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DISTORTION IN THE FINITE DETERMINATION RESULT FOR EMBEDDINGS OF LOCALLY FINITE METRIC SPACES INTO BANACH SPACES

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Abstract. Given a Banach space X and a real number $\alpha \geq 1$, we write: (1) $D(X) \leq \alpha$ if, for any locally finite metric space A, all finite subsets of which admit bilipschitz embeddings into X with distortions $\leq C$, the space A itself admits a bilipschitz embedding into X with distortion $\leq \alpha \cdot C$; (2) $D(X) = \alpha^+$ if, for every $\varepsilon > 0$, the condition $D(X) \leq \alpha + \varepsilon$ holds, while $D(X) \leq \alpha$ does not; (3) $D(X) \leq \alpha^+$ if $D(X) = \alpha^+$ or $D(X) \leq \alpha$. It is known that $D(X)$ is bounded by a universal constant, but the available estimates for this constant are rather large. The following results have been proved in this work: (1) $D((\oplus_{n=1}^{\infty} X_n)_p) \leq 1^+$ for every nested family of finite-dimensional Banach spaces $\{X_n\}_{n=1}^{\infty}$ and every $1 \leq p \leq \infty$. (2) $D((\oplus_{n=1}^{\infty} \ell_{\infty}^n)_p) = 1^+$ for $1 < p < \infty$. (3) $D(X) \leq 4^+$ for every Banach space X with no nontrivial cotype. Statement (3) is a strengthening of the Baudier–Lancien result (2008).

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1. Introduction. The study of bilipschitz embeddings of metric spaces into Banach spaces is a very active research area which has found many applications, not only within Functional Analysis, but also in Graph Theory, Group Theory, and Computer Science, see [**7**, **8**, **10**, **14**, **15**]. This paper contributes to the study of relations between the embeddability of an infinite metric space and its finite pieces. Let us recollect some necessary notions.

Definition 1.1. A metric space is called *locally finite* if each ball of finite radius in it has finite cardinality.

Definition 1.2.

(i) Let $0 \leq C < \infty$. A map $f : (A, d_A) \to (Y, d_Y)$ between two metric spaces is called *C-Lipschitz* if

$$\forall u, v \in A \quad d_Y(f(u), f(v)) \leq C d_A(u, v).$$

A map f is called *Lipschitz* if it is C-Lipschitz for some $0 \leq C < \infty$.

(ii) Let $1 \leq C < \infty$. A map, $f : A \to Y$, is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$\forall u, v \in A \quad rd_A(u, v) \leq d_Y(f(u), f(v)) \leq rCd_A(u, v). \tag{1}$$

A map f is a *bilipschitz embedding* if it is C-bilipschitz for some $1 \leq C < \infty$. The smallest constant C for which there exists $r > 0$ such that (1) is satisfied, is called the *distortion* of f.

We refer to [**6**, **14**] for unexplained terminology.

It has been known that the bilipschitz embeddability of locally finite metric spaces into Banach spaces is finitely determined in the following sense:

THEOREM 1.3 [**13**]. *Let A be a locally finite metric space whose finite subsets admit bilipschitz embeddings with uniformly bounded distortions into a Banach space X. Then, A also admits a bilipschitz embedding into X.*

To elaborate more, the argument of [**13**] leads to a stronger result which we state as Theorem 1.4. To formulate Theorem 1.4, it is convenient to introduce parameter $D(X)$ of a Banach space X. More specifically, given a Banach space X and a real number $\alpha \geq 1$, we write:

- $D(X) \leq \alpha$ if, for any locally finite metric space A, all finite subsets of which admit bilipschitz embeddings into X with distortions $\leq C$, the space A itself admits a bilipschitz embedding into X with distortion $\leq \alpha \cdot C$;
- $D(X) = \alpha$ if α is the least number for which $D(X) \leq \alpha$;
- $D(X) = \alpha^+$ if, for every $\varepsilon > 0$, the condition $D(X) \leq \alpha + \varepsilon$ holds, while $D(X) \leq \alpha$ does not;
- $D(X) = \infty$ if $D(X) \leq \alpha$ does not hold for any $\alpha < \infty$.

Further, we use inequalities like $D(X) < \alpha^+$ and $D(X) < \alpha$ with the natural meanings, for example, $D(X) < \alpha^+$ indicates that either $D(X) = \beta$ for some $\beta \leq \alpha$ or $D(X) = \beta^+$ for some $\beta < \alpha$.

THEOREM 1.4 [**13**]. *There exists an absolute constant $D \in [1, \infty)$, such that for an arbitrary Banach space X the inequality $D(X) \leq D$ holds.*

In the proof of Theorem 1.4 given in [**13**] as well as in the proofs of its special cases obtained in [**1**, **2**, **12**], the values of D implied by the argument are 'large'. For example, Baudier and Lancien in [**2**] worked out the numerical estimate provided by their proof and derived estimate $D(X) \leq 216$ for Banach spaces with no nontrivial cotype.

On the other hand, it is known that for some Banach spaces X the value of $D(X)$ is significantly smaller. In order to present relevant assertions, it is expedient to introduce the following definition.

DEFINITION 1.5. It is said that a Banach space X satisfies the *condition* **(U)** if each separable subset of an arbitrary ultrapower of X admits an isometric embedding into X.

The fact stated below is well known and its proof follows immediately from [**14**, Proposition 2.21]:

PROPOSITION 1.6. *If a Banach space X satisfies condition* **(U)**, *then $D(X) = 1$.*

Further, the next result due to Kalton and Lancien has to be cited in the context of the present work.

THEOREM 1.7 [**5**, Theorem 2.9]. $D(c_0) = 1^+$.

REMARK 1.8. Theorem 2.9 in [**5**] is stated in terms of locally compact metric spaces. However, the corresponding lower bound is proved also for locally finite metric spaces [**5**, page 256], yielding Theorem 1.7.

The purport of this work is to find upper estimates for $D(X)$ which are significantly stronger than the estimates implied by the proofs in [**1**, **2**, **12**, **13**]. Theorems 1.9, 1.12, 1.14, and their corollaries constitute the main results of the present paper.

