Canad. Math. Bull.Vol. **36** (1), 1993 pp. 87–95

WEIGHTED RESTRICTION FOR CURVES

JOSEPH D. LAKEY

ABSTRACT. We prove weighted norm inequalities for the Fourier transform of the form

$$\forall f \in \mathcal{S}(\mathbb{R}^d), \quad \Big(\int_{-\delta}^{\delta} |\hat{f}(\psi(t))|^q \, dt\Big)^{\frac{1}{q}} \leq C\Big(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, dx\Big)^{\frac{1}{p}},$$

where v is a nonnegative weight function on $\mathbb{R}^d$ and $\psi\colon [-1,1] \longrightarrow \mathbb{R}^d$ is a nondegenerate curve. Our results generalize unweighted (*i.e.* $v = 1$) restriction theorems of M. Christ, and two-dimensional weighted restriction theorems of C. Carton-Lebrun and H. Heinig.

1. **Introduction.** Restriction theorems for curves may be viewed as generalizations of Zygmund's two-dimensional spherical restriction theorem, which states that if $1 \leq p < 4/3$ and $1 \leq q \leq p'/3$, then

$$\forall f \in L^p(\mathbb{R}^2), \quad \Big(\int_{\Sigma_1} |\hat{f}(\theta)|^q \, d\sigma_1(\theta)\Big)^{\frac{1}{q}} \leq C\Big(\int_{\mathbb{R}^2} |f(x)|^p \, dx\Big)^{\frac{1}{p}}. \tag{1.1}$$

Here Σ_1 denotes the unit circle and σ_1 is arclength measure.

Extensions of Zygmund's result, where σ_1 is replaced by (affine) arclength measure on general plane curves, including degenerate curves, were proved by Sjölin [Sj]. More recently, Carton-Lebrun and Heinig have cleverly adapted Sjölin's techniques to obtain weighted restriction theorems for plane curves, where the L^p norm on the right hand side of (1.1) is replaced by a weighted L^p norm.

In another direction restriction to curves in d-dimensions has been investigated by Prestini [P1; P2] and Christ [Chr]. Prestini's main contribution was the reduction of the restriction problem (for nondegenerate curves) to estimating the (fractional) Vandermonde form $f \longmapsto \prod_{i=1}^d f(x_i) \prod_{1\leq i<j\leq d} |x_i - x_j|^{-\eta}$. Sharp "$L^p$ estimates" for the Vandermonde form were obtained by Christ [Chr], who also proved restriction theorems for certain degenerate curves with Euclidean arclength measure.

Our goal is to prove weighted restriction theorems of the form

$$\forall f \in \mathcal{S}(\mathbb{R}^d), \quad \Big(\int_{-\delta}^{\delta} \big|\hat{f}\big(\psi(t)\big)\big|^q \, dt\Big)^{\frac{1}{q}} \leq C\Big(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, dx\Big)^{\frac{1}{p}}, \tag{1.2}$$

where ψ is a nondegenerate curve in $\mathbb{R}^d$ and v is a nonnegative weight function. We prove weighted extensions of Christ's d-dimensional restriction theorem with weights

This work comprises part of the author's Ph.D. dissertation. The author is grateful for support received through the University of Maryland Department of Mathematics Dissertation Fellowship.

Received by the editors October 10, 1991 .

AMS subject classification: 42B10.

Key words and phrases: Fourier transform, restriction theorems, weighted norm inequalities.

https://doi.org/10.4153/CMB-1993-013-5 Published online by Cambridge University Press

satisfying conditions similar to those of Carton-Lebrun and Heinig. We also establish weighted restriction theorems where weights lie in certain Wiener amalgam spaces, and are especially adapted to restriction for compact curves. In particular we prove a new weighted extension of Zygmund's theorem. Our theorems are only proven for nondegenerate curves, although our techniques also work for curves of finite type as considered in [Chr]. Our "reference measure" is Euclidean arclength measure, as opposed to affine arclength measure. Our techniques are therefore not easily adaptable for proving weighted extensions of the deep work of Drury and Marshall concerning restriction to degenerate curves in higher dimensions.

