)$. Then $f \in \bar{K}$ if and only if, for each positive functional ϕ on $A(X)$,*

(1) $\phi(f) \leqq 0$ *whenever* $\phi(k) \leqq 0$ *for all* $k \in K$ *and*

(2) *for each* $x \in X_e$, $\phi(f) \leqq f(x)$ *whenever* $\phi(k) \leqq k(x)$ *for all* $k \in K$.

It is trivial that both (1) and (2) hold for functions in $\bar{K}$.

To prove the converse, we first show that $f \notin \bar{K}$ implies that, for some $a \in X_e$, $f \notin \overline{(K+N_a)}$. Suppose the contrary, i.e. suppose that, for each $x \in X_e$ and each $\varepsilon > 0$, we can find $k_x \in K$, $h_x \in N_x$ with $\| k_x + h_x - f \| < \varepsilon$. Then

$$k_x \geqq k_x + h_x > f - \varepsilon$$

and

$$k_x(x) = k_x(x) + h_x(x) < f(x) + \varepsilon.$$

We now appeal to a result of Edwards ((**4**), proposition 1). A specialization of this states that if $v \in A(X)$ and if $V \subseteq A(X)$ is a downward filtering family such that for each $x \in X_e$ there exists $u \in V$ with $u(x) < v(x)$, then V contains a function $u < v$. Applying this with $v = f + \varepsilon$, $V = \{k \in K: k > f - \varepsilon\}$ (V is downward filtering by (F)), we see that V contains an element k such that $k < f + \varepsilon$. But then $\| k - f \| < \varepsilon$, contradicting $f \notin \bar{K}$.

Let $a \in X_e$ be such that $f \notin \overline{(K+N_a)}$ and choose a continuous functional ψ on $A(X)$ and $c \in \mathbf{R}$ so that

$$\psi(f) < c \leqq \psi \mid \overline{(K+N_a)}.$$

ψ attains the value 0 on $K + N_a$; hence $\psi(f) < 0$. Therefore we can take $c = 0$.

Fix $h \in N_a$ and let $k \in K$, $\lambda > 0$. $0 \leqq \frac{1}{\lambda} \psi(k) + \psi(h)$. Letting $\lambda \to \infty$ we see that $\psi(h) \geqq 0$. By the Lemma, there exists $\alpha \in \mathbf{R}$ such that $\psi \leqq \alpha \varepsilon_a$. We distinguish two cases. If $\alpha \leqq 0$, we see that 1) is violated by $\phi = -\psi$. If $\alpha > 0$, 2) is violated by $\phi = \varepsilon_a - \alpha^{-1}\psi$. This completes the proof.

As noted by Bauer in (**1**), it is possible (in the continuous function case) to characterize the set on which (1) holds. This carries over to the present situation. Assuming for simplicity that K is closed, we define

$$K^* = \{f \in A(X): f \leqq k \text{ for some } k \in K\}.$$

K^* is a wedge in $A(X)$ containing K and a simple Hahn Banach argument shows that the closure of K^* is precisely the subset of $A(X)$ on which (1) is valid.

4. Two related results

Choquet and Deny deduce further results concerning wedges satisfying lattice conditions by applying the Kakutani-Stone theorem. We may use Edwards's generalization of this theorem to obtain analogues for affine functions. The proofs are elementary and will be omitted.

Proposition 1. *Let K be a closed wedge in $A(X)$ which satisfies both (F) and the corresponding condition with the inequality signs reversed. Then K is characterized by closed subsets P, N of X_e and by a strictly positive function u defined on a subset S of $X_e \times X_e$. K consists precisely of those functions $f \in A(X)$ with $f \geqq 0$ on P, $f \leqq 0$ on N, and $f(x) \geqq u(x, y) f(y)$ for all $(x, y) \in S$.*

Proposition 2. *Let K be a closed wedge in $A(X)$ satisfying (F) and such that K is not total in $A(X)$. Then there exist distinct points $x, y \in X_e$ and non-negative. constants a, b, not both zero, such that $ak(x) = bk(y)$ for all $k \in K$.*

REFERENCES

(1) H. Bauer, Cones convexes semi-réticulés de fonctions continues; théorèmes de représentation et de stabilité, Séminaire Brelot-Choquet-Deny, 8e année, exposé n° 5, Paris (1965).

(2) G. Choquet and J. Deny, Ensembles semi-réticulés et ensembles réticulés de fonctions continues, *J. Math. Pures Appl.* (9) **36** (1957), 179-189.

(3) D. A. Edwards, Minimum-stable wedges of semi-continuous functions, *Math. Scand.* **19** (1966), 15-26.

(4) D. A. Edwards, On uniform approximation of affine functions on a compact convex set, *Quart. J. Math. Oxford* (2) **20** (1969), 139-142.

(5) L. Nachbin, *Elements of approximation theory* (Van Nostrand Mathematical Studies No. 14, Princeton N.J., 1966).

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