Customarily, a family of finite-dimensional Banach spaces $\{X_n\}_{n=1}^{\infty}$ is said to be *nested* if X_n is a proper subspace of X_{n+1} for every $n \in \mathbb{N}$.

THEOREM 1.9. *Let* $1 \le p < \infty$. *If* $\{X_n\}_{n=1}^{\infty}$ *is a nested family of finite-dimensional Banach spaces, then* $D\left(\left(\oplus_{n=1}^{\infty} X_n\right)_p\right) \le 1^+$.

The main idea of our proofs of Theorems 1.9 and 1.14 is explained in Remark 2.1.

COROLLARY 1.10. *If* $1 \le p < \infty$, *then* $D(\ell_p) \le 1^+$.

REMARK 1.11. The problem of finiteness of $D(\ell_p)$, $p \neq 2, \infty$, was raised by Marc Bourdon and published in [**11**, Question 10.7]. A solution to this problem was found in [**1**, **13**], but in both of these papers the bounds on $D(\ell_p)$ are rather large numbers.

In some cases, the inequality in Theorem 1.9 can be reversed, as claimed by the forthcoming result:

THEOREM 1.12. *Let* $1 < p < \infty$, *then* $D\left(\left(\oplus_{n=1}^{\infty} \ell_\infty^n\right)_p\right) \ge 1^+$.

Together with the pertinent special case of Theorem 1.9 this leads to:

COROLLARY 1.13. *Let* $1 < p < \infty$, *then* $D\left(\left(\oplus_{n=1}^{\infty} \ell_\infty^n\right)_p\right) = 1^+$.

Our final goal is a significant improvement of the distortion estimate obtained in [**2**]. In this connection, the following outcome has been reached:

THEOREM 1.14. *Let* X *be a Banach space with no nontrivial cotype. Then,* $D(X) \le 4^+$.

2. Proof of theorem 1.9. Let $X = (\oplus_{n=1}^{\infty} X_n)_p$, $C \in [1, \infty)$, and let A be a locally finite metric space such that its finite subsets admit embeddings into X with distortion $\le C$. It has to be proved that, for each $\varepsilon > 0$, there exists a bilipschitz embedding of A into X with distortion $\le C + \varepsilon$. By the well-known fact (see [**14**, Proposition 2.21]), such a space A admits a bilipschitz embedding with distortion $\le C$ into any ultrapower of X. Thence, it is sufficient to show that, for any $\varepsilon > 0$, every locally finite metric subspace M of each ultrapower $X^{\mathcal{U}}$ admits a bilipschitz embedding into X with distortion $\le 1 + \varepsilon$. This can be accomplished by selecting an arbitrary $\varepsilon > 0$ and finding a bilipschitz embedding of a locally finite metric subspace M of $X^{\mathcal{U}}$ into X with distortion $\le 1 + \varphi(\varepsilon)$, where function φ is such that $\varphi(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

Without loss of generality, one may assume that $0 \in M$. Let $\{R_n\}_{n=1}^{\infty}$ be an increasing sequence of positive real numbers (we shall choose a sequence $\{R_n\}_{n=1}^{\infty}$ which is suitable for our purposes later). Consider subsets M_n of M defined by

$$M_n = \{x \in M : \ ||x|| \leq R_n\}.$$

Since M is a locally finite metric space, these sets are finite. Therefore, by the definition of an ultrapower, there exist bilipschitz embeddings of distortion $< 1 + \varepsilon$ of these sets into X. It follows immediately that, for each $n \in \mathbb{N}$, there exists $t(n) \in \mathbb{N}$ such that $t(n+1) \geq t(n)$, and the direct sum $(\oplus_{k=1}^{t(n)} X_k)_p$ admits a bilipschitz embedding of M_n with distortion $< 1 + \varepsilon$. Apart from that, since X_n, $n \in \mathbb{N}$, is a nested family of spaces, this implies that M_n admits a bilipschitz embedding with distortion $< 1 + \varepsilon$ into the space $Y_n := (\oplus_{k=m(n-1)+1}^{m(n)} X_k)_p$, where $m(0) = 0$ and $m(n) = m(n-1) + t(n)$. It is easy to see that Y_n is a nested family of finite-dimensional Banach spaces and that $X = (\oplus_{n=1}^{\infty} Y_n)_p$. We select and fix embeddings $E_n : M_n \to Y_n$ with distortion $< (1 + \varepsilon)$. Without loss of generality, it can be assumed that $E_n 0 = 0$ and

$$\forall x, y \in M_n \quad ||x - y|| \leq ||E_n x - E_n y|| < (1 + \varepsilon)||x - y||. \tag{2}$$

REMARK 2.1. Before we proceed, it seems beneficial to describe the main idea behind our proofs of Theorems 1.9 and 1.14. We have already introduced a sequence $\{E_n\}_{n=1}^{\infty}$ of embeddings of balls in M with increasing radii into X. Now, what remains is to find a low-distortion pasting technique for these maps. This is done by rather complicated formulae, namely, (6)–(8) and (22)–(24), which, in the case of ℓ_2-sums, reduce to what can be called an ε-normalization of the formula for the logarithmic spiral in the Euclidean plane: $\gamma_\varepsilon : (1, \infty) \to \mathbb{R}^2$, $\gamma_\varepsilon(t) = t(\cos(\varepsilon \ln t), \ \sin(\varepsilon \ln t))$. The curve γ_ε is a slight modification of the well-known example of a quasi-geodesic in $\mathbb{R}^2$ which is far from geodesic, see [**3**, p. 4].

One can view this pasting techniques as a transition from E_{2n} to E_{2n+2} along ε-normalized ℓ_p-versions of the logarithmic spiral. See (6)–(8) and (22)–(24). The low-distortion estimates for these embeddings are very close to the estimate, which shows that the map γ_ε has distortion $\leq (1 + \kappa(\varepsilon))$ with $(1 + \kappa(\varepsilon)) \downarrow 1$ as $\varepsilon \downarrow 0$.

To continue the proof, we opt for an increasing sequence $\{R_i\}_{i=1}^{\infty}$ of real numbers such that

$$R_1 = 1, \tag{3}$$

$$\varepsilon \ln(R_{2i}/R_{2i-1}) = \frac{\pi}{2}, \tag{4}$$

$$\frac{R_{2i+1}}{R_{2i}} \geq \frac{1}{\varepsilon}. \tag{5}$$

From this point on, we are going to consider the cases $1 \leq p \leq 2$ and $2 < p < \infty$ separately, mostly because in the case $1 \leq p \leq 2$ much simpler formulae can be used.

