COORDINATE SYSTEMS IN THE GENERAL RELATIVISTIC FRAMEWORK

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ABSTRACT. The treatment of the coordinate systems is briefly reviewed in the Newtonian mechanics, in the special theory of relativity, and in the general relativistic theory, respectively. Some reference frames and coordinate systems proposed within the general relativistic framework are introduced. With use of the ideas on which these coordinate systems are based, the proper reference frame comoving with a system of mass-points is defined as a general relativistic extension of the relative coordinate system in the Newtonian mechanics. The coordinate transformation connecting this and the background coordinate systems is presented explicitly in the post-Newtonian formalism. The conversion formulas of some physical quantities caused by this coordirate transformation are discussed. The concept of the rotating coordinate system is reexamined within the relativistic framework. A modification of the introduced proper reference frame is proposed as the basic coordinate system in the astrometry. The relation between the solar system barycentric coordinate system and the terrestrial coordinate system is given explicitly.

1. INTRODUCTION

The recent advances in the astrometry using the modern techniques such as the VLBI, the SLR, the LLR and so on, now require the method of analysis for such precise observations to be strict and correct up to the order of 10^{-8} at least. One of the factors to be considered in reply to this requirement is the introduction of the general relativistic treatments into the whole procedure of analysis. Among them the construction of the reference frames and the coordinate systems within the relativistic framework is important since all the position and the velocity of heavenly bodies and observers, which are the objects of the astrometry, are dependent on the reference frame and the coordinate system which are chosen.

Attention must be paid to the difference between the reference frame and the coordinate system. Mathematically the reference frame

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means a field of the set of base vectors while the coordinate system does the way to assign a set of numbers named the coordinates to each point in the space. In a flat space the set of base vectors is taken to be independent on the coordinates. In this case, to choose the reference frame is to choose the set of base vectors at the coordinate origin. This is no other than to give the coordinate system if the differences in the unit system are ignored. Therefore the reference frame has been regarded to be the same as the coordinate system by many authors. However the reference frame and the coordinate system should be discriminated in a distorted space. The former can be derived from the latter if the base vectors at a point are defined to be the tangent vectors of the coordinate grids of the chosen coordinate system at the same point. Such reference frame is called the reference frame accompanied with the coordinate system. In this paper we discuss only the coordinate systems and the reference frames accompanied with them.

As is shown in Table 1, the following six coordinate systems are frequently used in the astrometry:

1) the extragalactic coordinate system in which quasars may be located,
2) the galactic coordinate system to which the most star catalogs are referred,
3) the solar system barycentric coordinate system where the planetary and lunar ephemerides are obtained,
4) the terrestrial coordinate system where the motion of artificial satellites are discussed,
5) the terrestrial rotating coordinate system in which the station coordinates are written, and
6) the observer's coordinate system where the observables are measured by means of the non-gravitational physics.

One of the distinctive features of the group of coordinate systems is the hierarchy of the systems defining these coordinate systems. For example, the Earth is a member of the solar system and the solar system is a subsystem of the galaxy. In terms of the Newtonian mechanics, the coordinate system comoving with the smaller system is thought as a relative coordinate system when that comoving with the larger system is regarded as the absolute one. We must seek a general relativistic extension of the concept of relative coordinate system in order to define these coordinate systems correctly in the general relativistic sense.

Another distinctive feature is that there are two types of coordinate system; the one is the non-rotating coordinate system and the other is the rotating coordinate system. Although what the term 'rotation' means is well understood in the language of the Newtonian mechanics, its meaning seems not clear when the general relativity is introduced.

In the present paper, we try to solve these questions. Namely, we present a candidate for the basic coordinate system in the astrometry within the framework of post-Newtonian approximation of the general relativistic theory. We also give a general relativistic definition of the rotating coordinate system. To obtain a suitable unit system is essential in constructing the practical coordinate system. In this

Object	Coordinate System	Means of Observation
Universe	Cosmological	
Quasar	Extra-Galactic	VLBI
Star	Galactic	HIPPARCOS Photoelectric Meridian Circle
Planets/Moon	Solar System Barycentric	Planetary Mission/LLR
LAGEOS/GPS	Terrestrial	SLR/Satellite VLBI
Station	Terrestrial Rotating	Land Survey
Observer	Observer's	Atomic Clock/Laser Gyroscope

Table 1 Hierarchy of Coordinate Systems in Astrometry/Geodesy.