The author wishes to acknowledge his indebtedness to his thesis advisor, Professor John Benedetto, for his boundless patience and encouragement.

2. **Notation and background.** $\mathbb{R}^d$ denotes the Euclidean space of d-tuples $(x_1, \ldots, x_d)$ of real numbers. For a nonnegative measurable function v on $\mathbb{R}^d$, and $1 \leq p < \infty$, we consider the weighted L^p spaces

$$L^p_v = \left\{ f : \|f\|^p_{L^p_v} = \int_{\mathbb{R}^d} |f(x)|^p v(x)\, dx < \infty \right\}.$$

Given $f \in L^1(\mathbb{R}^d)$, the Fourier transform of f is

$$\hat{f}(y) = \int_{\mathbb{R}^d} e^{-2\pi i \langle x,y \rangle} f(x)\, dx,$$

where $\langle x, y \rangle = \sum_{j=1}^d x_j y_j$.

A (compact) curve in $\mathbb{R}^d$ is a continuous map $\psi\colon [-1,1] \longrightarrow \mathbb{R}^d$. For our purposes, ψ will be assumed to be smooth (or at the very least C^d where d is the dimension). We say that ψ is *nondegenerate* at $t \in (-1,1)$ if the vectors $\psi'(t), \psi''(t), \ldots, \psi^{(d)}(t)$ are linearly independent. We need a few additional definitions and lemmas for what follows.

In Section 4 we shall study restriction with weights satisfying certain local and global integrability conditions. Given exponents $1 < p, q < \infty$, we define the *Wiener amalgam spaces*,

$$W(L^p, l^q) = \left\{ f : \|f\|_{W(L^p,l^q)} = \left(\sum_{n \in \mathbb{Z}^d} \left(\int_{Q_n} |f(x)|^p\, dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}.$$

Here $\mathbb{Z}^d$ is the integer lattice $(n_1, \ldots, n_d)$ where $n_j \in \mathbb{Z}, j = 1, \ldots, d$, and $Q_n = n + [0,1)^d$. Similarly, we define

$$W(L^p, l^\infty) = \left\{ f : \|f\|_{W(L^p,l^\infty)} = \sup_{n \in \mathbb{Z}^d} \|f\chi_{Q_n}\|_{L^p} < \infty \right\}$$

and

$$W(L^\infty, l^q) = \left\{ f : \|f\|_{W(L^\infty,l^q)} = \left\| \{ \|f\chi_{Q_n}\|_{L^\infty} \} \right\|_{l^q} < \infty \right\}.$$

We shall use the following Hausdorff-Young theorem for Wiener amalgam spaces, see, *e.g.*, [FS]

THEOREM 2.1. *Whenever* $1 \le p,\ q \le 2$ *one has*

$$\forall f \in W(L^p, l^q), \quad \|\hat{f}\|_{W(L^{q'}, l^{p'})} \le C_{p,q} \|f\|_{W(L^p, l^q)}, \tag{2.1}$$

with constant $C_{p,q}$ *independent of* f.

The following lemma concerns rearrangement weighted norm inequalities for the Fourier transform, *e.g.*, [BH; JS; Mu].

LEMMA 2.2. *Given nonnegative functions* u, w *on* $\mathbb{R}^d$ *and exponents* $1 < \beta \le \alpha < \infty$ *satisfying*

$$\sup_{s>0} \Big(\int_0^s u^*(t)\, dt \Big)^{\frac{1}{\alpha}} \Big(\int_0^{\frac{1}{s}} (w^{-1})^*(t)^{\beta'-1}\, dt \Big)^{\frac{1}{\beta'}} < \infty. \tag{2.2}$$

One has

$$\forall F \in L^1 \cap L^p_w(\mathbb{R}^d), \quad \Big(\int |\hat{F}(y)|^\alpha u(y)\, dy \Big)^{\frac{1}{\alpha}} \le C \Big(\int_{\mathbb{R}^d} |F(x)|^\beta w(x)\, dx \Big)^{\frac{1}{\beta}}. \tag{2.3}$$

Here u^* denotes the decreasing rearrangement of u (see, *e.g.*, [StW, Chapter V]).