2.1. Spaces $\left(\oplus_{n=1}^{\infty} X_n\right)_p$, $1 \leq p \leq 2$. To construct an embedding $T : M \to X$ with needful properties, we employ the real-valued functions c_{2i-1} and s_{2i-1}, $i \in \mathbb{N}$ on M

defined by

$$c_{2i-1}(x) = \begin{cases} \cos^{2/p}(\varepsilon \ln(R_{2i-1}/R_{2i-1})) = 1 & \text{if } ||x|| \leq R_{2i-1} \\ \cos^{2/p}(\varepsilon \ln(||x||/R_{2i-1})) & \text{if } R_{2i-1} \leq ||x|| \leq R_{2i} \\ \cos^{2/p}(\varepsilon \ln(R_{2i}/R_{2i-1})) = 0 & \text{if } ||x|| \geq R_{2i} \end{cases} \tag{6}$$

$$s_{2i-1}(x) = \begin{cases} \sin^{2/p}(\varepsilon \ln(R_{2i-1}/R_{2i-1})) = 0 & \text{if } ||x|| \leq R_{2i-1} \\ \sin^{2/p}(\varepsilon \ln(||x||/R_{2i-1})) & \text{if } R_{2i-1} \leq ||x|| \leq R_{2i} \\ \sin^{2/p}(\varepsilon \ln(R_{2i}/R_{2i-1})) = 1 & \text{if } ||x|| \geq R_{2i}. \end{cases} \tag{7}$$

The equalities in the last lines of formulae (6) and (7) follow from (4). Consider the map $T : M \to X$ represented by

$$Tx = \begin{cases} c_1(x)E_2x + s_1(x)E_4x & \text{if } x \in M_3 \\ c_3(x)E_4x + s_3(x)E_6x & \text{if } x \in M_5\backslash M_3 \\ \cdots & \cdots \\ c_{2i-1}(x)E_{2i}x + s_{2i-1}(x)E_{2i+2}x & \text{if } x \in M_{2i+1}\backslash M_{2i-1} \\ \cdots & \cdots, \end{cases} \tag{8}$$

where we use the convention that a product of 0 and an undefined quantity is 0. Since $(c_{2i-1}(x))^p + (s_{2i-1}(x))^p = 1$ for all i and x, one derives applying (2), (8), $E_n0 = 0$, and $X = \left(\oplus_{n=1}^{\infty} Y_n\right)_p$, that

$$\forall x \in M \quad ||x|| \leq ||Tx|| < (1+\varepsilon)||x||. \tag{9}$$

What is demanded now is an estimate of the following form:

$$\forall x, y \in M \quad (1 - \psi(\varepsilon))||x - y|| \leq ||Tx - Ty|| < (1 + \xi(\varepsilon))||x - y||, \tag{10}$$

where functions ψ and ξ have positive values and are such that $\lim_{\varepsilon \downarrow 0} \psi(\varepsilon) = \lim_{\varepsilon \downarrow 0} \xi(\varepsilon) = 0$.

Obviously, it suffices to consider the case $||y|| \leq ||x||$. The simpler case $||y|| \leq \varepsilon||x||$ creates no difficulty because if this occurs, one obtains

$$(1-\varepsilon)||x|| \leq ||x|| - ||y|| \leq ||x-y|| \leq ||x|| + ||y|| \leq (1+\varepsilon)||x|| \tag{11}$$

and

$$\begin{aligned} (1 - \varepsilon(1+\varepsilon))||x|| \leq ||x|| - (1+\varepsilon)||y|| &\leq ||Tx|| - ||Ty|| \\ &\leq ||Tx - Ty|| \leq ||Tx|| + ||Ty|| \\ &\leq (1+\varepsilon)||x|| + (1+\varepsilon)||y|| \leq (1+\varepsilon)^2||x||. \end{aligned} \tag{12}$$

Combining (11) and (12), we get

$$\frac{1 - \varepsilon(1+\varepsilon)}{1+\varepsilon}||x-y|| \leq ||Tx - Ty|| \leq \frac{(1+\varepsilon)^2}{1-\varepsilon}||x-y||, \tag{13}$$

which is an estimate of the required form (10).

As a next step, set $R_0 = 0$. By virtue of condition (5) and inequality (13), it is enough to consider the case where

$$R_{2i-2} \le ||y|| \le ||x|| \le R_{2i+1}, \quad i = 1, 2, \dots . \tag{14}$$

It should be pointed out that since functions c_{2i-1} and s_{2i-1} are constant on intervals of the form $[R_{2j}, R_{2j+1}]$, there are many trivial cases. Out of the remaining ones, we deal first with the case $R_{2i-1} \le ||y|| \le ||x|| \le R_{2i}$.

For simplicity of notation in the following calculations, it is handy to use c for c_{2i-1}, s for s_{2i-1}, E for E_{2i}, and F for E_{2i+2}. With this in mind, one has:

$$\begin{aligned} ||Tx - Ty||^p &= ||c(x)Ex - c(y)Ey||^p + ||s(x)Fx - s(y)Fy||^p \\ &= ||c(x)(Ex - Ey) + (c(x) - c(y))Ey||^p \\ &\quad + ||s(x)(Fx - Fy) + (s(x) - s(y))Fy||^p. \end{aligned} \tag{15}$$

Consider each of the summands in the last line separately. To begin with, the Mean Value Theorem yields

$$\begin{aligned} c(x) - c(y) &= \cos^{2/p}(\varepsilon \ln(||x||/R_{2i-1})) - \cos^{2/p}(\varepsilon \ln(||y||/R_{2i-1})) \\ &= \frac{2}{p} \cos^{\frac{2}{p}-1}(\varepsilon \ln(\tau/R_{2i-1})) \cdot (-\sin(\varepsilon \ln(\tau/R_{2i-1}))) \cdot \varepsilon \frac{1}{\tau}(||x|| - ||y||). \end{aligned} \tag{16}$$

for some number τ satisfying $\tau \in (||y||, ||x||)$. Now, recall that $1 \le p \le 2$ and hence $\frac{2}{p} - 1 \ge 0$. Therefore,

$$||(c(x) - c(y))Ey|| \le \frac{2}{p} \cdot \varepsilon \frac{1}{\tau}(||x|| - ||y||) \cdot (1 + \varepsilon)||y|| \le 2\varepsilon(1 + \varepsilon)||x - y||. \tag{17}$$

