A NEW LOOK AT THE KUMMER SURFACE

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Dedicated to H. S. M. Coxeter on his sixtieth birthday

Kummer's surface has the base surface F of a certain net $\mathfrak{N}$ of quadrics in [5] for a non-singular model. All the quadrics of $\mathfrak{N}$ have a common self-polar simplex Σ, and $\mathfrak{N}$ can, in a double-infinity of ways, be based on a quadric Ω_1 and two quadrics that Ω_1 reciprocates into each other. F is invariant under harmonic inversions in the vertices and opposite bounding primes of Σ and (§2) contains 32 lines. In §3 it is shown, conversely, that those quadrics for which a given simplex is self-polar and which contain a line of general position constitute a net of this kind.

Each quadric of $\mathfrak{N}$ is shown, in §5, to contain 32 tangent planes of F. This is confirmed by another argument in §6, where it is explained how this fact establishes an involution of pairs of conjugate directions at any point of F; this involves a system of asymptotic curves. That these are intersections of F with quadrics for which, too, Σ is self-polar is seen in §7; that they correspond to the traditional asymptotic curves on K will be proved in §12.

After describing a less-specialized surface Φ in §8, the rudiments of the geometry on F are alluded to in §9; §10 introduces the Weddle surface W as a projection of F and avails itself of the classical birational correspondence between W and K. The association between the 15 quadric line cones through F and the 15 separations of the nodes of K into complementary octads is explained in §11. The details of the mapping of curves, of lower orders, on F and K into each other are discussed in §§12–15. §§16–17 end the paper with a brief note on tetrahedroids.

1. The following pages concern matters related to the geometry of the surface F, in projective space [5], common to the three quadrics

$$(1.1) \qquad \Omega_0 \equiv \sum x_i^2 = 0, \qquad \Omega_1 \equiv \sum a_i x_i^2 = 0, \qquad \Omega_2 \equiv \sum a_i^2 x_i^2 = 0,$$

where the summations over i run from 0 to 5 and no two of the six coefficients a_i are equal. It is sometimes convenient to speak of the quadric $\Omega_\kappa = 0$ merely as "the quadric Ω_κ."

The first appearance of these three equations is in Klein's early paper **(11)** on linear and quadratic complexes. He uses his mapping of the lines of [3] by the points of Ω_0, so that the lines of a general quadratic complex are mapped on the threefold $\Omega_0 = \Omega_1 = 0$. The equations (1.1) are all on p. 223 of **(11)**,

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where Klein derives $\Omega_2 = 0$, to determine the singular lines of the complex, by an appeal to Plücker's formula (**12**, p. 296) determining the singular lines of any complex, the term *singular line* being introduced and defined by Plücker in this place. This happened all but a century ago; but geometers have now, for little short of half the century, deemed the *locus classicus* for this figure in [5], and for the insight it affords into the theory of the Kummer surface, to be the exposition, as authoritatively magisterial as it is undeniably idiosyncratic, in Baker's book (**3**). He naturally takes the mapping of the quadratic complex on $\Omega_0 = \Omega_1 = 0$ from Klein, but he derives the relation $\Omega_2 = 0$ which the singular lines must satisfy by a geometrical argument.

Although it is common knowledge, at least with readers of Klein or Baker, that F is in birational correspondence with a Kummer surface K—note, in particular, lines 10–12 on p. 219 of (**3**) and their substantiation on p. 221—there seems to have been scarcely any exploration of the details of this correspondence. But it has now been known (**8**) for a decade that the projection of F from any one of its 32 lines onto a solid is a Weddle surface W; this knowledge opens another way to scrutinize the correspondence between K and the non-singular model F of both K and W, since the birational correspondence between W and K is notorious (**2**, p. 65; **5**, p. 237; **10**, p. 166; **14**, pp. 158 *et seq.*).

2. Suppose now that Ω, Ω' are two non-singular quadrics in general position in [5]; they have a unique common self-polar simplex Σ, and Σ is self-polar also for the polar reciprocal Ω'' of Ω with respect to Ω'. The octavic surface F common to Ω, Ω', Ω'' shares with them the property of being invariant under the six harmonic inversions h_i in the vertices and opposite bounding primes of Σ. The h_i are commutative, and generate an elementary abelian group G of order 32; of its 31 subgroups of order 16 one, call it G^+, is singled out by this representation; G^+ consists of the identity and the 15 harmonic inversions $h_i h_j$ in the edges $X_i X_j$ and opposite bounding solids of Σ. The coset G^- of G^+ consists of the six h_i themselves and of the 10 harmonic inversions, like

$$h_0 h_1 h_2 \equiv h_3 h_4 h_5,$$

in opposite planes of Σ. If Σ is the simplex of reference, h_i replaces x_i by $-x_i$ while leaving the other five coordinates unaltered.

The polarizations now to occur in the next paragraph are all with respect to Ω'.

The polar plane α'' of a plane α on Ω is on Ω''; in general, α and α'' are skew and meet Ω' in non-singular conics. But if α meets Ω' in lines l, m which intersect at P, then α'', common to the polar solids L, M of l, m, passes, with both L and M, through P and so, as it is in the polar prime of P, meets Ω' in lines l'', m'' through P. This shows, incidentally, that P is on F (**3**, p. 219). Lastly, should α meet Ω' in a repeated line λ, it must lie in the polar solid Λ and so α'', lying in Λ and containing λ, also meets Ω' in λ repeated. And λ is on F. So, therefore, is any line obtainable from λ by applying the projectivities of G.

