Bull. Aust. Math. Soc. **108** (2023), 244–253
doi:10.1017/S0004972722001289

A PAIR OF EQUATIONS IN EIGHT PRIME CUBES AND POWERS OF 2

XUE HAN and HUAFENG LIU✉

(Received 14 September 2022; accepted 3 October 2022; first published online 14 December 2022)

Abstract

In this paper, we show that every pair of sufficiently large even integers can be represented as a pair of eight prime cubes and k powers of 2. In particular, we prove that $k = 335$ is admissible, which improves the previous result.

2020 *Mathematics subject classification*: primary 11P32; secondary 11P05, 11P55.

Keywords and phrases: circle method, Goldbach–Linnik problem, powers of 2.

1. Introduction

In 1951 and 1953, Linnik [5, 6] showed that every large even integer N can be represented in the form of two primes and a bounded number of powers of 2, namely

$$N' = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{k'}}. \tag{1.1}$$

Later, Liu *et al.* [8] proved that $k' = 54000$ is acceptable in (1.1). After many improvements, up to now, the best result is $k' = 8$ established by Pintz and Ruzsa [14]. In 2013, Kong [3] first considered the simultaneous representation of pairs of positive even integers as sums of two primes and powers of 2, that is,

$$\begin{cases} N_1' = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{k'}}, \\ N_2' = p_3 + p_4 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{k'}}. \end{cases}$$

She proved that these equations are solvable for a pair of sufficiently large positive even integers N_1' and N_2' satisfying $N_2' \gg N_1' > N_2'$ for $k' = 63$ unconditionally, and for $k' = 31$ under the generalised Riemann hypothesis (GRH). Subsequently, Kong and Liu [4] improved the value of k' to 34 unconditionally and to 18 under the GRH.

In 2001, based on the works of Linnik [5, 6] and Gallagher [2], Liu and Liu [7] proved that every large even integer N can be written as a sum of eight cubes of primes

This work is supported by the National Natural Science Foundation of China (Grant No. 12171286).

and a bounded number of powers of 2, namely

$$N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}.$$

So far, the best result for this equation is $k = 30$ obtained by Zhu [19].

As a generalisation, in 2013, Liu [11] first considered the simultaneous representation of pairs of positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$ in the form

$$\begin{cases} N_1 = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \\ N_2 = p_9^3 + p_{10}^3 + \cdots + p_{16}^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \end{cases} \tag{1.2}$$

where k is a positive integer. Liu [11] proved that the equations in (1.2) are solvable for $k = 1432$. This number k was improved successively to $k = 1364$, $k = 658$ and $k = 609$ by Platt and Trudgian [15], Zhao [17] and Liu [9], respectively. We make a further improvement on the value of k in (1.2) by establishing the following result.

THEOREM 1.1. *For $k = 335$, the equations in (1.2) are solvable for every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$.*

To prove Theorem 1.1, we apply the circle method in combination with some new arguments of Kong and Liu [4]. To apply the circle method, similarly to [4], we divide $[0, 1]^2$ into three arcs, which means we can avoid the limitation of two arcs in Liu [9] after applying integral transforms (see Section 4 for details), resulting in the sharper k in (1.2).

NOTATION 1.2. Throughout this paper, the letter p, with or without a subscript, always represents a prime. Both N_1 and N_2 denote sufficiently large positive even integers, $e(x) = \exp(2\pi ix)$ and $n \sim N$ means $N < n \leq 2N$. The letter ϵ denotes a positive constant which is arbitrarily small but may not be the same at different occurrences.

2. Outline of the proof

In this section, we give an outline for the proof of Theorem 1.1. To apply the circle method, we let, for $i = 1, 2$,

$$P_i = N_i^{1/9-2\epsilon}, \quad Q_i = N_i^{8/9+\epsilon}, \quad L = \frac{\log(N_1/\log N_1)}{\log 2}.$$