Similarly, it can be demonstrated that

$$||(s(x) - s(y))Ey|| \le 2\varepsilon(1 + \varepsilon)||x - y||. \tag{18}$$

Inequalities (15), (17), and (18) lead to:

$$\begin{aligned} ((\max\{c(x) - 2\varepsilon(1 + \varepsilon), 0\})^p &+ (\max\{s(x) - 2\varepsilon(1 + \varepsilon), 0\})^p)||x - y||^p \\ &\le ||Tx - Ty||^p \\ &\le (1 + \varepsilon)^p((c(x) + 2\varepsilon)^p + (s(x) + 2\varepsilon)^p)||x - y||^p. \end{aligned} \tag{19}$$

Notice that

$$\lim_{\varepsilon \downarrow 0}((\max\{c(x) - 2\varepsilon(1 + \varepsilon), 0\})^p + (\max\{s(x) - 2\varepsilon(1 + \varepsilon), 0\})^p) = 1$$

and

$$\lim_{\varepsilon \downarrow 0}(1 + \varepsilon)^p((c(x) + 2\varepsilon)^p + (s(x) + 2\varepsilon)^p) = 1.$$

due to the fact that $c^p(x) + s^p(x) = 1$. Thus, inequality (19) provides the desired estimate (10).

To complete the proof, consider the case where $||y|| \in [R_{2i-2}, R_{2i-1}]$ and $||x|| \in [R_{2i-1}, R_{2i}]$. Then, $c_{2i-1}(y) = \cos^{2/p}(\varepsilon \ln(R_{2i-1}/R_{2i-1}))$, and, therefore, proceeding as in

(16) and as in the first inequality in (17), we get

$$||(c(x)-c(y))Ey|| \leq \frac{2}{p}\cdot\varepsilon\frac{1}{\tau}(||x||-R_{2i-1})\cdot(1+\varepsilon)||y||.$$

for some number $\tau \in (R_{2i-1}, ||x||)$. Hence,

$$||(c(x)-c(y))Ey|| \leq 2\varepsilon(1+\varepsilon)||x-y||$$

in this case, too. Likewise, one can check that (18) holds as well. The other subcases of

$$R_{2i-2} \leq ||y|| \leq ||x|| \leq R_{2i+1}$$

can be treated in the same manner.

2.2. Spaces $\left(\oplus_{n=1}^{\infty} X_n\right)_p, p>2$. The maps used in the case $1 \leq p \leq 2$ are not suitable for $p>2$ because the power of cosine in (16) becomes negative and a nontrivial estimate does not come out in this way. To get around this problem, functions c_{2i-1} and s_{2i-1}, $i \in \mathbb{N}$ will be chosen differently.

We start by introducing the functions $f_p : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ and $g_p : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ by

$$f_p(t) = \frac{\cos t}{(\cos^p t + \sin^p t)^{\frac{1}{p}}}, \qquad g_p(t) = \frac{\sin t}{(\cos^p t + \sin^p t)^{\frac{1}{p}}}. \tag{20}$$

It is clear that

$$(f_p(t))^p + (g_p(t))^p = 1. \tag{21}$$

Now, define c_{2i-1} and s_{2i-1}, $i \in \mathbb{N}$, as follows:

$$c_{2i-1}(x) = \begin{cases} f_p(\varepsilon \ln(R_{2i-1}/R_{2i-1})) = 1 & \text{if } ||x|| \leq R_{2i-1} \\ f_p(\varepsilon \ln(||x||/R_{2i-1})) & \text{if } R_{2i-1} \leq ||x|| \leq R_{2i} \\ f_p(\varepsilon \ln(R_{2i}/R_{2i-1})) = 0 & \text{if } ||x|| \geq R_{2i} \end{cases} \tag{22}$$

$$s_{2i-1}(x) = \begin{cases} g_p(\varepsilon \ln(R_{2i-1}/R_{2i-1})) = 0 & \text{if } ||x|| \leq R_{2i-1} \\ g_p(\varepsilon \ln(||x||/R_{2i-1})) & \text{if } R_{2i-1} \leq ||x|| \leq R_{2i} \\ g_p(\varepsilon \ln(R_{2i}/R_{2i-1})) = 1 & \text{if } ||x|| \geq R_{2i}. \end{cases} \tag{23}$$

The equalities in the last lines of formulae (22) and (23) can be derived from (4). Similar to the construction of the previous section, let us introduce the map $T : M \to X$ by

$$Tx = \begin{cases} c_1(x)E_2x + s_1(x)E_4x & \text{if } x \in M_3 \\ c_3(x)E_4x + s_3(x)E_6x & \text{if } x \in M_5 \backslash M_3 \\ \cdots & \cdots \\ c_{2i-1}(x)E_{2i}x + s_{2i-1}(x)E_{2i+2}x & \text{if } x \in M_{2i+1} \backslash M_{2i-1} \\ \cdots & \cdots \end{cases} \tag{24}$$

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In this equation R_i, E_i, and M_i have the same meaning as in our argument for $1 \le p \le 2$. The equation (21) implies that $(c_{2i-1}(x))^p + (s_{2i-1}(x))^p = 1$ for all i and x. Therefore,

$$\forall x \in M \quad ||x|| \le ||Tx|| \le (1+\varepsilon)||x||. \tag{25}$$

If $||y|| \le \varepsilon||x||$, the desired estimate (10) can be proved in exactly the same way as in the case $1 \le p \le 2$. For the same reason as in the case $1 \le p \le 2$, it suffices to consider the case where $R_{2i-1} \le ||y|| \le ||x|| \le R_{2i}$. For simplicity of notation in what follows, we use c for c_{2i-1}, s for s_{2i-1}, E for E_{2i}, and F for E_{2i+2}. Having said so, we write

$$\begin{aligned} ||Tx - Ty||^p &= ||c(x)Ex - c(y)Ey||^p + ||s(x)Fx - s(y)Fy||^p \\ &= ||c(x)(Ex - Ey) + (c(x) - c(y))Ey||^p \\ &\quad + ||s(x)(Fx - Fy) + (s(x) - s(y))Fy||^p. \end{aligned} \tag{26}$$