Background Coordinate System ⇨ Local Coordinate System

Non-Rotating Coordinate System ➡ Rotating Coordinate System

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paper, however, we restrict ourselves to the discussion on the features of the coordinate systems being independent on the choice of the unit system. The system of units is discussed from the general relativistic point of view in another paper of the authors (Fukushima et al., 1986).

In Sections 2, 3, and 4, an overview on the coordinate systems is given in the Newtonian mechanics, in the special theory of relativity, and in the general relativistic theory, respectively. Also in Section 4, some reference frames and coordinate systems proposed in the general relativistic theory are reviewed. In Section 5, the proper reference frame (PRF) of a system of masspoints is introduced as a general relativistic extension of the relative coordinate system stated above. In Section 6, the coordinate transformation defining the PRF is explicitly given. In Section 7, the conversion formulas of some physical quantities due to this coordinate transformation are obtained in the post-Newtonian framework. In Section 8, the concept of rigidly rotating coordinate system is reexamined in the relativistic framework. In Section 9, a modification of the PRF defined in Section 5 is proposed as the basic coordinate system in the astrometry. The difference between this modified coordinate system and the PRF is investigated extensively. In Section 10, the coordinate transformation and other transformation formulas are given to describe the relation between the solar system barycentric coordinate system and the terrestrial coordinate system.

As for the relativistic quantities, we use the same notations as those in the textbook of Misner et al. (1970, hereafter cited as MTW).

2. COORDINATE SYSTEMS IN THE NEWTONIAN MECHANICS

In the Newtonian mechanics, the time is absolute. In other words, it is assumed that the character of the time does not depend upon the chosen coordinate system. Then the time interval of any pair of events is constant in whichever coordinate system it is measured. On the other hand, the space is relative. Namely the positional relation of the pair of events depends on the chosen coordinate system. From these it is clear that no coordinate transformation is allowed to make the new time coordinate depend on the old space coordinates. Consequently the term 'coordinate system' is used only in the sense it determines the way to assign a set of three numbers (named the 3-dimensional spatial coordinates) to each point in the space.

It is known that there exist a group of special coordinate systems where the space is homogeneous and isotropic globally in the Newtonian mechanics. Such coordinate systems are called the inertial coordinate systems. Many people roughly call them the inertial frames. The inertial coordinate systems are important because the laws of the Newtonian mechanics are never changed in whichever inertial coordinate system they are expressed. This is the Galilei's principle of relativity.

The space in the Newtonian mechanics is measured by the Euclidean distance. The coordinate transformation relating one inertial coordinate system to another is limited to be a uniform parallel translation (the Galilean transformation) plus a time-independent spatial rotation. Such a transformation is called the static affine transformation.

As any spatial coordinate transformation which conserves the Euclidean distance is composed to be a combination of the translation and the rotation, all coordinate systems are classified to the three categories; 1) the inertial coordinate system, 2) the non-rotating accelerated coordinate system which is obtained from the inertial coordinate system via a time-dependent parallel translation only, 3) the rigidly rotating accelerated coordinate system which is obtained from the non-rotating one by applying a time-dependent rigid rotation.

Practically the inertial coordinate system cannot be realized in the laboratory. Therefore we must know the relation between the inertial and the non-inertial coordinate systems. In the Newtonian mechanics, a sufficiently general coordinate transformation is given as

$$x^j = x^j{}_0(t) + x^{\hat{m}} R^j{}_{\hat{m}}(t) \qquad (2\text{-}1)$$

where x^j and $x^{\hat{m}}$ are the space coordinates of the inertial and the non-inertial coordinate systems, respectively. The term $x^j{}_0(t)$ represents the time-dependent translational motion of the space origin of the non-inertial coordinate system expressed in the inertial coordinate system. The matrix R is the matrix of the rigid rotation, which is obtained from

$$dR^j{}_{\hat{m}}/dt = -\Omega^j{}_i(t) R^i{}_{\hat{m}} \qquad (2\text{-}2)$$

where Ω is the angular velocity tensor of the rigid rotation, which is anti-symmetric and is dependent on the time coordinate only. Usually the rotational matrix R is expressed in terms of the Euler's angles (Goldstein, 1980). The conversion formulas of various physical quantities caused by this coordinate transformation are seen in any textbook on the classical mechanics (Landau and Lifshitz, 1973, for example).