Take, now, the quadrics

$$\Omega_0 \equiv \sum x_i^2 = 0, \qquad \Omega_1 \equiv \sum a_i x_i^2 = 0,$$

and the polar reciprocal of Ω_0 with respect to Ω_1, namely

$$\Omega_2 \equiv \sum a_i^2 x_i^2 = 0.$$

Write $$f(\theta) \equiv (\theta - a_0)(\theta - a_1)(\theta - a_2)(\theta - a_3)(\theta - a_4)(\theta - a_5)$$
$$\equiv \theta^6 - e_1\theta^5 + e_2\theta^4 - e_3\theta^3 + e_4\theta^2 - e_5\theta + e_6,$$

and $$s_\kappa \equiv \sum a_i^\kappa / f'(a_i).$$

Then $s_0 = s_1 = s_2 = s_3 = s_4 = 0$ and $s_5 = 1$. These facts are for immediate use, but two others are added, for later use, giving s_6 and s_7. Since

$$\theta^6 - f(\theta) \equiv e_1\theta^5 - e_2\theta^4 + \ldots,$$
$$a_i^6 = e_1 a_i^5 - e_2 a_i^4 + \ldots,$$

so that $s_6 = e_1$; and since

$$\theta^7 - f(\theta) \equiv e_1\theta^6 - e_2\theta^5 + \ldots,$$
$$a_i^7 = e_1 a_i^6 - e_2 a_i^5 + \ldots,$$

so that $s_7 = e_1^2 - e_2$.

Whatever signs may be prefixed to the six square roots, the line λ whose parametric form is

(2.1) $$x_i \sqrt{f'(a_i)} = t + a_i$$

lies on all three quadrics, and so on F. Moreover, as Baker points out in a note appended at the end (pp. 266–267) of the 1940 reprint of **(3)**, the plane given in terms of homogeneous parameters u: v: w by

$$x_i \sqrt{f'(a_i)} = u + va_i + wa_i^2,$$

while lying wholly on Ω_0, meets Ω_1 in the line $w = 0$, i.e. λ, repeated. One may add that its polar plane

$$x_i \sqrt{f'(a_i)} = ua_i^{-1} + v + wa_i$$

with respect to Ω_1 lies wholly on Ω_2 and meets Ω_1 in the line $u = 0$, i.e. λ again, repeated. When different signs are prefixed to the six square roots in (2.1), there arise 32 lines on F, transforms of each other under the operations of G.

3. It is now apparent that the base surface F of the net of quadrics

$$\xi\Omega_0 + \eta\Omega_1 + \zeta\Omega_2 = 0$$

contains 32 lines. But *the converse is also true*; if the surface of intersection of three linearly independent quadrics having a common self-polar simplex Σ in [5] contains a line of general position (i.e. not meeting any plane face of Σ),

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then the net of quadrics can, in reference to Σ, be based on $\Omega_0, \Omega_1, \Omega_2$. For suppose that $\sum\alpha_i x_i^2 = 0$ contains the join λ of y and z:

$$(3.1) \qquad \sum\alpha_i y_i^2 = \sum\alpha_i y_i z_i = \sum\alpha_i z_i^2 = 0.$$

Since

$$\begin{vmatrix} y_i^2 & y_j^2 & y_k^2 \\ y_i z_i & y_j z_j & y_k z_k \\ z_i^2 & z_j^2 & z_k^2 \end{vmatrix} = p_{ij} p_{ik} p_{jk}$$

where $p_{ij} = y_i z_j - y_j z_i$, it follows on eliminating, say, α_4 and α_5 from (3.1) that

$$\alpha_0 p_{04} p_{05} + \alpha_1 p_{14} p_{15} + \alpha_2 p_{24} p_{25} + \alpha_3 p_{34} p_{35} = 0,$$

p_{45} not vanishing because λ does not meet $x_4 = x_5 = 0$. If, then, three linearly independent quadrics

$$(3.2) \qquad \sum\alpha_i x_i^2 = 0, \qquad \sum\beta_i x_i^2 = 0, \qquad \sum\gamma_i x_i^2 = 0$$

all contain λ,

$$(3.3) \quad p_{04} p_{05} : p_{14} p_{15} : p_{24} p_{25} : p_{34} p_{35} = (123) : -(023) : (013) : -(012)$$

where (ijk) is the determinant $|\alpha_i \beta_i \gamma_k|$. And there are 15 such sets of relations, any pair of suffixes being eligible to play the part of 4 and 5 here.

Now there is a standard condition (**13**, p. 401) for six points $(\alpha_i, \beta_i, \gamma_i)$ in a plane to be on a conic; one of several alternative forms for it is

$$(012)(035)(134)(245) + (345)(301)(412)(520) = 0.$$

But, in the present context, by analogues of (3.3), the condition holds because

$$(012)/(412) = p_{43} p_{45}/p_{03} p_{05}, \qquad (413)/(013) = -p_{02} p_{05}/p_{42} p_{45},$$

$$(035)/(435) = -p_{41} p_{42}/p_{01} p_{02}, \qquad (452)/(052) = -p_{01} p_{03}/p_{41} p_{43}.$$

Hence, by choosing three linearly independent quadrics (3.2) suitably, one may take

$$\alpha_i : \beta_i : \gamma_i = 1 : a_i : a_i^2.$$

4. The net $\mathfrak{N}$ of quadrics $\xi\Omega_0 + \eta\Omega_1 + \zeta\Omega_2 = 0$ through F includes a system Γ, of index 2, of quadrics

$$\theta^2\Omega_0 - 2\theta\Omega_1 + \Omega_2 \equiv \sum(\theta - a_i)^2 x_i^2 = 0$$

which are reciprocals of each other in pairs with respect to the quadrics of $\mathfrak{N}$ outside Γ; indeed

$$u^2\Omega_0 - 2u\Omega_1 + \Omega_2 = 0 \quad \text{and} \quad v^2\Omega_0 - 2v\Omega_1 + \Omega_2 = 0$$

are polar reciprocals with respect to

$$uv\Omega_0 - (u + v)\Omega_1 + \Omega_2 = 0.$$

One ignores the six critical values a_i of u or v so that the quadrics remain non-singular.