Examine each of the summands in the last line separately. Notice that $c(x) - c(y) = F(||x||) - F(||y||)$, where

$$\begin{aligned} F(r) &= \frac{G(r)}{B(r)}, \\ G(r) &= \cos(\varepsilon \ln(r/R_{2i-1})) \\ B(r) &= (\cos^p(\varepsilon \ln(r/R_{2i-1})) + \sin^p(\varepsilon \ln(r/R_{2i-1})))^{1/p} \end{aligned}$$

By the Mean Value Theorem,

$$F(||x||) - F(||y||) = \frac{G'(\tau)B(\tau) - G(\tau)B'(\tau)}{(B(\tau))^2}(||x|| - ||y||). \tag{27}$$

for some $\tau \in (||y||, ||x||)$. Obviously (recall that $p > 2$),

$$2^{-\frac{p}{2}+1} \le \cos^p t + \sin^p t \le 1,$$

and hence

$$2^{-\frac{1}{2}+\frac{1}{p}} \le B(\tau) \le 1.$$

In addition,

$$G'(\tau) = -\sin(\varepsilon \ln(\tau/R_{2i-1}))\varepsilon\frac{1}{\tau},$$

whence

$$|G'(\tau)| \le \frac{\varepsilon}{\tau}.$$

By plain calculations,

$$\begin{aligned} B'(\tau) = \frac{1}{p}(B(\tau))^{1-p}\Big(&p\cos^{p-1}(\varepsilon \ln(\tau/R_{2i-1})) \cdot (-\sin(\varepsilon \ln(\tau/R_{2i-1}))) \cdot \frac{\varepsilon}{\tau} \\ &+ p\sin^{p-1}(\varepsilon \ln(\tau/R_{2i-1})) \cdot \cos(\varepsilon \ln(\tau/R_{2i-1})) \cdot \frac{\varepsilon}{\tau}\Big), \end{aligned}$$

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which implies

$$|B'(\tau)| \leq \left(2^{-\frac{1}{2}+\frac{1}{p}}\right)^{1-p} \left(\frac{\varepsilon}{\tau} + \frac{\varepsilon}{\tau}\right).$$

Using the obvious bound $|G(\tau)| \leq 1$, one arrives at

$$\left|\frac{G'(\tau)B(\tau) - G(\tau)B'(\tau)}{(B(\tau))^2}\right| \leq \frac{\frac{\varepsilon}{\tau} + 2^{\frac{(p-1)(p-2)}{2p}} \cdot 2\frac{\varepsilon}{\tau}}{2^{2(\frac{1}{p}-\frac{1}{2})}} = C(p)\frac{\varepsilon}{\tau},$$

where $C(p)$ is some constant depending on p only. Since $\tau \in (||y||, ||x||)$, it can be established that

$$||(c(x) - c(y))Ey|| \leq C(p)\frac{\varepsilon}{\tau}(||x|| - ||y||) \cdot (1+\varepsilon)||y|| \leq \varepsilon(1+\varepsilon)C(p)||x - y||.$$

Likewise, it can be shown that

$$||(s(x) - s(y))Ey|| \leq \varepsilon(1+\varepsilon)C(p)||x - y||.$$

Combining the preceding inequalities with (26), one concludes that the next estimate is valid.

$$\begin{aligned}((\max\{c(x) - \varepsilon(1+\varepsilon)C(p), 0\})^p \\ + (\max\{s(x) - \varepsilon(1+\varepsilon)C(p), 0\})^p)||x - y||^p \\ \leq ||Tx - Ty||^p \\ \leq (1+\varepsilon)^p((c(x) + \varepsilon C(p))^p + (s(x) + \varepsilon C(p))^p)||x - y||^p.\end{aligned} \tag{28}$$

Clearly, (21) implies that $c^p(x) + s^p(x) = 1$, whence

$$\lim_{\varepsilon\downarrow 0}((\max\{c(x) - \varepsilon(1+\varepsilon)C(p), 0\})^p + (\max\{s(x) - \varepsilon(1+\varepsilon)C(p), 0\})^p) = 1$$

and

$$\lim_{\varepsilon\downarrow 0}(1+\varepsilon)^p((c(x) + \varepsilon C(p))^p + (s(x) + \varepsilon C(p))^p) = 1.$$

Thus, the inequality (28) is of the desired type (10). □

3. Proof of theorem 1.12. *Proof.* By the well-known observation of Fréchet [**4**, p. 161] (see also [**14**, Proposition 1.17]), all finite metric spaces admit isometric embeddings into $X = \left(\oplus_{n=1}^{\infty}\ell_{\infty}^{n}\right)_p$. Therefore, to prove Theorem 1.12, a construction of a locally finite metric space A which is not isometric to a subset of X (for $1 < p < \infty$) is needed.

The following notation for X will be employed. Each element $x \in X$ is a sequence $x = \{x_n\}_{n=1}^{\infty}$, where $x_n \in \ell_{\infty}^{n}$. The norm of x in X will be denoted by $||x||_X$. By the definition of direct sums one has

$$||x||_X = \left(\sum_{n=1}^{\infty} ||x_n||_{\infty}^{p}\right)^{\frac{1}{p}}, \tag{29}$$

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where $||x_n||_\infty$ is the norm in ℓ_∞^n (with slight abuse of notation, we use the same notation for all n). Denoting the norm of ℓ_p by $||\cdot||_p$, the right-hand side of (29) can be written as $||\{||x_n||_\infty\}_{n=1}^\infty||_p$.

At this stage, some simple geometric properties of X are needed. Consider triples of points $x, y, z \in X$ satisfying

$$||x - z||_X = ||x - y||_X + ||y - z||_X. \tag{30}$$

Let $x = \{x_n\}$, $y = \{y_n\}$, $z = \{z_n\}$, where $x_n, y_n, z_n \in \ell_\infty^n$ are the components of x, y, and z, respectively.