3. COORDINATE SYSTEMS IN THE SPECIAL THEORY OF RELATIVITY

In the special theory of relativity, the time is no longer absolute. The coordinate system as well as the coordinate transformation becomes to be four dimensional inevitably. Any type of coordinate transformation between the coordinate systems is allowed in the special theory of relativity. It should be noted, however, that the absoluteness of the inertial coordinate systems still remains. Namely, there exist a number of inertial coordinate systems also in the special theory of relativity. They are transformed from one to another by only a limited type of coordinate transformation, as the principle of special relativity tells. Such a coordinate transformation is called the Poincaré transformation. The Poincaré transformation is a combination of a uniform parallel translation, a constant boost, and a time-independent, rigid space rotation (MTW, Box 2.4). Here the word 'boost' denotes the pure Lorentz transformation. The Poincaré transformation is expressed as

$$x^\mu = x^\mu{}_0 + x^{\tilde{\alpha}} L^\mu{}_{\tilde{\alpha}} \qquad (3\text{-}1)$$

where x^μ and $x^{\tilde{\alpha}}$ are the 4-dimensional coordinates of the same event in the old and the new inertial coordinate systems, respectively, and L is the general Lorentz matrix. The coordinates of the origin of the old coordinate system x^{μ}_{0} are constant in the new coordinate system. The general Lorentz matrix in the above expression is the function of six parameters (three Euler's angles and three components of the boost velocity) only and is never dependent on the coordinates.

If the spatial rotation is ignored, the Lorentz matrix is given as

$$L^{0}_{0} = \{1-(\mathbf{v}/c)^{2}\}^{-1/2}, \tag{3-2}$$

$$L^{m}_{0} = L^{0}_{m} = (v^{m}/c)\{1-(\mathbf{v}/c)^{2}\}^{-1/2}, \tag{3-3}$$

$$L^{i}_{m} = [\ \underset{\sim}{1} + \mathbf{v} \otimes \mathbf{v}\ \{1-(\mathbf{v}/c)^{2}\}^{-1/2} \ /\ [\ 1 + \{1-(\mathbf{v}/c)^{2}\}^{1/2}\ ]/c^{2}\ ]^{i}_{m} \tag{3-4}$$

where c is the speed of light, **v** is the coordinate velocity of the origin of the old inertial coordinate system evaluated in the new inertial coordinate system, **a** $\otimes$ **b** is the dyadic of vectors **a** and **b**, and $\underset{\sim}{1}$ is the 3-dimensional unit tensor.

The effects of Poincaré transformation on the physical quantities such as the velocity and so on are seen in any text book on the special relativity (Møller, 1952, for example). Among them, the conversion formulas of the time interval and the space distance are important in the astrometry and the geodesy. For example, the time-space component (3-3) produces the aberration in terms of the astronomy (MTW, Box 2.4). The time-time component (3-2) shows that the moving clock ticks slower than the clock at rest. This is the well-known Lorentzian time dilatation (Landau and Lifshitz, 1962, Section 3). Also the space-space component (3-4) shows that the coordinate grids of a moving coordinate system suffer a deformation consisting of the shear and the anisotropic expansion. The isotropic part of the deformation is the well-known Lorentzian contraction (Landau and Lifshitz, 1962, Section 4).

4. COORDINATE SYSTEMS IN THE GENERAL RELATIVISTIC THEORY

In the general relativistic theory, there exists no inertial coordinate system. This is because 1) the geometric character of the spacetime (i.e. the metric tensor) is not proper to the spacetime itself but dependent on the physical phenomena in the spacetime; 2) the metric tensor has ten freedoms so that it is never transformed into the Minkowskian metric tensor all over the spacetime by any choice of a coordinate transformation which has only four freedoms. The metric tensor is transformed to be a Minkowskian metric tensor only at one event by a suitable choice of the coordinate system. Thus any inertial coordinate system is local in the general relativity. Here the term 'local' means being valid only at one event in the spacetime. That is, the inertial coordinate system is no more than a reference frame in the general relativity. Such a reference frame is called a locally Lorentzian frame. Some people call it a locally Minkowskian frame.

In place of the inertial coordinate systems, any coordinate system can be chosen as the basic coordinate system to construct a dynamical theory in the general relativistic theory. This is because whatever choice of the coordinate system does not change the expression of the physical laws. This is the well-known principle of general relativity. In Table 2, we summarize the basic items relating with the coordinate systems in the Newtonian Mechanics, in the special theory of relativity, and in the general relativistic theory.