When the quadrics of $\mathfrak{N}$ are mapped by the points (ξ, η, ζ) of a plane π the map γ of Γ is the conic $\eta^2 = 4\zeta\xi$. Six points $(a_i^2, -2a_i, 1)$ on γ map cones; the 15 vertices of the hexagram H of tangents to γ at these points map line-cones whose vertices are the 15 edges of Σ and compose the Jacobian curve of $\mathfrak{N}$. Points on a side of H distinct from the five vertices map cones sharing a common vertex at a vertex of Σ; one of them belongs to Γ. Two points A, B on γ, neither being a point $(a_i^2, -2a_i, 1)$, map quadrics of Γ that are polar reciprocals with respect to that quadric mapped by the pole of AB.

The envelope of the quadrics of Γ is $\Omega_0 \Omega_2 = \Omega_1^2$, a quartic primal Q with F for a double surface. Since its equation is

$$\sum_{i,j} (a_i - a_j)^2 x_i^2 x_j^2 = 0,$$

it also has nodes at the vertices of Σ, the nodal cone at X_i being

$$a_i^2\Omega_0 - 2a_i \Omega_1 + \Omega_2 \equiv \sum_j (a_i - a_j)^2 x_j^2 = 0,$$

one of the six cones of Γ. The quartic threefold along which this cone touches Q is the cone of chords of F through X_i; every other generator of this nodal cone has four-point intersection with Q at X_i. The threefolds along which the quadrics mapped by A and B touch Q form its complete intersection with that quadric of $\mathfrak{N}$ which reciprocates these two quadrics into each other:

$$(u^2\Omega_0 - 2u\Omega_1 + \Omega_2)(v^2\Omega_0 - 2v\Omega_1 + \Omega_2) - \{uv\Omega_0 - (u + v)\Omega_1 + \Omega_2\}^2 \equiv (u - v)^2(\Omega_0 \Omega_2 - \Omega_1^2).$$

These remarks on the net of quadrics in [5] manifestly apply to analogous nets in spaces of any dimension; there, too, one will encounter the system Γ of quadrics that are polar reciprocals with respect to members of the net outside Γ with its envelope Q having nodes at the vertices of Σ and $\Omega_0 = \Omega_1 = \Omega_2 = 0$ as a double locus. Baker, when obtaining the normals to a cyclide by projection, encountered the net in [4]. His curious oversight has puzzled several readers when, momentarily oblivious of the fact that Ω_0 and Ω_2 are polar reciprocals with respect to Ω_1, he remarks (**3**, p. 185) that their common curve "lies also on the polar reciprocal" of Ω_0 in regard to Ω_1!

5. The equations $\Omega_0 = \Omega_1 = \Omega_2 = 0$, since the rank of the matrix of their coefficients is 3, have $6 - 3$ linearly independent solutions as linear homogeneous equations for the six x_i^2. Thus every point y of F has its coordinates satisfying equations

$$y_i^2 f'(a_i) = \xi + \eta a_i + \zeta a_i^2 \qquad (i = 0, 1, 2, 3, 4, 5). \tag{5.1}$$

The tangent plane T of F at the point y is

$$\sum x_i y_i = \sum a_i x_i y_i = \sum a_i^2 x_i y_i = 0,$$

so that the whole of T is obtained by varying p: q: r in

$$x_i y_i = (p + qa_i + ra_i^2)/f'(a_i) \tag{5.2}$$

or, by (5.1), in

$$x_i = (p + qa_i + ra_i^2)y_i/(\xi + \eta a_i + \zeta a_i^2). \tag{5.3}$$

At all such points x,

$$\begin{aligned}\xi\Omega_0 + \eta\Omega_1 + \zeta\Omega_2 &= \sum(\xi + \eta a_i + \zeta a_i^2)x_i^2 \\ &= \sum(p + qa_i + ra_i^2)^2 y_i^2/(\xi + \eta a_i + \zeta a_i^2) \\ &= \sum(p + qa_i + ra_i^2)^2/f'(a_i) = 0,\end{aligned}$$

so that T is on the quadric $\xi\Omega_0 + \eta\Omega_1 + \zeta\Omega_2 = 0$. Each quadric containing F therefore also contains, in general, 32 tangent planes of F.

6. That the tangent plane T, at a point y on F, lies on one of the quadrics through F may also be seen as follows.

The linear system S, based on six linearly independent members, of quadrics for which Σ is self-polar cuts T in a linear system of conics that includes the six repeated lines of intersection of T with the bounding primes of Σ. The equations of these lines are, by (5.3), $p + qa_i + ra_i^2 = 0$, so that they are all tangents of the conic $\Gamma(y)$ whose equation is $q^2 = 4rp$; the conic in which T meets any quadric of S is therefore outpolar to $\Gamma(y)$. If this quadric is one of those containing F, it meets T, if it does not contain it wholly, in a pair of lines through y which, by virtue of the outpolarity, are conjugate for $\Gamma(y)$. Thus the quadrics of $\mathfrak{N}$ cut T in a pencil of line-pairs in involution, and one of them contains T.

The tangents from y, where p: q: r = ξ: η: ζ, to $\Gamma(y)$ are

$$(\zeta p - \xi r)^2 = (\xi q - \eta p)(\eta r - \zeta q);$$

they are the focal rays of the involution, and each of them lies with F on a pencil of quadrics that meet T in the line repeated. Thus two among the single-infinity of pairs of conjugate tangential directions at a point of F are peculiar in being self-conjugate and may be designated asymptotic; their existence implies that of a system of asymptotic curves, two of these passing through a point of general position on F.