The following two lemmas are due to M. Christ.

LEMMA 2.3 ([CHR; LEMMA 2.1]). *Let* $\psi\colon [-1,1] \longrightarrow \mathbb{R}^d$ *be nondegenerate at* $t = 0$. *Then there is a* $\delta > 0$ *and* $C > 0$ *such that in the region* $E = \{x : 0 < x_1 < x_d < \cdots < x_d < \delta\}$, *we have the Jacobian estimate*

$$\left| \frac{\partial y}{\partial x} \right| \ge C \prod_{1 \le i < j \le d} (x_j - x_i) \tag{2.4}$$

where $y = \sum_{j=1}^d \psi(x_i)$ *is a nonsingular change of variables on* E.

LEMMA 2.4 [CHR; PROPOSITION 2.2]. *For* $0 \le \eta$, *one has*

$$\forall f \ge 0, \quad \int \prod_{j=1}^d f(x_j) \prod_{i<j} |x_i - x_j|^{-\eta}\, dx_1 \cdots dx_d \le C \|f\|^d_{L^p(\mathbb{R})} \tag{2.5}$$

if and only if

$$\eta < \frac{2}{d}, \text{ and } \frac{1}{p} + \frac{\eta(d-1)}{2} = 1. \tag{2.6}$$

3. **Restriction with weights.** Our first result is

THEOREM 3.1. *Given a smooth curve $\psi\colon[-1,1]\longrightarrow\mathbb{R}^d$ for which ψ is non-degenerate at $t=0$. For some $\delta>0$ fixed, and $f\in C^\infty(-\delta,\delta)$, set*

$$Af(\gamma)=\int_{-\delta}^{\delta}\exp\bigl(-2\pi i\langle\psi(t),\gamma\rangle\bigr)f(t)\,dt. \tag{3.1}$$

Suppose that

$$\frac{d+2}{d+1}<q<\infty,\quad 1<p\leq\frac{d(d+1)q}{(d+1)(d-1)q+2}, \tag{3.2}$$

and $v\in L^1_{\mathrm{loc}},+(\mathbb{R}^d)$ satisfies

$$\sup_{s\geq c}\Bigl[s^{-\frac{2p'}{d(d+1)q}}\int_0^s(v^{1-p'})^*(t)\,dt\Bigr]<\infty \tag{3.3}$$

for some $c>0$. Then if δ is sufficiently small and $C>0$ is large enough one has

$$\forall f\in C^\infty(-\delta,\delta),\quad \|v^{-\frac{1}{p}}Af\|_{L^{p'}(\mathbb{R}^d)}\leq C\|f\|_{L^{q'}(-\delta,\delta)}. \tag{3.4}$$

PROOF. First we wish to obtain an expression involving Af which looks like a Fourier transform. Since ψ is nondegenerate at $t=0$, we may choose $\delta>0$ such that after a nonsingular change of coordinates (on $\mathbb{R}^d$ and $[-\delta,\delta]$), ψ has the form $\psi(t)=\bigl(t,\psi_2(t),\ldots,\psi_d(t)\bigr)$ where $\psi_j(t)=t^j\bigl(1+O(t)\bigr)$ as $t\longrightarrow 0$, *cf.*, [Chr]. For simplicity (that is, in order to apply Lemma 2.4) we shall prove the result for $f\chi_{[0,\delta)}$, since the estimate for $f\chi_{(-\delta,0]}$ is essentially the same.

Now we decompose $[0,\delta)^d$. For $\sigma\in S_d$ (the symmetric group on d letters), let $E_\sigma=\{x\in\mathbb{R}^d:0\leq x_{\sigma(1)}<\cdots<x_{\sigma(d)}<\delta\}$. Then $[0,\delta)^d=\cup_{\sigma\in S_d}E_\sigma$ almost everywhere. Furthermore, the E_σ's are pairwise disjoint, since for $x\in[0,\delta)^d$ satisfying $x_i\neq x_j$ for all $i\neq j$, arranging the x_i's in increasing order determines a unique element $\tau\in S_d$.