For $i = 1, 2$, we define the major arcs $\mathfrak{M}_i$ and minor arcs $C(\mathfrak{M}_i)$ as

$$\mathfrak{M}_i = \bigcup_{1\leq q_i\leq P_i} \bigcup_{\substack{1\leq a_i\leq q_i \\ (a_i,q_i)=1}} \mathfrak{M}_i(a_i, q_i), \quad C(\mathfrak{M}_i) = [0, 1]\backslash\mathfrak{M}_i, \tag{2.1}$$

where

$$\mathfrak{M}_i(a_i, q_i) = \left\{\alpha_i \in [0, 1] : \left|\alpha_i - \frac{a_i}{q_i}\right| \leq \frac{1}{q_i Q_i}\right\}$$

and

$$1 \leq a_i \leq q_i \leq Q_i, \quad (a_i, q_i) = 1.$$

https://doi.org/10.1017/S0004972722001289 Published online by Cambridge University Press

Note that the major arcs $\mathfrak{M}_i(a_i, q_i)$ are mutually disjoint since $2P_i \le Q_i$. We further define

$$\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{M}_2 = \{(\alpha_1, \alpha_2) \in [0,1]^2 : \alpha_1 \in \mathfrak{M}_1, \alpha_2 \in \mathfrak{M}_2\}, \tag{2.2}$$

$$C(\mathfrak{M}) = [0,1]^2 \backslash \mathfrak{M}. \tag{2.3}$$

As in [16], let $\delta = 10^{-4}$ and

$$U_i = \Big(\frac{N_i}{16(1+\delta)}\Big)^{1/3}, \quad V_i = U_i^{5/6}.$$

For $i = 1, 2$, we set

$$S(\alpha_i, U_i) = \sum_{p \sim U_i} (\log p) e(p^3 \alpha_i), \quad T(\alpha_i, V_i) = \sum_{p \sim V_i} (\log p) e(p^3 \alpha_i), \tag{2.4}$$

$$G(\alpha_i) = \sum_{4 \le v \le L} e(2^v \alpha_i), \quad \mathscr{E}_\lambda = \{(\alpha_1, \alpha_2) \in [0,1]^2 : |G(\alpha_1 + \alpha_2)| \ge \lambda L\}.$$

Let

$$R(N_1, N_2) = \sum \log p_1 \log p_2 \cdots \log p_{16}$$

be the weighted number of solutions of (1.2) in $(p_1, p_2, \ldots, p_{16}, v_1, v_2, \ldots, v_k)$ with

$$p_1, p_2, p_3, p_4 \sim U_1, \quad p_5, p_6, p_7, p_8 \sim V_1,$$
$$p_9, p_{10}, p_{11}, p_{12} \sim U_2, \quad p_{13}, p_{14}, p_{15}, p_{16} \sim V_2,$$
$$4 \le v_j \le L, \quad j = 1, 2, \ldots, k.$$

Then we rewrite $R(N_1, N_2)$ as

$$\begin{aligned} R(N_1, N_2) = \Big(\iint_{\mathfrak{M}} + \iint_{C(\mathfrak{M}) \cap \mathscr{E}_\lambda} + \iint_{C(\mathfrak{M}) \backslash \mathscr{E}_\lambda} \Big) & S^4(\alpha_1, U_1) T^4(\alpha_1, V_1) S^4(\alpha_2, U_2) T^4(\alpha_2, V_2) \\ & \times G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \, d\alpha_1 \, d\alpha_2 \\ := R_1(N_1, N_2) + R_2(N_1, N_2) + R_3(N_1, N_2). & \end{aligned}$$

In Section 3, we first give some lemmas. In Section 4, we shall estimate $R_i(N_1, N_2)$ for $i = 1, 2, 3$ and complete the proof of Theorem 1.1.

3. Auxiliary lemmas

Let

$$\begin{aligned} C(q, a) &= \sum_{\substack{m=1 \\ (m,q)=1}}^{q} e\Big(\frac{am^3}{q}\Big), \quad B(n, q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^8(q, a) e\Big(-\frac{an}{q}\Big), \\ A(n, q) &= \frac{B(n, q)}{\varphi^8(q)}, \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q). \end{aligned} \tag{3.1}$$

https://doi.org/10.1017/S0004972722001289 Published online by Cambridge University Press

LEMMA 3.1. *Let* $\mathscr{A}(N_i, k) = \{n_i \geq 2 : n_i = N_i - 2^{\nu_1} - 2^{\nu_2} - \cdots - 2^{\nu_k}\}$ *with* $k \geq 35$. *Then, for* $N_1 \equiv N_2 \equiv 0 \pmod 2$,

$$\sum_{\substack{n_1 \in \mathscr{A}(N_1,k) \\ n_2 \in \mathscr{A}(N_2,k) \\ n_1 \equiv n_2 \equiv 0 \pmod 2}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) \geq 0.1596600336 L^k.$$