When y is on a line λ of F, the involution in T is parabolic, its focal rays coinciding with λ, which is the only asymptotic direction at y; the 32 lines are the envelope of the asymptotic curves. The quadric that contains the tangent planes at the 32 points (2.1) is, by (5.1),

$$t^2\Omega_0 + 2t\Omega_1 + \Omega_2 = 0.$$

This provides another identification of those quadrics of $\mathfrak{N}$ that are mapped in π by Γ: not only are they the quadrics reciprocal to each other with respect to quadrics of $\mathfrak{N}$; they are those quadrics through F such that the tangent planes of F lying on them have their contacts with F on its lines.

Any quadric of S that contains a tangent line to F at a point y on the section B_i by $x_i = 0$ also contains its harmonic inverse in h_i; the two tangent lines are harmonic to the tangent of B_i and the join of the contact to X_i, so that these last two lines are the focal rays of the relevant involution. Since B_i has, at every point, an asymptotic direction for tangent, it is itself an asymptotic curve; the six B_i are *principal* asymptotic curves on F.

7. The six quadrics $\Omega_r \equiv \sum a_i{}^r x_i{}^2 = 0$ are linearly independent because the Vandermonde determinant of their coefficients is not zero; they therefore span the system S. Since Ω_3 and Ω_5 are polar reciprocals with respect to Ω_4, the net of quadrics $\rho\Omega_3 + \sigma\Omega_4 + \tau\Omega_5 = 0$ has the same properties as $\mathfrak{N}$; its base surface F' contains 32 lines.

The tangent to the curve of intersection of F and $\rho\Omega_3 + \sigma\Omega_4 + \tau\Omega_5 = 0$ is the line common to the tangent plane of F and the prime

$$\sum (\rho a_i{}^3 + \sigma a_i{}^4 + \tau a_i{}^5) x_i y_i = 0.$$

This line is, by (5.2),

$$\sum a_i{}^3(\rho + \sigma a_i + \tau a_i{}^2)(p + q a_i + r a_i{}^2)/f'(a_i) = 0,$$
$$p\tau + q(\sigma + \tau e_1) + r(\rho + \sigma e_1 + \tau e_1{}^2 - \tau e_2) = 0.$$

It is a tangent of $q^2 = 4rp$ when

$$(\sigma + \tau e_1)^2 = \tau(\rho + \sigma e_1 + \tau e_1{}^2 - \tau e_2),$$
$$\sigma^2 + \sigma\tau e_1 + \tau^2 e_2 = \tau\rho,$$

and this independently of which point of the curve is in question. So the asymptotic curves on F are its intersections with quadrics

$$(\sigma^2 + \sigma\tau e_1 + \tau^2 e_2)\Omega_3 + \tau(\sigma\Omega_4 + \Omega_5) = 0,$$
$$\sigma^2\Omega_3 + \sigma\tau(e_1\Omega_3 + \Omega_4) + \tau^2(e_2\Omega_3 + \Omega_5) = 0.$$

It is found, on substituting for x_i from (2.1), that this quadric touches the line where $t + e_1 + \sigma\tau^{-1} = 0$. Since the envelope of these quadrics is

$$(e_1\Omega_3 + \Omega_4)^2 = 4\Omega_3(e_2\Omega_3 + \Omega_5),$$

this quartic primal meets F in the 32 lines. There is a corresponding result concerning the 2^n lines on the surface common to certain $n - 2$ quadrics in $[n]$; for $n = 4$ it is stated on p. 189 of **(3)**.

When x is at any of the points y on F given by (5.1),

$$\Omega_3 = \zeta, \qquad \Omega_4 = \eta + e_1\zeta, \qquad \Omega_5 = \xi + e_1\eta + (e_1{}^2 - e_2)\zeta;$$

hence the quadric $\rho\Omega_3 + \sigma\Omega_4 + \tau\Omega_5 = 0$ contains those points of F at which the

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tangent planes are on $\xi\Omega_0 + \eta\Omega_1 + \zeta\Omega_2 = 0$ provided that the symmetric bilinear form

$$\rho\zeta + \sigma\eta + \tau\xi + e_1(\sigma\zeta + \tau\eta) + (e_1^2 - e_2)\tau\zeta$$

is zero, and conversely.

8. The surface of intersection of three linearly independent quadrics in [5], having for its prime sections canonical curves of genus 5, was alluded to by Enriques. In the special circumstance of the three quadrics having a common self-polar simplex Σ the surface Φ has Humbert canonical curves (**7**) for its sections by the six bounding primes of Σ and contains an involution J of sets of 32 points. The points of a set are paired in perspective from each vertex of Σ; the join of the vertex X_i ($i = 0, 1, 2, 3, 4, 5$) to any point P of Φ meets Φ again at P_i, the image of P in the harmonic inversion h_i. If P is in $x_i = 0$, the bounding prime of Σ opposite to its vertex X_i, it coincides with P_i, and $X_i P$ is a tangent to Φ at P; the branch curve β_i of h_i on Φ is the section by $x_i = 0$. β_i and β_j have eight intersections—those points on Φ satisfying $x_i = x_j = 0$. There are no points common to three branch curves: the equations

$$x_0 = x_1 = x_2 = \sum\alpha_i x_i^2 = \sum\beta_i x_i^2 = \sum\gamma_i x_i^2 = 0$$

have no common non-zero solution unless $|\alpha_3 \beta_4 \gamma_5| = 0$; and it is precisely such relations, all 20 of them, that are disallowed. There is not to be any quadric cone with a plane vertex containing Φ.