Notice that for $\sigma\in S_d$ fixed,

$$\bigl(Af(\gamma)\bigr)^d=d!\int_{E_\sigma}\exp\Bigl(-2\pi i\langle\gamma,\sum_{i=1}^d\psi(x_i)\rangle\Bigr)\prod_{i=1}^d f(x_i)\,dx_1\cdots dx_d. \tag{3.5}$$

If δ is small enough (depending on ψ), then by Lemma 2.4 the change of variables $y_j=\sum_{i=1}^d\psi_j(x_i)$ is admissible on E_σ. Thus we may rewrite

$$\bigl(Af(\gamma)\bigr)^d=d!\Bigl[\prod_{i=1}^d f(x_i)\chi_D\Bigl|\frac{\partial x}{\partial y}\Bigr|\Bigr]^\wedge(\gamma) \tag{3.6}$$

where D is the image of $E=E_{id}$ under the change of variables $x\longrightarrow y$ above (here integration is with respect to the y variable). Since $y=\bigl(\sum\psi_1(x_i),\ldots,\sum\psi_d(x_i)\bigr)$ we have $|y|\leq M=d^{3/2}\cdot\sup_{t\in[0,\delta)}|\psi(t)|$. We now argue as in [C-L,H] in order to apply Lemma 2.2, where $\hat{F}(\gamma)=\bigl(Af(\gamma)\bigr)^d/d!$, $u=v^{1-p'}$, $\alpha=p'/d$, and $w(y)=w(|y|)=\bigl(\chi_{[0,M]}(|y|)\bigr)^{-1}$.

Let $\Omega_d(M)$ be the volume of a ball of radius M in $\mathbb{R}^d$. Since $w^{-1}(y) = \chi_{[0,M]}(|y|)$, one has

$$(w^{-1})^*(t) \equiv \inf\{s > 0 : \left|\{y \in \mathbb{R}^d : |w^{-1}| > s\}\right| \le t\} = \begin{cases} 0, & t > \Omega_d(M) \\ 1, & 0 < t \le \Omega_d(M) \end{cases}.$$

Thus if we take $c = \left(\Omega_d(M)\right)^{-1}$, condition (2.2) of Lemma 2.2 is equivalent to

$$\sup_{s>c}\Big(s^{-\frac{p'}{d\beta'}} \int_0^s (v^{1-p'})^*(t)\,dt\Big) < \infty. \tag{3.7}$$

Suppose for now that (3.7) holds. By Lemma 2.2 one then gets the Fourier transform norm inequality

$$\Big(\int |\hat{F}(\gamma)|^{\frac{p'}{d}} v^{1-p'}(\gamma)\,d\gamma\Big)^{\frac{d}{p'}} \le C\Big(\int |F(y)|^\beta w(y)\,dy\Big)^{\frac{1}{\beta}}, \tag{3.8}$$

provided $\beta \le p'/d$. "Unchanging" variables $y \longmapsto x$ and applying Lemma 2.3 we get

$$\begin{aligned} \|v^{-\frac{1}{p}} Af\|_{L^{p'}}^d &\le C\Big(\int_{E_\sigma} \Big|\prod_{i=1}^d f(x_i)\Big|^\beta \Big|\frac{\partial y}{\partial x}\Big|^{1-\beta} dx_1 \cdots dx_d\Big)^{\frac{1}{\beta}} \\ &\le C\Big(\int_{E_{id}} g(x_i) \prod_{i<j} (x_j - x_i)^{1-\beta}\,dx_1 \cdots dx_d\Big)^{\frac{1}{\beta}}, \end{aligned} \tag{3.9}$$

where $|f(x_i)|^\beta = g(x_i)$.

Finally applying Lemma 2.4 gives

$$\|v^{-\frac{1}{p}} Af\|_{L^{p'}}^d \le C\|g\|_{L^r}^{\frac{d}{\beta}} = C\|f\|_{L^{\beta r}}^d \equiv C\|f\|_{L^{q'}}^d, \tag{3.10}$$

provided

$$\begin{aligned} &(\beta - 1) < \frac{2}{d}, \text{ and} \\ &\frac{1}{r} + \frac{(\beta - 1)(d-1)}{2} = 1. \end{aligned} \tag{3.11}$$

Here we have defined $q' = \beta r$. It remains to show that (3.2) and (3.3) imply (3.11), (3.7), and the condition $\beta \le p'/d$ used in obtaining (3.8).