PROOF. From (5.9) of [12] and Lemma 2.3 of [18], for $p \geq 13$ and $p \equiv 1 \pmod 3$,

$$1 + A(n, p) \geq 1 - \frac{(2\sqrt{p} + 1)^8}{(p-1)^7},$$

and

$$\prod_{p \geq 17} (1 + A(n_i, p)) \geq 0.8206744593.$$

Then,

$$\begin{aligned}\prod_{p \geq 13} (1 + A(n_i, p)) &= (1 + A(n_i, 13)) \times \prod_{p \geq 17} (1 + A(n_i, p)) \\ &\geq 0.4233091149 \times 0.8206744593 \\ &\geq 0.3473989790 := C.\end{aligned}$$

Noting that $\mathfrak{S}(n_i) = 2(1 - 1/2^8) \prod_{p>3} (1 + A(n_i, p))$ and putting $q = \prod_{3<p<12} = 385$,

$$\begin{aligned}&\sum_{\substack{n_1 \in \mathscr{A}(N_1,k) \\ n_2 \in \mathscr{A}(N_2,k) \\ n_1 \equiv n_2 \equiv 0 \pmod 2}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) \\ &\quad \geq \left(2\left(1 - \frac{1}{2^8}\right)C\right)^2 \sum_{\substack{n_1 \in \mathscr{A}(N_1,k) \\ n_2 \in \mathscr{A}(N_2,k) \\ n_1 \equiv n_2 \equiv 0 \pmod 2}} \prod_{i=1}^{2} \prod_{3<p_i<12} (1 + A(n_i, p_i)) \\ &\quad \geq \left(2\left(1 - \frac{1}{2^8}\right)C\right)^2 \sum_{1 \leq j_1 \leq q} \sum_{1 \leq j_2 \leq q} \sum_{\substack{n_1 \in \mathscr{A}(N_1,k) \\ n_2 \in \mathscr{A}(N_2,k) \\ n_1 \equiv n_2 \equiv 0 \pmod 2 \\ n_1 \equiv j_1 \pmod q \\ n_2 \equiv j_2 \pmod q}} \prod_{i=1}^{2} \prod_{3<p_i<12} (1 + A(j_i, p_i)) \\ &\quad \geq \left(2\left(1 - \frac{1}{2^8}\right)C\right)^2 \sum_{1 \leq j_1 \leq q} \sum_{1 \leq j_2 \leq q} \prod_{i=1}^{2} \prod_{3<p_i<12} (1 + A(j_i, p_i)) \sum_{\substack{n_1 \in \mathscr{A}(N_1,k) \\ n_2 \in \mathscr{A}(N_2,k) \\ n_1 \equiv n_2 \equiv 0 \pmod 2 \\ n_1 \equiv j_1 \pmod q \\ n_2 \equiv j_2 \pmod q}} 1.\end{aligned}$$

https://doi.org/10.1017/S0004972722001289 Published online by Cambridge University Press

Considering the inner sum,

$$S := \sum_{\substack{n_1\in\mathscr{A}(N_1,k)\\ n_2\in\mathscr{A}(N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2\\ n_1\equiv j_1 \pmod q\\ n_2\equiv j_2 \pmod q}} 1 = \sum_{\substack{4\le v_j\le L,1\le j\le k,i=1,2\\ 2^{v_1}+2^{v_2}+\cdots+2^{v_k}\equiv N_i \pmod 2\\ 2^{v_1}+2^{v_2}+\cdots+2^{v_k}\equiv N_i-j_i \pmod q}} 1.$$

Since $N_1 \equiv N_2 \equiv 0 \pmod 2$,

$$\begin{cases} 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \equiv N_1 \pmod 2 \\ 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \equiv N_2 \pmod 2 \end{cases}$$

is equivalent to

$$2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \equiv N_1 \pmod 2.$$

Additionally, if $N_2 \equiv N_1 + t \pmod q$ and $j_2 \equiv j_1 + t \pmod q$ with $1 \le t \le q$,

$$\begin{cases} 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \equiv N_1 - j_1 \pmod q \\ 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \equiv N_2 - j_2 \pmod q \end{cases}$$

is equivalent to

$$2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \equiv N_1 - j_1 \pmod q.$$