Although Φ does not lie on any quadric cone with a plane vertex, it lies on 15 cones Λ_{ij} with line vertices $X_i X_j$. A solid of either of the two generating systems on Λ_{ij} meets the other quadrics through Φ in a pencil of quadric surfaces and so cuts Φ in an elliptic quartic curve. Either system of generating solids affords a pencil of such curves on Φ, which thus contains 30 pencils of elliptic quartics falling into 15 opposite pairs. Two quartics belonging one to each of two opposite pencils compose a prime section of Φ and have four common points in the plane common to their two solids; this quadrangle of intersections has, if Λ_{ij} is the cone to which the solids belong, two diagonal points at X_i and X_j and the third at the intersection of its plane with that bounding solid of Σ opposite the edge $X_i X_j$.

Not only is the whole of Φ compounded of J; so is its intersection with any quadric for which Σ is self-polar and which does not contain the whole of Φ. Since Σ is self-polar for six linearly independent quadrics of which three contain Φ, the whole system is spanned by the net with Φ for its base surface and by a complementary net with base surface, say, Φ'; the two nets have no common quadric, Φ and Φ' no common point. Thus there is on Φ a net of curves, of order 16, of which each member is compounded of J; it includes the repeated sections of Φ by the faces of Σ, these six sections together forming the Jacobian of the net of curves. Since, in [5], the canonical series on the complete curve of intersection of four surfaces, of respective orders n_1, n_2, n_3, n_4, is cut (**4**, p. 239) by surfaces of order $n_1 + n_2 + n_3 + n_4 - 6$, the canonical series on the curve of order 16 common to Φ and a quadric is cut by quadrics; the canonical sets consist of 32 points and the curve has genus $\frac{1}{2}(32 + 2) = 17$. On each curve of the net there is a pencil of canonical sets which belong to J. Since the curve is projected, from any edge of Σ onto the opposite bounding solid, into an elliptic quartic covered four times, 15 of its 17 linearly independent abelian integrals of the first kind can be taken as elliptic integrals.

9. The properties of Φ are shared by F; but F has additional properties consequent upon its containing lines. These, transforms of one another under the projectivities of G, consist of a line λ, 6 lines λ_i, 15 lines λ_{ij}, and, since, for example, $\lambda_{012} \equiv \lambda_{345}$, 10 lines λ_{ijk}. A suffix confers no privilege: each line is equivalent to any other, G being transitive on the 32 lines. The intersection of λ with $x_i = 0$, being invariant under h_i, is on λ_i; the plane $\lambda\lambda_i$ contains X_i. Each line meets six others, its transforms by the h_i; for instance, λ_{012} meets λ_{12}, λ_{02}, λ_{01}, λ_{45}, λ_{35}, λ_{34}. As λ does not meet any bounding solid of Σ, it does not meet λ_{ij}; still less does it meet λ_{ijk}.

The plane joining $X_i X_j$ to a point P on F meets F further in P_i, P_j, P_{ij}; X_i and X_j are two of the diagonal points of the quadrangle $PP_i P_j P_{ij}$. As P varies on F the plane describes a line-cone L_{ij} with vertex $X_i X_j$; the two systems of generating solids on L_{ij} cut F in pencils of elliptic quartics, but now, in contrast to the uniformity on Φ, four quartics of each pencil are quadrilaterals: for example, the solid joining $X_0 X_1$ to λ also contains λ_0, λ_1, λ_{01}. Two quartics compose a prime section of F when they belong one to each of a pair of opposite pencils, and this is so whether or not either, or both, curves are composite; when both are composite, the prime section consists of the eight lines of a double-four. For example: λ, λ_0, λ_1, λ_{01} and λ_2, λ_{02}, λ_{12}, λ_{012} are composite members of opposite pencils on L_{01}; they compose the double-four

$$\begin{matrix} \lambda & \lambda_{12} & \lambda_{02} & \lambda_{01} \\ \lambda_{012} & \lambda_0 & \lambda_1 & \lambda_2 \end{matrix}$$

To build a double-four one can use any of the 32 lines and any three of its six transversals, so that there are $32 \times 20/8 = 80$ double-fours on F.

10. The Weddle surface W that is **(8)** the projection of F from λ onto a solid Π has its six nodes N_i at the intersections of Π with the planes λX_i; N_i, the projection of X_i from λ, is also the projection of every point on λ_i. Just as the points of F are collinear in pairs with X_i, so the points of W are collinear in pairs with N_i; just as joining a point of F to the vertices of Σ generates a closed set of 32 points on F, so, in accordance with Baker's discovery **(1)**, joining a point of W to its nodes generates a closed set of 32 points on W. This indeed is the true explanation and *raison d'être* of Baker's property.

The line $N_i N_j$ on W is the projection of λ_{ij} on F; the lines λ_{ijk} on F are projected into lines common to planes spanned by complementary triads of nodes of W: for example, $\lambda_{012} \equiv \lambda_{345}$ becomes the line common to $N_0 N_1 N_2$ and $N_3 N_4 N_5$. The twisted cubic through the nodes of W is the locus of intersections of Π with tangent planes to F at points of λ.

This projection of F into W gives, when compounded with a (1, 1) correspondence between W and K, a (1, 1) mapping of the surfaces F and K onto each other. The passage between W and K has been charted frequently: as convenient and succinct a record as any is the table on p. 167 of Hudson's book **(10)**. So it appears that the 16 nodes of K are mapped on F by the 16 lines λ_i and λ_{ijk}, while the conics of contact of K with its 16 tropes are mapped on F by the 16 lines λ_{ij} and λ. This accords precisely with Hudson's nomenclature for the nodes and tropes **(10**, p. 16**)**; indeed, any account of K must inevitably evolve a notation of this kind **(14**, p. 161**)**. The penultimate entry in Hudson's table implies that the section of F by a prime through λ (which is of order 7 and, since the tangent planes to F at the points of λ generate a cubic cone, trisecant to λ) maps a sextic curve on K through those 10 nodes not on the conic mapped by λ. There is a web of these sextics on K; they are the curves discussed on p.157 of **(10)**, and since λ belongs to $80 \times 8/32 = 20$ double-fours on F, the web of sextics on K includes 20 members composed of three conics.