LEMMA 3.1. *For any triple $x, y, z \in X$ of pairwise distinct vectors satisfying* (30)*, the vector $\{||x_n - y_n||_\infty\}_{n=1}^\infty \in \ell_p$ is a positive multiple of $\{||y_n - z_n||_\infty\}_{n=1}^\infty \in \ell_p$.*

Proof. Assume the contrary. Recall that $1 < p < \infty$. Using the fact that for $u, v \in \ell_p$, the inequality $||u + v||_p \le ||u||_p + ||v||_p$ is strict if u and v are nonzero and are not positive multiples of each other, one derives that the ℓ_p-norm of the vector $\{||x_n - y_n||_\infty + ||y_n - z_n||_\infty\}_{n=1}^\infty$ is strictly less than

$$\left\|\{||x_n - y_n||_\infty\}_{n=1}^\infty\right\|_p + ||\{||y_n - z_n||_\infty\}_{n=1}^\infty||_p = ||x - y||_X + ||y - z||_X.$$

On the other hand, by the triangle inequality in ℓ_∞^n,

$$\left\|\{||x_n - y_n||_\infty + ||y_n - z_n||_\infty\}_{n=1}^\infty\right\|_p \ge \left\|\{||x_n - z_n||_\infty\}_{n=1}^\infty\right\|_p = ||x - z||_X.$$

This contradicts (30). □

The next definition will be used in the sequel.

DEFINITION 3.2. A *metric ray* in a metric space (A, d_A) is a sequence $r = \{r_i\}_{i=0}^\infty$ of points such that the sequence $d_A(r_i, r_0)$ is strictly increasing and, for $i < j < k$, the following equality holds:

$$d_A(r_i, r_k) = d_A(r_i, r_j) + d_A(r_j, r_k). \tag{31}$$

For all of the metric rays in Banach spaces considered in this paper, it will be assumed that

$$r_0 = 0. \tag{32}$$

Consider subspaces $X_k = \left(\oplus_{n=1}^k \ell_\infty^n\right)_p$ in X and the natural projections $P_k : X \to X_k$ defined by $P(\{x_n\}_{n=1}^\infty) = \{x_n\}_{n=1}^k$.

LEMMA 3.3. *For each metric ray $r = \{r_i\}_{i=0}^\infty$ in X and each $\varepsilon \in (0, 1)$, there is $k \in \mathbb{N}$ such that the natural projection $P_k : X \to X_k$ satisfies*

$$||P_k r_i - r_i||_X \le \varepsilon ||r_i||_X \quad for \;\; every \;\; i = 0, 1, \ldots \tag{33}$$

Under the assumption $r_0 = 0$, a number k satisfying this condition can be determined from the number $\varepsilon > 0$ and the vector r_1.

Proof. Let $r_i = \{r_{in}\}_{n=1}^\infty$, where $r_{in} \in \ell_\infty^n$. With the help of Definition 3.2 and Lemma 3.1, one derives that for $i < j < k$, the vector $\{||r_{jn} - r_{in}||_\infty\}_{n=1}^\infty \in \ell_p$ is a positive multiple

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of $\{||r_{kn} - r_{jn}||_\infty\}_{n=1}^\infty$. Using the fact that $r_{0n} = 0$ for every n, it can be easily obtained that any vector of the form $\{||r_{jn} - r_{in}||_\infty\}_{n=1}^\infty$ is a positive multiple of $\{||r_{1n}||_\infty\}_{n=1}^\infty$, and any vector of the form $\{||r_{in}||_\infty\}_{n=1}^\infty$ is also a positive multiple of $\{||r_{1n}||_\infty\}_{n=1}^\infty$. Now, pick $k \in \mathbb{N}$ such that $||P_k r_1 - r_1||_X \leq \varepsilon ||r_1||_X$. This means that $||\{||r_{1n}||_\infty\}_{n=k+1}^\infty||_p \leq \varepsilon ||\{||r_{1n}||_\infty\}_{n=1}^\infty||_p$. The fact that $\{||r_{in}||_\infty\}_{n=1}^\infty$ is a positive multiple of $\{||r_{1n}||_\infty\}_{n=1}^\infty$ leads to $||\{||r_{in}||_\infty\}_{n=k+1}^\infty||_p \leq \varepsilon ||\{||r_{in}||_\infty\}_{n=1}^\infty||_p$, or $||P_k r_i - r_i||_X \leq \varepsilon ||r_i||_X$, as required. □

In order to complete the proof of Theorem 1.12, we introduce a locally finite metric space A which does not admit an isometric embedding into X.

To begin with, let $\{N_t\}_{t=1}^\infty$ be an increasing sequence of positive integers so that $\lim_{t\to\infty} N_t = \infty$. Consider the set $S \subset \ell_\infty$ consisting of all sequences, for which the first coordinate is a nonnegative integer, the next N_1 coordinates are nonnegative integer multiples of 3, the next N_2 coordinates are nonnegative integer multiples of 3^2, the next N_3 coordinates are nonnegative integer multiples of 3^3, and so on. Clearly, S is countable. In addition, it is not difficult to see that S is locally finite implying that all of its subsets are also locally finite.

Further, let $\{I_t\}_{t=0}^\infty$ be a partition of $\mathbb{N}$, where $I_0 = \{1\}$, $I_1 = \{2, \ldots, 1 + N_1\}$, and $I_t = \{1 + N_1 + \cdots + N_{t-1} + 1, \ldots, 1 + N_1 + \cdots + N_{t-1} + N_t\}$ for $t \geq 2$. The definition of S can be rewritten as: a sequence $\{s_i\}_{i=1}^\infty \in \ell_\infty$ is in S if and only if each s_i is a nonnegative integer multiple of 3^t for $i \in I_t$.