To say that (3.2) implies (3.11), what we mean, precisely, is that if β and r are defined in terms of each other as in (3.11), then $\beta - 1 < \frac{2}{d}$. The relationships between q, β, and r yield $q' = \beta r = 2\beta/\big(2 - (\beta - 1)(d-1)\big)$. This is equivalent to $q = 2\beta'/(d+1)$, or

$$\beta = \Big(q\Big(\frac{d+1}{2}\Big)\Big)' = \frac{q(d+1)}{q(d+1) - 2}.$$

Now the restriction on q in (3.2) implies $\beta < (d+2)/d$, which gives (3.11). Next we show that $\beta \le p'/d$, or $p \le (\beta d)'$. By (3.2),

$$\begin{aligned} 1 < p &\le \frac{d(d+1)q}{\big((d+1)(d-1)q + 2\big)} = \frac{d(d+1)q}{d(d+1)q - \big((d+1)q - 2\big)} \\ &= \frac{d(\frac{d+1}{2}q)'}{d(\frac{d+1}{2}q)' - 1} = \Big(d\Big(\frac{d+1}{2}q\Big)'\Big)' = (d\beta)'. \end{aligned}$$

Finally, since $q = 2\beta'/(d+1)$, the supremum in (3.3) is essentially the same as the one in (3.7). This completes the proof. ∎

COROLLARY 3.2. *Under the hypotheses of Theorem 3.1,*

$$\forall g \in \mathcal{S}(\hat{\mathbb{R}}^d), \quad \left(\int_{-\delta}^{\delta} \left|\hat{g}\big(\psi(t)\big)\right|^q dt\right)^{\frac{1}{q}} \leq C\|g\|_{L^p_v}. \tag{3.12}$$

PROOF. We use a simple duality argument. First take, say, $g \in \mathcal{S}(\mathbb{R}^d)$ and $f \in C(-\delta,\delta)$. Then by Fubini, Hölder, and the theorem, one has

$$\begin{aligned}\int_{-\delta}^{\delta} f(t)\big(\hat{g}\circ\psi(t)\big)\, dt &= \int_{\mathbb{R}^d} g(\gamma)Af(\gamma)\, d\gamma \\ &\leq \|g\|_{L^p_v}\|(Af)v^{-\frac{1}{p}}\|_{L^{p'}} \leq C\|g\|_{L^p_v}\|f\|_{L^{q'}(-\delta,\delta)}.\end{aligned}$$

Dividing by $\|f\|_{L^{q'}(-\delta,\delta)}$ and taking the supremum over f gives the result. ∎

REMARK 3.3. Checking the exponents in the theorem shows that we are reduced to the nondegenerate case of the theorem of Carton-Lebrun and Heinig when $d = 2$ and to the nondegenerate case of Christ's theorem when $v = 1$.

4. **Weights in Wiener amalgam spaces.** In this section we present a result similar to Corollary 3.2. Our theorem involves a different class of weights, although the techniques are more or less the same. The weight condition is membership in a Wiener amalgam space, and in particular, the condition does not involve rearrangements. The conditions on the exponents arise from applications of the Hausdorff-Young theorem for Wiener amalgam spaces, along with Lemmas 2.3 and 2.4.