This bisection of the lines on F, lines of one half mapping nodes and those of the other half conics on K, is the same as that associated with the separation of G into its normal subgroup G^+ with its coset G^-; G^- transposes the two halves, so that its operations correspond to the 16 polarities—six null polarities and 10 reciprocations in quadrics—that transform the points of K into its tangent planes, whereas the operations of G^+, conserving each half, correspond to the 16 collineations—identity and 15 biaxial harmonic inversions—for which K is invariant. Familiar properties of K, to say nothing of unfamiliar ones, are mirrored faithfully on F. Just as each node of K lies in six tropes and each trope contains six nodes, so each line on F meets six of the opposite half. Two lines on F that have, on F, a common transversal map two nodes on the same conic or two conics through the same node; if two lines on F do have a common transversal, they have two: if two nodes of K lie on one of its conics, they lie together on two. A double-four on F maps four nodes and four conics, the conics lying in the faces of the tetrahedron whose vertices are the nodes: this is a Rosenhain tetrad, and it is known **(10**, p. 78**)** that there are 80 of them. But how many tetrahedra of tropes are there none of whose vertices is a node? The map of one such is λ, λ_{01}, λ_{23}, λ_{45} with the pairs of suffixes composing one of the 15 synthemes; hence, as is also known **(10**, p. 79**)** in the geometry of K, there are $16 \times 15/4 = 60$ of these Göpel tetrahedra.

11. The cones L_{ij} and their vertices $X_i X_j$ are associated with subgroups g_{ij} of index 2 in G^+. The two reguli on, say, the section of L_{45} by $x_4 = x_5 = 0$ are transposed by each of h_0, h_1, h_2, h_3; each system of solids on L_{45} is invariant under the operations

$$I, \quad h_0 h_1, \quad h_0 h_2, \quad h_0 h_3, \quad h_1 h_2, \quad h_1 h_3, \quad h_2 h_3, \quad h_4 h_5,$$

and these constitute g_{45}. The products of h_4 and of h_5 with any one of h_0, h_1, h_2, h_3 make up the coset of g_{45} in G^+ and transpose the two systems of solids.

The projectivities of g_{45} transform any line on F into one of a set, closed under g_{45}, of eight; there are answering divisions of the nodes of K into complementary octads, of the tropes of K into complementary octahedra. The vertices of each octad are coupled, as are the faces of each octahedron: the solid spanned by the lines on F that map a couple, whether of nodes or of conics, contains $X_4 X_5$.

The conics in the tropes of either octahedron are mapped by lines whose joins to $X_4 X_5$ are solids of the same system; there are four of these solids, one for each couple of tropes, and each of them contains also two lines that map a couple of nodes; each trope of an octahedron contains two nodes of one octad, and therefore four of the complementary octad.

Save for the 32 lines the curves of lowest order on F are the elliptic quartics e in the generating solids of the cones L_{ij}. Each e is skew to any of those 16 lines on F whose join to $X_i X_j$ belongs to the same system as the solid containing e, whereas e intersects each of the other 16 lines. The curve η on K mapped by e therefore passes through all the nodes of one octad but not through any of the complementary octad; it meets the tropes of one octahedron only at four nodes and so is also a quartic, elliptic, of course, with e. There are 30 pencils of curves η on K, each including four pairs of conics, and the pencils consist of 15 associated pairs, "association" implying that the octads on the two curves are complementary (**10**, p. 149). It is these curves η that are the contacts with the 30 systems of quadrics of which Zeuthen (**15**, p. 157) shows K to be the envelope.

12. If p is a point of general position on K, the elliptic quartics through it afford 15 pairs of tangent lines to K at p; each of these pairs is, in the traditional sense, a pair of conjugate directions (**6**, p. 688). Now p is mapped on F by P, the pairs of directions on K at p by pairs of directions on F at P. But each of these latter pairs is a pair of lines of intersection of the tangent plane to F at P with a quadric $\xi\Omega_0 + \eta\Omega_1 + \zeta\Omega_2 = 0$, indeed with L_{ij}:

$$a_i a_j \Omega_0 - (a_i + a_j)\Omega_1 + \Omega_2 = 0.$$

It follows from §6 that these are 15 of those pairs, of directions on F, that were called conjugate; hence the asymptotic curves on K are mapped by the curves that have been called asymptotic on F.

13. The prime sections f of F are canonical curves of genus 5; as f meets every line on F once, it maps a curve ϕ through every node of K and meeting each trope once apart from the six nodes in it. As a trope touches K at every point on its conic that is not a node, ϕ is also octavic, and of genus 5. These curves ϕ are those octavics on K—that pass through all the nodes and of which six are linearly independent—that conclude Hudson's short catalogue of curves on K (**10**, p. 159). If f is composite, so is ϕ; one instance, with f including λ, has already been mentioned, ϕ then being a conic and a sextic (of genus 3) through the 10 nodes not on the conic. A more thorough-going decomposition occurs when f is a double-four; ϕ then consists of the conics in the four faces of a Rosenhain tetrahedron.

Since the prime sections f map octavic curves on K, the plane sections k of K are mapped by octavics (of genus 3) on F. Thus there is on F a web U of curves C, skew to the lines λ_i, λ_{ijk} because k need not contain a node, but having λ, λ_{ij} all for chords because k meets each conic twice. Of course, certain plane sections of K do contain nodes, and the curves C which map them include the corresponding lines; for example, the section of K by a trope is a repeated conic and is mapped by a repeated line, say λ, with its six transversals mapping the nodes on the conic. Another instance worth noting is the trinodal quartic section of K by the plane of three nodes not in the same trope; there are $16.15.6/3.2.1 = 240$ such planes—the faces of the 60 Göpel tetrahedra—and the map C consists of three skew lines, which have no common transversal, and a rational normal quintic of which they are chords.