Finally, a subset $A \subset S$ is taken to be the union of metric rays $r(j)$, $j \in \mathbb{N}$, constructed as described below. For each $j \in \mathbb{N}$ pick $n_1(j) \in I_1$, $n_2(j) \in I_2$, etc. This can and will be performed in such a way that the next condition is satisfied

$$\forall t \in \mathbb{N} \quad \forall n \in I_t \quad \exists j \in \mathbb{N} \quad n = n_t(j). \tag{34}$$

After this, the collection $\{r(j)\}_{j=1}^\infty$ of metric rays, where $r(j) = \{r_t(j)\}_{t=0}^\infty$, is defined as follows:

(A) $r_0(j) = 0 \in \ell_\infty$ (for every $j \in \mathbb{N}$).
(B) $r_1(j)$ is the unit vector $(1, 0, \ldots, 0, \ldots) \in \ell_\infty$ (for every $j \in \mathbb{N}$).
(C) For $t \geq 2$, let $r_t(j)$ be the vector which has $1 + 3 + \cdots + 3^{t-1}$ as its first coordinate, $3 + \cdots + 3^{t-1}$ as its $n_1(j)$ coordinate, ..., $3^{t-2} + 3^{t-1}$ as its $n_{t-2}(j)$ coordinate, 3^{t-1} as its $n_{t-1}(j)$ coordinate, while all the other coordinates are 0.

It can be noticed that each $r(j)$ is a metric ray and that, for every t and j, the vector $r_t(j)$ is in the set S described above.

The set A is locally finite, since it is a subset of S. Suppose that A admits an isometric embedding $E : A \to X$. Without loss of generality, assume that $E(0) = 0$ (recall that $0 \in A$). Clearly, isometries map metric rays onto metric rays. It will be proved by applying Lemma 3.3 in the case where $\varepsilon \in (0, 1)$ is sufficiently small, that the existence of such isometric embedding leads to a contradiction.

Namely, select $\varepsilon \in (0, 1)$ in such a way that

$$3^{t-1} - 2\varepsilon 3^t \geq 3^{t-2}, \tag{35}$$

for every $t \in \mathbb{R}$. Here, condition (35) is written in the form in which it will be used. Applying Lemma 3.3 to the ray $\{Er_t(j)\}_{t=0}^\infty$, we conclude that there is $k \in \mathbb{N}$ such that

$$||P_k Er_t(j) - Er_t(j)||_X \leq \varepsilon ||Er_t(j)||_X = \varepsilon ||r_t(j)||_\infty, \tag{36}$$

for every t, where the equality holds due to the fact that E is an isometry mapping 0 to 0. The last statement of Lemma 3.3 implies that k depends only on the vector $Er_1(j)$, and therefore does not depend on j (by condition (B)).

Set $m = \dim X_k$, where, as before, $X_k = P_k X$. It is common knowledge that there exists an absolute constant C such that, for any $\delta > 0$, the cardinality of a δ-separated set inside a ball of radius R in an m-dimensional Banach space does not exceed $(CR/\delta)^m$. See [**14**, Lemma 9.18].

Denote by B_t the ball of A of radius 3^t centered at 0. Then, $P_k EB_t$ is contained in the ball of radius 3^t of X_k. Hence, the mentioned fact on δ-separated sets implies that the cardinality of a 3^{t-2}-separated set in $P_k EB_t$ does not exceed $(9C)^m$. By showing that the construction of A implies that $P_k EB_t$ contains a 3^{t-2}-separated set of cardinality N_{t-1}, one obtains a contradiction, because $\{N_t\}_{t=1}^{\infty}$ is indefinitely increasing.

To achieve this goal, remark that for any $t \in \mathbb{N}$, the vector $r_t(j)$ is in B_t and even in the ball of radius $1 + 3 + 3^2 + \cdots + 3^{t-1}$. Combining conditions (34) and (C), it is concluded that the set of all vectors $\{r_t(j)\}_{j=1}^{\infty}$ contains a subset of cardinality N_{t-1} which is 3^{t-1}-separated.

Applying inequality (36) to any two images $Er_t(j_1)$ and $Er_t(j_2)$ of elements of this subset, what follows can be reached:

$$\begin{aligned} ||P_k Er_t(j_1) - P_k Er_t(j_2) - (Er_t(j_1) - Er_t(j_2))||_X \\ \leq ||P_k Er_t(j_1) - Er_t(j_1)||_X + ||P_k Er_t(j_2) - Er_t(j_2)||_X \\ \leq \varepsilon(||r_t(j_1)||_\infty + ||r_t(j_2)||_\infty), \end{aligned}$$

and, as a result,

$$\begin{aligned} ||P_k Er_t(j_1) - P_k Er_t(j_2)||_X \\ \geq ||Er_t(j_1) - Er_t(j_2)||_X - \varepsilon(||r_t(j_1)||_\infty + ||r_t(j_2)||_\infty) \\ \geq 3^{t-1} - 2\varepsilon 3^t \overset{(35)}{\geq} 3^{t-2}, \end{aligned}$$

which confirms that $P_k EB_t$ contains a 3^{t-2}-separated set of cardinality N_{t-1}. This proves the theorem. □

4. Proof of theorem 1.14. *Proof.* To prove Theorem 1.14 it suffices to show that, given an $\varepsilon > 0$, every locally finite metric space admits a bilipschitz embedding into X with distortion $\leq (4 + \varepsilon)$.

As in [**2**], we use the existence inside X of a subspace which is close to $\left(\oplus_{n=1}^{\infty} \ell_\infty^n\right)$, where the direct sum is not an ℓ_p-sum, but just a finite-dimensional decomposition with small decomposition constant. The existence of such a sum is derived from the Maurey–Pisier theorem [**9**] (see also [**14**, Theorems 2.55 and 2.56]) by the line of reasoning which goes back to Mazur, see [**6**, p. 4].

Since our argument is a modification of the one contained in [**6**], the needed details of the construction used there are presented below for the reader's convenience.

Definition 4.1. Let $\lambda \in (0, 1]$. A subspace $N \subset X^*$ is called *λ-norming over a subspace $Y \subset X$* if

$$\forall y \in Y \ \sup\{|f(y)| : f \in N, \ ||f|| \leq 1\} \geq \lambda ||y||.$$

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LEMMA 4.2. *For any* $\lambda \in (0, 1)$ *and any finite-dimensional subspace* $Y \subset X$ *there exists a finite-dimensional subspace* $N \subset X^*$ *which is* λ*-norming over* Y.