Therefore, when $N_1 \equiv N_2 \equiv 0 \pmod 2$, $N_2 \equiv N_1 + t \pmod q$ and $j_2 \equiv j_1 + t \pmod q$,

$$S \ge \sum_{\substack{4\le v_1,v_2,\dots,v_k\le L\\ 2^{v_1}+2^{v_2}+\cdots+2^{v_k}\equiv N_1 \pmod 2\\ 2^{v_1}+2^{v_2}+\cdots+2^{v_k}\equiv N_1-j_1 \pmod q}} \Big(\frac{L}{\rho(3q)} + O(1)\Big)^k \sum_{\substack{4\le v_1,v_2,\dots,v_k\le \rho(3q)\\ 2^{v_1}+2^{v_2}+\cdots+2^{v_k}\equiv a_j \pmod{3q}}} 1,$$

where the natural number $a_j \in [1, 3q]$ satisfies the conditions $a_j \equiv N_1 \pmod 3$ and $a_j \equiv N_1 - j_1 \pmod q$, and $\rho(q)$ denotes the smallest positive integer ρ such that $2^\rho \equiv 1 \pmod q$.

Noting that

$$S \ge \frac{1}{3q}\Big(\frac{L}{\rho(3q)} + O(1)\Big)^k \sum_{t=0}^{3q-1} e\Big(\frac{ta_j}{3q}\Big)\Big(\sum_{1\le s\le \rho(3q)} e\Big(\frac{t2^s}{3q}\Big)\Big)^k,$$

we get

$$\begin{aligned} S &\ge \frac{1}{3q}\Big(\frac{L}{\rho(3q)} + O(1)\Big)^k (\rho(3q)^k - (3q-1)(\max)^k) \\ &= \frac{L^k}{3q}\Big(1 - (3q-1)\Big(\frac{\max}{\rho(3q)}\Big)^k\Big) + O(L^{k-1}), \end{aligned}$$

where

$$\max = \max\Big\{\Big|\sum_{1\le s\le \rho(3q)} e\Big(\frac{j2^s}{3q}\Big)\Big| : 1 \le j \le 3q-1\Big\}.$$

https://doi.org/10.1017/S0004972722001289 Published online by Cambridge University Press

Since $3q = 1155$ and $\rho(3q) = 60$, with the help of a computer,

$$\max = 30\ldots, \quad (3q-1)\Big(\frac{\max}{\rho(3q)}\Big)^{50} < 10^{-10}.$$

Therefore,

$$S \geq \frac{(1-10^{-10})L^k}{3q} + O(L^{k-1}).$$

By a numerical calculation,

$$\max_{1\leq t\leq q}\Big(\sum_{1\leq j_1\leq q}\prod_{3<p_1<12}(1+A(j_1,p_1))\prod_{3<p_2<12}(1+A(j_1+t,p_2))\Big) \geq 384.9999769.$$

Then,

$$\begin{aligned}\sum_{\substack{n_1\in\mathscr{A}(N_1,k)\\ n_2\in\mathscr{A}(N_2,k)\\ n_1\equiv n_2\equiv 0 \ (\mathrm{mod}\ 2)}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) &\geq 384.9999769\Big(2\Big(1-\frac{1}{2^8}\Big)C\Big)^2\frac{(1-10^{-10})L^k}{3q}\\ &\geq 0.1596600336L^k. \end{aligned}$$

□

LEMMA 3.2 [12, Lemma 2.1]. *Let $\mathfrak{M}_i$, $S(\alpha_i,U_i)$ and $T(\alpha_i,V_i)$ be defined as in (2.1) and (2.4), respectively. For $N_i/2\leq n_i\leq N_i$,*

$$\int_{\mathfrak{M}_i} S^4(\alpha_i,U_i)T^4(\alpha_i,V_i)e(-n_i\alpha_i)\,d\alpha_i = \frac{1}{3^8}\mathfrak{S}(n_i)\mathfrak{J}(n_i) + O(N_i^{13/9}L^{-1}),$$

where $\mathfrak{S}(n_i)$ is defined as in (3.1) and satisfies $\mathfrak{S}(n_i)\gg 1$ for $n_i\equiv 0 \ (\mathrm{mod}\ 2)$, and $\mathfrak{J}(n_i)$ is defined as

$$\mathfrak{J}(n_i) := \sum_{\substack{m_1+m_2+\cdots+m_8=n_i\\ U_i^3<m_1,m_2,m_3,m_4\leq 8U_i^3\\ V_i^3<m_5,m_6,m_7,m_8\leq 8V_i^3}} (m_1m_2\ldots m_8)^{-2/3}$$

and satisfies $N_i^{13/9}\ll\mathfrak{J}(n_i)\ll N_i^{13/9}$.