Now apply an operation of G^-, so transforming U into a web V on F of curves D, also octavics of genus 3 which, in contrast to the curves C, are skew to the lines λ, λ_{ij} but have λ_i, λ_{ijk} for chords; they map a web of curves of genus 3 on K with nodes at every node of K and, not meeting any conic save at nodes, of order 12. If it is h_i that one applies to the map C of the section of K by a plane α, the outcome is the map D of contacts of K with those of its tangent planes that pass through the pole α_i of α in that null polarity which reciprocates K into itself and corresponds to h_i. The locus of these contacts is the intersection of K with the first polar of α_i, a cubic surface through all the nodes of K. Hence, as no two planes α have the same pole, V maps the curves in which K is cut by the whole web of its first polars.

14. Each of U and V includes 16 curves composed of six lines and a repeated transversal; call these the special members of the two webs. Since any irreducible curve of either web has six lines of each special curve of the other as chords, and is skew to their transversal, two curves C, D have 12 intersections. In the particular instance of their being harmonic inverses in h_i, eight of these are in $x_i = 0$; the remaining four are collinear in pairs with X_i, C and D being projected from the plane of these two chords into the same plane quartic.

It will be seen below that any two curves C, D make up the intersection of F with a quadric; as they are both of genus 3 and have 12 intersections, the composite curve $C + D$ has (**9**, p. 402; **4**, p. 215) the genus $3 + 3 + 12 - 1 = 17$ that the intersection of F with a quadric is known to have.

One quadric cutting F in any pair C, D that are both special is salient: it is a cone whose vertex is the space spanned by the repeated components. If these are skew, they span a solid; the cone is a pair of primes both of which meet F in double-fours. An example is

$$C: \quad 2\lambda + \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5,$$

$$D: \quad 2\lambda_{012} + \lambda_{12} + \lambda_{02} + \lambda_{01} + \lambda_{45} + \lambda_{35} + \lambda_{34},$$

which together make up the double-fours

$$\begin{matrix} \lambda & \lambda_{12} & \lambda_{02} & \lambda_{01} \\ \lambda_{012} & \lambda_0 & \lambda_1 & \lambda_2 \end{matrix} \quad \text{and} \quad \begin{matrix} \lambda & \lambda_{45} & \lambda_{35} & \lambda_{34} \\ \lambda_{012} & \lambda_3 & \lambda_4 & \lambda_5 \end{matrix}$$

in primes which both contain the solid $\lambda\lambda_{012}$. If, on the other hand, the repeated components are incident, let C be as above, with

$$D: \qquad 2\lambda_0 + \lambda + \lambda_{01} + \lambda_{02} + \lambda_{03} + \lambda_{04} + \lambda_{05}.$$

Then C, D compose the intersection of F with that quadric cone whose vertex is the plane $\lambda\lambda_0$ and which contains the five vertices of Σ other than X_0 (which it contains already in its own vertex). There is one such cone, just as there is one conic through five coplanar points; its generating solid $\lambda\lambda_0 X_i$ contains both λ_i and λ_{0i}.

Either U or V can be spanned by four of its special curves; for example

$$\begin{aligned} &2\lambda_0 + \lambda + \lambda_{01} + \lambda_{02} + \lambda_{03} + \lambda_{04} + \lambda_{05}, \\ &2\lambda_1 + \lambda_{01} + \lambda + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15}, \\ &2\lambda_2 + \lambda_{02} + \lambda_{12} + \lambda + \lambda_{23} + \lambda_{24} + \lambda_{25}, \\ &2\lambda_{012} + \lambda_{12} + \lambda_{02} + \lambda_{01} + \lambda_{45} + \lambda_{35} + \lambda_{34} \end{aligned}$$

are linearly independent, so spanning V. For, the third curve, since it does not include λ_{01}, is not in the pencil spanned by the first two while the fourth, lacking λ common to the other three, is not in the net spanned by them. It now follows that any curve of either U or V makes up the complete intersection, of F and a quadric, with any special member of the other web; and then that any C and any D are on a quadric not containing the whole surface. Among the quadrics through C and D is always one that circumscribes Σ, and each web is cut on F by quadrics through any member of the other.

15. The prime sections f of F, as observed in §13, map octavic curves ϕ of genus 5 through all the nodes of K; two of these, ϕ and ϕ', meet in those eight points mapped on F by its intersections with the solid common to the primes containing f and f'.

The postulation (**9**, p. 537) of ϕ for quartic surfaces Φ is 28, so that it lies on 7 linearly independent ones. These cut, apart from the 24 common points of ϕ and ϕ', the canonical series (**9**, p. 534) of freedom 4 on ϕ', so that only 5 more linear conditions are necessary, once Φ already contains ϕ, for it to contain ϕ' also. Hence there are two linearly independent Φ, and so a pencil, through both ϕ and ϕ'; K is itself one of these. Any two octavics ϕ, ϕ' form the complete intersection of K with a quartic surface through all its nodes; and, more particularly, there is a quartic surface touching K all along ϕ. This is no novelty, but such results are generally obtained by using Θ-functions.

Any quadric is, in many ways, a linear combination of prime pairs; hence the intersection of F and any quadric not containing the whole surface maps the intersection of K with a quartic Ψ through all 16 nodes. It is when the quadric is itself a prime pair that the curve common to K and Ψ consists of octavics ϕ, ϕ'; when the quadric is a repeated prime, Ψ touches K all along a curve ϕ.