THEOREM 4.1. *Let $\psi\colon [-1,1] \longrightarrow \mathbb{R}^d$ be a C^∞ curve which is nondegenerate at $t = 0$. Let $v \in L^1_{\mathrm{loc}},{+}(\mathbb{R}^d)$ and let p, q, r, s, α, β be exponents satisfying the following:*

$$v^{1-p'} \in W(L^{r'}, l^{s'}), \quad 1 \leq r,\ s \leq \infty, \tag{4.1}$$

$$\left(\frac{p'}{d}\right) r \geq 2 \text{ and } \left(\frac{p'}{d}\right) s \geq 2, \tag{4.2}$$

$$\beta \geq \left(\left(\frac{p'}{d}\right)s\right)', \tag{4.3}$$

$$\begin{cases} 0 \leq \beta - 1 < \frac{2}{d}, \\ \frac{1}{\alpha} + \frac{(\beta-1)(d-1)}{2} = 1, \end{cases} \tag{4.4}$$

and

$$q' = \beta\alpha. \tag{4.5}$$

Then for $C > 0$ large enough and $\delta > 0$ small enough,

$$\forall g \in \mathcal{S}(\hat{\mathbb{R}}^d), \quad \left(\int_{-\delta}^{\delta} \left|\hat{g}\big(\psi(y)\big)\right|^q\right)^{\frac{1}{q}} \leq C\|g\|_{L^p_v}. \tag{4.6}$$

PROOF. The theorem will follow in the same manner that Corollary 3.2 follows from Theorem 3.1 once we show that

$$\forall f \in C(-\delta,\delta), \quad \|v^{-\frac{1}{p}} Af\|_{L^{p'}(\mathbb{R}^d)} \le C\|f\|_{L^{q'}(-\delta,\delta)},$$

where Af is as in (3.1). As in the proof of Theorem 3.1 we establish the formula (3.6), then look for conditions which imply

$$\Big(\int_{\mathbb{R}^d} |\hat{F}(\gamma)|^{\frac{p'}{d}} v^{1-p'}(\gamma)\, d\gamma\Big)^{\frac{d}{p'}} \le C\Big(\int_{\mathbb{R}^d} |F(y)|^{\beta}\chi_D(y)\, dy\Big)^{\frac{1}{\beta}}. \tag{4.7}$$

Subsequent change of variables gives the result. To show how the Wiener amalgam space conditions give (4.7) we proceed as follows.

Let $V = v^{1-p'}$ and suppose that $V \in W(L^{r'}, l^{s'})$. We'll argue for the case where $1 < r, s < \infty$, and the other cases will follow by similar arguments. We have

$$\begin{aligned}
\Big(\int_{\mathbb{R}^d} |\hat{F}(\gamma)|^{\frac{p'}{d}} v^{1-p'}(\gamma)\, d\gamma\Big) &= \Big(\sum_{n\in\mathbb{Z}^d}\int_{Q_n} |\hat{F}(\gamma)|^{\frac{p'}{d}} v^{1-p'}(\gamma)\, d\gamma\Big)\\
&\le \sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n} |\hat{F}(\gamma)|^{\frac{p'r}{d}}\Big)^{\frac{1}{r}}\Big(\int_{Q_n} V^{r'}(\gamma)\, d\gamma\Big)^{\frac{1}{r'}}\\
&\le \Big(\sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n} |\hat{F}(\gamma)|^{\frac{p'r}{d}}\Big)^{\frac{s}{r}}\Big)^{\frac{1}{s}}\Big(\sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n} V^{r'}(\gamma)\, d\gamma\Big)^{\frac{s'}{r'}}\Big)^{\frac{1}{s'}}\\
&= \|V\|_{W(L^{r'},l^{s'})}\Big(\sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n} |\hat{F}(\gamma)|^{\frac{p'r}{d}}\Big)^{\frac{s}{r}}\Big)^{\frac{1}{s}}.
\end{aligned} \tag{4.8}$$

Now set $t = p'r/d$ and $u = p's/d$. That is, $s/r = u/t$ and $s = ud/p'$. Then the right hand side of (4.8) is

$$\|V\|_{W(L^{r'},l^{s'})}\Big(\sum_{n\in\mathbb{Z}^d}\Big(\int_{Q_n} |\hat{F}(\gamma)|^{t}\Big)^{\frac{u}{t}}\Big)^{\frac{1}{u}\frac{p'}{d}} = \|V\|_{W(L^{r'},l^{s'})}\|\hat{F}\|^{\frac{p'}{d}}_{W(L^{t},l^{u})}.$$