Then a general coordinate transformation which can connect any pair of coordinate systems should be investigated. However a ridiculous coordinate transformation or an extraordinary coordinate system is not only rarely proposed but of little use. Rather the generalization of the coordinate transformations and the coordinate systems stated in the preceding two sections is useful for the practical purpose, and has been done in fact. To see this, we illustrate in the followings some reference frames and coordinate systems which has been proposed in the general relativistic framework. We remark that this is never a comprehensive survey and we apologize the authors of the books we referred if we make any mistake in understanding their works.

4.1. Natural Frame and Proper Frame

In his textbook of the astrometry, Murray gives a locally Lorentzian frame attached to the observer in the spacetime with a spherically symmetric metric in order to express the observed direction of the incident light (Murray, 1983). The frame is named the natural frame when the observer is at rest, and is named the proper frame when the observer is moving relative to the frame. The relation between the natural and the proper frames is the boost of the observer's velocity. The resulting transformation formula of the base vectors is just the same as that of the Lorentz transformation given in the previous section. The relation between the natural frame and the background coordinate system of a spherically symmetric spacetime is obtained by the coordinate transformation which transforms the spherically symmetric metric to the Minkowskian metric. The tetrad (a set of four base vectors) of the natural frame is expressed as

$$e^{0}{}_{\underset{\sim}{0}} = (-g_{00})^{-1/2},\ e^{0}{}_{\underset{\sim}{m}} = e^{m}{}_{\underset{\sim}{0}} = 0,\ e^{i}{}_{\underset{\sim}{m}} = (g_{mm})^{-1/2}\delta^{i}{}_{m} \qquad (4\text{-}1)$$

where $e^{\mu}{}_{\underset{\sim}{\alpha}}$ is the μ-th component of the α-th base vector expressed in the background coordinate system. The coordinate transformation producing the tetrad contains neither a boost nor a spatial rotation.

We note that both natural and proper frames are reference frames and are never coordinate systems. Namely they can not be used in describing the phenomena which cover a finite region.

4.2. Local Quasi-Cartesian Coordinate System

In his text book on the general relativity, Will provides an asymptotically Minkowskian coordinate system of the solar system to give a basic coordinate system where his post-Newtonian formalism is developed

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Item / Theory	Newtonian Mechanics	Special Relativity	General Relativity
Time	Absolute	Relative	Relative
Space	Relative	Relative	Relative
Principle of Relativity	Galilean	Special	General
Metric	Euclidean	Minkowskian	Riemannian
Basic Coordinate System	Inertial	Inertial	Any
Coordinate Transformation	Static Affine	Poincaré	General
Transformation Matrix	Rotational	Lorentz	General
Coordinate System = Reference Frame ?	Yes	Yes	No

Table 2 Comparison of Basic Items relating to Coordinate System

(Will, 1981). He starts a discussion from the background coordinate system named 'the universe rest frame', which is defined as the coordinate system where the universe appears almost isotropic. He assumes that the metric tensor consists of the Robertson-Walker part and the perturbation part. Then he eliminates the former by a coordinate transformation like (4-1). The resulting coordinate system is called a local quasi-Cartesian coordinate system by Will.

This procedure assures the existence of the coordinate system where the metric approaches to the Minkowskian metric sufficiently far from the origin. Such a coordinate system is frequently called the parametrized post-Newtonian (PPN) coordinate system. In other words, the PPN coordinate system is the coordinate system where the metric tensor is expressed as the PPN metric tensor (Will, Section 4.2).

4.3. Fermi Coordinate System and Optical Coordinate System

Synge explains a different approach to construct a coordinate system in the general relativity (Synge, 1960). First he thinks that a reference frame is equivalent with a tetrad. Then he shows that a tetrad transported by the Fermi-Walker transportation law along the world line of an observer leads to the correct relativistic generalization of the Newtonian concept of the non-rotating reference frame comoving with the observer. It is noted that the tetrad of Murray's natural frame is not a non-rotating tetrad defined by Synge. The Fermi coordinate system he gives there is not a reference frame. It consists of the Fermi-Walker transported tetrad and three space coordinate axes which are the space-like geodesics starting from the space coordinate origin with the initial direction being same as the corresponding base vector.

Also he proposes the optical coordinate system as a candidate for the practical coordinate system. It is different from the Fermi coordinate system only in the point that the space coordinates are defined by means of the null geodesics in place of the space-like geodesics. However this coordinate system is not so practical as he thought because the expression of the equation of motion for the slowly moving bodies is too complicated in this coordinate system.