LEMMA 3.3 [18, Lemma 2.6]. *For $(1-\delta)N_i\leq n_i\leq N_i$,*

$$\mathfrak{J}(n_i) > 1.42432055N_i^{13/9}.$$

LEMMA 3.4. *We have $\mathrm{meas}(\mathscr{E}_\lambda)\ll N_i^{-E(\lambda)}$ with $E(0.9570253) > \frac{8}{9}+10^{-10}$.*

PROOF. This is (2.7) in Lemma 2.1 of Zhao [17]. □

LEMMA 3.5 [17, Lemma 2.5]. *Let $\mathfrak{M}_i$ and $S(\alpha_i,U_i)$ be defined as in (2.1) and (2.4), respectively. We have*

$$\max_{\alpha_i\in C(\mathfrak{M}_i)}|S(\alpha_i,U_i)|\ll N_i^{11/36+\epsilon}.$$

https://doi.org/10.1017/S0004972722001289 Published online by Cambridge University Press

LEMMA 3.6. *Let $S(\alpha_i, U_i)$ and $T(\alpha_i, V_i)$ be defined as in (2.4). We have*

$$\int_0^1 |S(\alpha_i, U_i)T(\alpha_i, V_i)|^4 \, d\alpha_i \le 0.134694091 N_i^{13/9}.$$

PROOF. The idea of the proof is similar to that of Lemma 2.6 in Liu and Lü [13]. However, we take $\nu = 100552$ obtained by Elsholtz and Schlage-Puchta [1] instead of 147185.22 obtained by Liu [10]. This leads to a better upper bound.

Here we only consider the case $i = 1$ since the case $i = 2$ can be proved similarly. From (2.7) of Ren [16] and Proposition 2 of Elsholtz and Schlage-Puchta [1],

$$\sum_{N_1/9<l\le N_1} r^2(l) \le \vartheta(0) \le (\nu + o(1))U_1 V_1^4 L^{-8},$$

where $\nu = 100552$, $r(n)$ denotes the number of representations of n as $p_1^3 + p_2^3 + p_3^3 + p_4^3$ with $p_1, p_2 \sim U_1$, $p_3, p_4 \sim V_1$ and $\vartheta(0)$ denotes the number of solutions of the equation $p_1^3 + p_2^3 + p_3^3 + p_4^3 = p_5^3 + p_6^3 + p_7^3 + p_8^3$ with $p_1, p_2, p_5, p_6 \sim U_1, p_3, p_4, p_7, p_8 \sim V_1$.

Therefore,

$$\begin{aligned}\int_0^1 |S(\alpha_1, U_1)T(\alpha_1, V_1)|^4 \, d\alpha_1 &\le (\log(2U_1))^4(\log(2V_1))^4\vartheta(0)\\ &\le 0.134694091 N_1^{13/9}. \qquad \square\end{aligned}$$

4. Proof of Theorem 1.1

To prove Theorem 1.1, we first estimate $R_1(N_1, N_2)$. By Lemmas 3.1, 3.2 and 3.3,

$$\begin{aligned}R_1(N_1, N_2) &= \iint_{\mathfrak{M}} S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)\\ &\quad \times G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)\, d\alpha_1\, d\alpha_2\\ &\ge \Big(\frac{1}{3^8}\Big)^2 \sum_{\substack{n_1\in\mathscr{A}(N_1,k)\\ n_2\in\mathscr{A}(N_2,k)}} \mathfrak{S}(n_1)\mathfrak{S}(n_2)\mathfrak{J}(n_1)\mathfrak{J}(n_2) \qquad (4.1)\\ &\ge \frac{0.1596600336 \times (1.42432055)^2}{3^{16}}(N_1N_2)^{13/9}L^k\\ &\ge 7.524395606 \times 10^{-9}(N_1N_2)^{13/9}L^k,\end{aligned}$$

where $\mathfrak{M}$ is defined by (2.2).