Applying Theorem 2.1, it follows that the left hand side of (4.7) is bounded by $\|V\|_{W(L^{r'},l^{s'})} \cdot \|F\|^{p'/d}_{W(L^{u'},l^{t'})}$, provided (4.2) holds. But

$$\begin{aligned}
\|F\|_{W(L^{u'},l^{t'})} &= \Big(\sum_n\Big(\int_{Q_n} |F|^{u'}\Big)^{\frac{t'}{u'}}\Big)^{\frac{1}{t'}}\\
&\le \Big(\sum_n\Big(\int_{Q_n} |F|^{\beta}\Big)^{\frac{t'}{\beta}}\Big)^{\frac{1}{t'}}\\
&\le \#\{n : Q_n \cap D \ne \emptyset\} \sup_n\Big(\int_{Q_n} |F|^{\beta}\Big)^{\frac{1}{\beta}}\\
&\le C_D\|F\|_{L^{\beta}}.
\end{aligned}$$

The first inequality follows from Hölder and (4.3), and the second follows from the fact that F is actually supported in D. Thus (4.7) holds as long as (4.1)–(4.3) hold.

The remainder of the argument is the same as in the proof of Theorem 3.1. That is, we unchange variables and apply Lemmas 2.4 and 2.5, which give (4.6) provided (4.4) and (4.5) hold (in Lemma 2.4, we take $\alpha = p, \beta - 1 = \eta$, and $q' = \beta\alpha$). This proves the theorem. ∎

It appears to be a rather gruesome task to check all of the conditions of Theorem 4.1. The following example gives a generalization of Zygmund's restriction theorem when $d = 2$.

EXAMPLE 4.2. Let $V = v^{1-p'} \in W(L^1, l^\infty)$ (the theorem still applies to such V). In view of (4.1), we have $r' = 1$ and $s' = \infty$. Now (4.4) implies $\alpha = 2/(3-\beta)$ where $1 \leq \beta < 2$. But for $\beta < 2$, condition (4.3) implies

$$2 > \beta \geq \left(\frac{p'}{2}\right)' = \frac{p}{2-p},$$

so that $1 \leq p < 4/3$. Notice then that (4.2) is automatically satisfied. In view of (4.5) the corresponding condition on q is

$$q' = \beta\alpha = \frac{2\beta}{3-\beta} \geq \frac{2(\frac{p}{2-p})}{3-(\frac{p}{2-p})} = \frac{2p}{3(2-p)-p} = \frac{p}{3-2p}.$$

That is, $q \leq \big(2p/(3-2p)\big)' = p'/3$. Thus we have

COROLLARY 4.3. *Let $1 \leq p < 4/3$ and $q \leq p'/3$, and $v \geq 0$ such that $v^{1-p'} \in W(L^1, l^\infty)$ be given. Then for $\delta > 0$ sufficiently small,*

$$\forall g \in \mathcal{S}(\mathbb{R}^2), \quad \left(\int_{-\delta}^{\delta} |\hat{g}(\cos t, \sin t)|^q \, dt\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^2} |g(x)|^p v(x) \, dx\right)^{\frac{1}{p}}.$$

REMARK 4.4. This corollary shows that existence of restrictions of elements of weighted L^p to the unit circle depends only in a weak manner on the local behavior of the weight. In fact, this result extends the known class of functions whose Fourier transforms have well-defined restrictions to the unit circle. For example, consider the weight $v(x_1, x_2) = |(\sin x_1)|^\alpha$, and the function $f(x) = \sum_{n=1}^{\infty} |x_1 - n|^{-\frac{1}{\gamma}} \chi_{[n, n+n^{-\gamma}]}(x_1) \cdot \chi_{[-1,1]}(x_2)$. For appropriate choices of α and γ, one can get $f \notin L^p(\mathbb{R}^2)$ for any $1 \leq p < 4/3$ but $v^{1-p'} \in W(L^1, l^\infty)$ and $f \in L^p_v(\mathbb{R}^2)$ for some such p, so that the Fourier transform of f will restrict to the unit circle.

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Department of Mathematics
The University of Texas
Austin, Texas 78712
U.S.A.