Conversely, the intersection of K with any quartic Ψ through all 16 nodes is mapped by the intersection of F with a quadric, for the linear systems of curves, on K and on F, have the same freedom or dimension. There are $35 - 16 = 19$ linearly independent Ψ and, K being among them, they cut a system of freedom 17 on K. And there are 21 linearly independent quadrics in [5]; as F is on three of these, they cut a system also of freedom 17. This system includes the net, cut on F by the quadrics $\rho\Omega_3 + \sigma\Omega_4 + \tau\Omega_5 = 0$, every one of whose individual curves is invariant under G and which includes all the asymptotic curves; the six principal asymptotic curves are repeated members of this net.

16. When a Kummer surface is specialized so as to be a tetrahedroid K_1, four of the Göpel tetrads collapse to lie in faces f_1, f_2, f_3, f_4 of a tetrahedron T; each f_j meets K_1 in a pair of conics γ_j, γ'_j whose intersections are the nodes of K_1 in f_j, and each vertex v_j of T is a concurrence of four tropes of a collapsed Göpel tetrahedron. The tropes through v_j meet the opposite face f_j in the common tangents of γ_j and γ'_j; the six nodes in any one of these tropes, lying in pairs on its intersections with the three faces of T through v_j, are in involution on the conic of contact with K_1.

These particularities are mapped on the corresponding surface F_1 in [5]; one expects a curve D to consist of, say, λ, λ_{01}, λ_{23}, λ_{45} and of two conics each meeting all four lines. There will be four such curves D, accounting for λ and every λ_{ij}, as well as four C accounting for all λ_i, λ_{ijk}. The presence of conics on F_1 is effected by the splitting of certain rational normal quintics (see §13) into two conics and a line intersecting both of them.

The projective feature of F_1 consequent on the specialization is, simply, that the intersections of any of its 32 lines with its six transversals are paired in an involution. As the line (2.1) meets its transversals where $t = -a_i$, one can now take

$$a_0 + a_1 = a_2 + a_3 = a_4 + a_5 = 0$$

and assert that F_1 is defined by

$$\begin{aligned}
\Omega_0 &\equiv x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2, \\
\Omega_1 &\equiv a(x_0^2 - x_1^2) + b(x_2^2 - x_3^2) + c(x_4^2 - x_5^2), \\
\Omega_2 &\equiv a^2(x_0^2 + x_1^2) + b^2(x_2^2 + x_3^2) + c^2(x_4^2 + x_5^2).
\end{aligned}$$

The existence of eight conics on F is patent, since the plane

$$x_0 + ix_1 = x_2 + ix_3 = x_4 + ix_5 = 0$$

with i, not now in demand as an index of summation, denoting the complex square root of -1, lies on both Ω_0 and Ω_2 and so meets F_1 in the conic in which it meets Ω_1. The same holds for the other seven planes that occur on sign changes of i. Baker (**3**, p. 218) remarks that a quadratic complex whose singular surface is a tetrahedroid is given by such an equation $\Omega_1 = 0$, but he does not draw any deduction from this, nor so much as mention the consequent form of Ω_2. F_1 is invariant not only under G but also under the transpositions of x_0 with x_1, x_2 with x_3, x_4 with x_5, and any projectivities that result from combining these (cf. **10**, p. 217).

17. The surface F_4 that is the non-singular model of a quadruple tetrahedroid K_4 may also be given by convenient equations. The nodes in a trope of K_4 are paired in four different involutions on their conic, as are the vertices of a regular hexagon on its circumcircle; the coefficients a_i can then be the sixth roots of unity. If $\omega = \exp(2\pi i/3)$, F_4 is the intersection of the quadrics

$$\begin{aligned}\Omega_0 &\equiv x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,\\ \Omega_1 &\equiv x_0^2 - \omega x_1^2 + \omega^2 x_2^2 - x_3^2 + \omega x_4^2 - \omega^2 x_5^2 = 0,\\ \Omega_2 &\equiv x_0^2 + \omega^2 x_1^2 + \omega x_2^2 + x_3^2 + \omega^2 x_4^2 + \omega x_5^2 = 0,\end{aligned}$$

and is invariant not only under G but also under the projectivity c of period 6 that imposes the cyclic permutation (012345) on the x_i. Thus F_4 is invariant under a group G^* of 192 projectivities having G for a normal subgroup: for $h_i c = c h_{i+1}$, suffixes being reduced, where necessary, modulo 6. The involution c^3 commutes, since $h_i c^3 = c^3 h_{i+3}$, with $h_0 h_3$, $h_1 h_4$, $h_2 h_5$; these four mutually commutative involutions generate an elementary abelian group, normal in G^*, of order 8.

As with F_1, so analogously with F_4; the 8 planes

$$x_0^2 + x_3^2 = x_1^2 + x_4^2 = x_2^2 + x_5^2 = 0$$

are on both Ω_0 and Ω_2, so that the conics in which they meet Ω_1 are on F_4. The other 24 conics on F_4 are in the following planes:

(i) $x_0^2 - x_3^2 = x_1^2 - \omega x_2^2 = x_4^2 - \omega x_5^2 = 0$, common to $\Omega_1 = 0$ and $\Omega_0 = \Omega_2$;

(ii) $x_1^2 - x_4^2 = x_2^2 - \omega x_3^2 = x_5^2 - \omega x_0^2 = 0$, common to $\Omega_1 = 0$ and $\Omega_0 = \omega\Omega_2$;

(iii) $x_2^2 - x_5^2 = x_3^2 - \omega x_4^2 = x_0^2 - \omega x_1^2 = 0$, common to $\Omega_1 = 0$ and $\Omega_0 = \omega^2\Omega_2$.

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