Next, we estimate $R_2(N_1, N_2)$. By (2.1) and (2.3),

$$C(\mathfrak{M}) \subset \{(\alpha_1, \alpha_2) : \alpha_1 \in C(\mathfrak{M}_1), \alpha_2 \in [0, 1]\} \cup \{(\alpha_1, \alpha_2) : \alpha_1 \in [0, 1], \alpha_2 \in C(\mathfrak{M}_2)\}.$$

https://doi.org/10.1017/S0004972722001289 Published online by Cambridge University Press

From Lemma 3.5 and the trivial bounds of $G(\alpha_i)$ and $T(\alpha_i, V_i)$,

$$\begin{aligned} R_2(N_1,N_2) &= \iint_{C(\mathfrak{M})\cap\mathscr{E}_\lambda} S^4(\alpha_1,U_1)T^4(\alpha_1,V_1)S^4(\alpha_2,U_2)T^4(\alpha_2,V_2) \\ &\qquad \times G^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)\,d\alpha_1\,d\alpha_2 \\ &\ll L^k\Big(\iint_{\substack{(\alpha_1,\alpha_2)\in C(\mathfrak{M}_1)\times[0,1]\\ |G(\alpha_1+\alpha_2)|\geq\lambda L}} + \iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]\times C(\mathfrak{M}_2)\\ |G(\alpha_1+\alpha_2)|\geq\lambda L}}\Big) \\ &\qquad S^4(\alpha_1,U_1)T^4(\alpha_1,V_1)S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)\,d\alpha_1\,d\alpha_2 \\ &\ll L^kN_1^{10/9}N_1^{11/9+\epsilon}\iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |G(\alpha_1+\alpha_2)|\geq\lambda L}} |S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)|\,d\alpha_1\,d\alpha_2 \\ &\qquad + L^kN_2^{10/9}N_2^{11/9+\epsilon}\iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |G(\alpha_1+\alpha_2)|\geq\lambda L}} |S^4(\alpha_1,U_1)T^4(\alpha_1,V_1)|\,d\alpha_1\,d\alpha_2. \end{aligned} \tag{4.2}$$

Let $\varpi = \alpha_1 + \alpha_2$. By the periodicity of $G(\alpha)$,

$$\begin{aligned} &\iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |G(\alpha_1+\alpha_2)|\geq\lambda L}} |S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)|\,d\alpha_1\,d\alpha_2 \\ &\qquad\qquad = \int_0^1 |S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)|\Big(\int_{\substack{\varpi\in[\alpha_2,1+\alpha_2]\\ |G(\varpi)|\geq\lambda L}} d\varpi\Big)\,d\alpha_2. \end{aligned}$$

By Lemmas 3.4 and 3.6,

$$\iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |G(\alpha_1+\alpha_2)|\geq\lambda L}} |S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)|\,d\alpha_1\,d\alpha_2 \ll N_2^{13/9}N_1^{-8/9-10^{-10}}. \tag{4.3}$$

Similarly,

$$\iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |G(\alpha_1+\alpha_2)|\geq\lambda L}} |S^4(\alpha_1,U_1)T^4(\alpha_1,V_1)|\,d\alpha_1\,d\alpha_2 \ll N_1^{13/9}N_2^{-8/9-10^{-10}}. \tag{4.4}$$

From (4.2)–(4.4),

$$\begin{aligned} R_2(N_1,N_2) &\ll N_1^{10/9}N_1^{11/9+\epsilon}N_2^{13/9}N_1^{-8/9-10^{-10}}L^k + N_2^{10/9}N_2^{11/9+\epsilon}N_1^{13/9}N_2^{-8/9-10^{-10}}L^k \\ &\ll (N_1N_2)^{13/9}L^{k-1}, \end{aligned} \tag{4.5}$$

where $N_2 \gg N_1 > N_2$.

https://doi.org/10.1017/S0004972722001289 Published online by Cambridge University Press

Finally, we estimate $R_3(N_1, N_2)$. By Lemma 3.6 and the definition of $\mathscr{E}_\lambda$,

$$
\begin{aligned}
R_3&(N_1, N_2)\\
&= \iint_{C(\mathfrak{M})\setminus\mathscr{E}_\lambda} S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)\\
&\quad \times G^k(\alpha_1+\alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)\, d\alpha_1\, d\alpha_2 \\
&\le (\lambda L)^k \int_0^1 |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)|\, d\alpha_1 \int_0^1 |S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)|\, d\alpha_2\\
&\le 0.0181424982\lambda^k (N_1N_2)^{13/9}L^k.
\end{aligned}
\tag{4.6}
$$