Proof. The existence of such a subspace can be established as follows. Let $\{x_i\}_{i=1}^m$ be an $(1-\lambda)$-net in the unit sphere of Y and let N be the linear span of functionals x_i^* satisfying the conditions $||x_i^*|| = 1$ and $x_i^*(x_i) = 1$. The verification that N is λ-norming is immediate. □

Let $\varepsilon \in (0, 1)$ and $\{\varepsilon_i\}_{i=1}^\infty$ be positive numbers satisfying

$$\prod_{i=1}^{\infty}(1-\varepsilon_i) > 1-\varepsilon. \tag{37}$$

Denote by (M, d_M) the locally finite metric space which will be embedded into X. Pick a point $O \in M$ and set

$$M_n = \{x \in M : \ d_M(x, O) \le R_n\},$$

where $\{R_n\}_{n=1}^\infty$ is the sequence defined in (3)–(5). Let $c(n)$ be the cardinality of M_n. As a consequence of Fréchet's observation, M_n admits an isometric embedding E_n into $\ell_\infty^{c(n)}$. Further, the Maurey–Pisier theorem states that the space X contains a subspace Y_1 such that there is a linear map $S_1 : Y_1 \to \ell_\infty^{c(1)}$ satisfying

$$||y|| \le ||S_1 y|| \le (1+\varepsilon)||y||.$$

Consider a finite-dimensional subspace $N_1 \subset X^*$ so that N_1 is $(1-\varepsilon_1)$-norming over Y_1 and set

$$W_1 = (N_1)_\top := \{x \in X : \ \forall x^* \in N_1 \ \ x^*(x) = 0\}.$$

It is easy to derive from the definition of cotype that W_1 has no nontrivial cotype. Applying the Maurey–Pisier theorem once more, one finds a subspace $Y_2 \subset W_1$ and a linear map $S_2 : Y_2 \to \ell_\infty^{c(2)}$ satisfying

$$||y|| \le ||S_2 y|| \le (1+\varepsilon)||y||.$$

Now, take $N_2 \subset X^*$ as a finite-dimensional subspace which contains N_1 and is $(1-\varepsilon_2)$-norming over $\mathrm{lin}(Y_1 \cup Y_2)$, and set $W_2 = (N_2)_\top$.

We continue in an obvious way. In the nth step, we find a subspace

$$Y_n \subset W_{n-1} = (N_{n-1})_\top,$$

and a linear map $S_n : Y_n \to \ell_\infty^{c(n)}$ satisfying

$$||y|| \le ||S_n y|| \le (1+\varepsilon)||y||.$$

It is clear that, for $u \in W_n$ and $v \in (N_n)_\top$, the inequality below is true

$$||u+v|| \ge (1-\varepsilon_n)||u||. \tag{38}$$

It is easy to see that $\{Y_i\}_{i=1}^\infty$ form a finite-dimensional decomposition of the closed linear span of $\bigcup_{i=1}^\infty Y_i =: Y$. Writing a sum of the form $\sum_{i=1}^\infty y_i$, we mean that $y_i \in Y_i$.

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We introduce the following norm on Y:

$$\left\|\sum_{i=1}^{\infty} y_i\right\|_a = \max\left\{\left\|\sum_{i=1}^{\infty} y_i\right\|_X, \quad \max\{||S_j y_j|| + ||S_k y_k|| : \ j,k \in \mathbb{N}\}\right\}. \tag{39}$$

Let us show that the norm $||\cdot||_a$ is $\dfrac{4(1+\varepsilon)}{1-\varepsilon}$-equivalent to $||\cdot||_X$. In fact, it is clear that

$$\left\|\sum_{i=1}^{\infty} y_i\right\|_X \leq \left\|\sum_{i=1}^{\infty} y_i\right\|_a.$$

On the other hand, inequality (38) yields

$$(1-\varepsilon_k)\left\|\sum_{i=1}^{k} y_i\right\|_X \leq \left\|\sum_{i=1}^{\infty} y_i\right\|_X$$

and

$$(1-\varepsilon_{k-1})\left\|\sum_{i=1}^{k-1} y_i\right\|_X \leq \left\|\sum_{i=1}^{\infty} y_i\right\|_X.$$

By the triangle inequality,

$$||y_k||_X \leq \left(\frac{1}{1-\varepsilon_k} + \frac{1}{1-\varepsilon_{k-1}}\right)\left\|\sum_{i=1}^{\infty} y_i\right\|_X.$$

The stated above equivalence of $\|\cdot\|_a$ and $\|\cdot\|_X$ now follows from $||S_k y_k|| \leq (1+\varepsilon)||y_k||$ and (37).

Observe that $\mathrm{lin}\{Y_j \cup Y_k\}$ with the norm $||\cdot||_a$ is isometric to $\ell_\infty^{c(j)} \oplus_1 \ell_\infty^{c(k)}$. Consider M as a subset of ℓ_∞ such that $O \in M$ coincides with $0 \in \ell_\infty$. This implies that the argument used to prove Theorem 1.9 in the case $p = 1$ can be applied to get an embedding of distortion $\leq (1+\varepsilon)$ of M into $(Y, ||\cdot||_a)$. Indeed, let us define an embedding $T : M \to Y$ by the formula (8) (we use $p = 1$ in (6) and (7)). Now we can see that if Tx and Ty are in the same sum of the form $\ell_\infty^{c(j)} \oplus_1 \ell_\infty^{c(k)}$, the desired estimate can be obtained in the same way as in the final part of Section 2.1. On the other hand, if Tx and Ty are not both in the same direct sum of the form $\ell_\infty^{c(j)} \oplus_1 \ell_\infty^{c(k)}$, then $||y|| \leq \varepsilon||x||$. In this case the estimate also goes through in exactly the same way as in (11)–(13).

To summarize, an embedding of M into $(Y, ||\cdot||_a)$ with distortion $\leq (1+\varepsilon)$ exists. Combining this fact with the established above equivalence between $||\cdot||_X$ and $||\cdot||_a$ on Y, one obtains an embedding into X with distortion $\leq \frac{4(1+\varepsilon)^2}{1-\varepsilon}$. With $\varepsilon \downarrow 0$, the result stated in Theorem 1.14 is proved. □

5. An open problem. In our opinion the most interesting open problem related to this study is:

PROBLEM 5.1. Do there exist Banach spaces X with $D(X) > 1^+$?

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