Putting (4.1), (4.5) and (4.6) together,

$$
\begin{aligned}
R(N_1, N_2) &> R_1(N_1, N_2) - R_3(N_1, N_2) + O((N_1N_2)^{13/9}L^{k-1})\\
&> (7.524395606 \times 10^{-9} - 0.0181424982\lambda^k)(N_1N_2)^{13/9}L^k,
\end{aligned}
$$

where $\lambda = 0.9570253$. Then we can deduce that

$$R(N_1, N_2) > 0$$

provided that $k \ge 335$. Thus, we complete the proof of Theorem 1.1.

Acknowledgement

The authors would like to thank the referee for useful comments.

References

[1] C. Elsholtz and J. C. Schlage-Puchta, ‘The density of integers representable as the sum of four prime cubes’, *Acta Arith.* **192** (2020), 363–369.

[2] P. X. Gallagher, ‘Primes and powers of 2’, *Invent. Math.* **29** (1975), 125–142.

[3] Y. F. Kong, ‘On pairs of linear equations in four prime variables and powers of 2’, *Bull. Aust. Math. Soc.* **87** (2013), 55–67.

[4] Y. F. Kong and Z. X. Liu, ‘On pairs of Goldbach–Linnik equations’, *Bull. Aust. Math. Soc.* **95** (2017), 199–208.

[5] Y. V. Linnik, ‘Prime numbers and powers of two’, *Tr. Mat. Inst. Steklova* **38** (1951), 151–169 (in Russian).

[6] Y. V. Linnik, ‘Addition of prime numbers with powers of one and the same number’, *Mat. Sb. (N.S.)* **32** (1953), 3–60 (in Russian).

[7] J. Y. Liu and M. C. Liu, ‘Representation of even integers by cubes of primes and powers of 2’, *Acta Math. Hungar.* **91** (2001), 217–243.

[8] J. Y. Liu, M. C. Liu and T. Z. Wang, ‘The number of powers of 2 in a representation of large even integers (II)’, *Sci. China Ser. A* **41** (1998), 1255–1271.

[9] Y. H. Liu, ‘Two results on Goldbach–Linnik problems for cubes of primes’, *Rocky Mountain J. Math.* **52** (2022), 999–1007.

[10] Z. X. Liu, ‘Density of the sums of four cubes of primes’, *J. Number Theory* **132** (2012), 735–747.

[11] Z. X. Liu, ‘On pairs of quadratic equations in primes and powers of 2’, *J. Number Theory* **133** (2013), 3339–3347.

[12] Z. X. Liu and G. S. Lü, ‘Eight cubes of primes and powers of 2’, *Acta Arith.* **145** (2010), 171–192.

[13] Z. X. Liu and G. S. Lü, 'Two results on powers of 2 in Waring–Goldbach problem', *J. Number Theory* **131** (2011), 716–736.
[14] J. Pintz and I. Z. Ruzsa, 'On Linnik's approximation to Goldbach's problem II', *Acta Math. Hungar.* **161** (2020), 569–582.
[15] D. J. Platt and T. S. Trudgian, 'Linnik's approximation to Goldbach's conjecture, and other problems', *J. Number Theory* **153** (2015), 54–62.
[16] X. M. Ren, 'Sums of four cubes of primes', *J. Number Theory* **98** (2003), 156–171.
[17] X. D. Zhao, 'Goldbach–Linnik type problems on cubes of primes', *Ramanujan J.* **57** (2022), 239–251.
[18] X. D. Zhao and W. X. Ge, 'Eight cubes of primes and 204 powers of 2', *Int. J. Number Theory* **16** (2020), 1547–1555.
[19] L. Zhu, 'Goldbach–Linnik type problems on eight cubes of primes', *Rocky Mountain J. Math.* **52** (2022), 1127–1139.

XUE HAN, School of Mathematics and Statistics,
Shandong Normal University, Jinan 250358, Shandong, PR China
e-mail: han_xue@stu.sdnu.edu.cn

HUAFENG LIU, School of Mathematics and Statistics,
Shandong Normal University, Jinan 250358, Shandong, PR China
e-mail: hfliu_sdu@hotmail.com