4.4. Proper Reference Frame

Misner et al. define the Fermi coordinate system in a clear way (MTW, Section 13.6). They extend the concept of the Fermi coordinate system to the case the tetrad suffers a spatial rotation also. The resulting coordinate system is named the proper reference frame (PRF) although it is a coordinate system. The non-rotating PRF is just the same as the Fermi coordinate system. It is remarked that the PRF is not an extension of the proper frame of Murray. Misner et al. also give a precise way to construct the coordinate grids of the PRF. Furthermore they give the approximate form of the metric tensor in the PRF.

The PRF is the most appropriate coordinate system for the observer resting at its space origin, because 1) the time coordinate is the proper time of the clock he carries, 2) the spatial distance from him is the proper length, say, measured indirectly by the laser ranging system, and 3) the spatial direction of incident light he observes is referred to the 3-dimensional gyroscope comoving with him. Note that the PRF defined by Misner et al. is to be comoving with a massless observer.

5. PROPER REFERENCE FRAME OF A SYSTEM OF MASSPOINTS

As is stated in Subsection 4.4, the proper reference frame (PRF) seems a suitable coordinate system on which the astrometry is constructed. However there is a problem that the PRF is defined only when it comoves with a massless observer. The coordinate systems comoving with a massive body such as the Earth or comoving with a system of masspoints such as the solar system are required in the astrometry.

The Will's approach seen in Subsection 4.2 becomes a great help to extend the concept of the PRF into the case it comoves with a massive body. Namely we divide the metric tensor of the background spacetime as

$$g = g^{*} + g^{+} \tag{5-1}$$

where g is the full metric tensor, g^{+} is the direct contribution of the body with which the desired coordinate system is comoving, and g^{*} is the rest part. Then we construct a PRF in the same manner as Misner et al. did, while g^{*} is used in place of g. In other words, we define the PRF of a massive body as that of a massless observer locating at the barycenter of the body in a fictitious spacetime whose metric tensor is given by g^{*}. The introduction of such an imaginary spacetime is permitted because we use this spacetime only to obtain a coordinate system, and because any coordinate system can be chosen as the basic coordinate system whether it is realistic or fictitious. Clearly the coordinate system defined above is an extension of the PRF of a massless observer since g^{*} coincides with g when the mass of the body reduces to zero.

Furthermore this approach matches well to the assumption that the planets and the sun move on geodesics (MTW, Section 40.9). This assumption is restated as 'The worldline of a massive body in the background spacetime is obtained as a geodesic in a fictitious spacetime whose metric is given as g^{*}'. Note that this assumption is the base of the post-Newtonian many-body equations of motion widely used to create the latest planetary and lunar ephemerides (MTW, Exercise 39.15; Standish and Williams, 1982; JHD, 1984). This correspondence of our definition of the PRF to the geodesic motion assumption shows how g should be separated into g^{*} and g^{+} in constructing the PRF of a massive body.

In the following two sections, we present the coordinate transformation which defines the PRF mathematically and give the resulting conversion formulas of some physical quantities.

6. MATHEMATICAL FORMULATION OF THE PROPER REFERENCE FRAME

The coordinate transformation from the background coordinate system to the PRF of a massive body (abridged as the central body) is given as

$$x^{\mu} = x^{\mu}_{0}(x^{\tilde{0}}) + x^{\tilde{m}}\, e^{\mu}_{\tilde{m}}(x^{\tilde{0}}) + \delta x^{\mu}(x^{\tilde{\alpha}}) \tag{6-1}$$

where x^{μ} and $x^{\tilde{\alpha}}$ are the coordinates of the same event in the background coordinate system and in the PRF of the central body, respectively, $e^{\mu}_{\tilde{\alpha}}$ is the tetrad of the PRF, and δx^{μ} is the deviation of a space-like geodesic from a straight line. The explicit expression of (6-1) is given in the following five stages.

6.1. Metric Tensor

First of all, the expressions of the metric tensor which will be used in the following subsections are given. The post-Newtonian expansion of the metric tensor which does not rotate in the Newtonian limit is written in the background coordinate system as

$$\begin{cases} g_{00} = -1 + 2\phi/c^2 + 2\psi/c^4, \\ g_{0j} = (\mathbf{g}/c^3)_j, \qquad g_{ij} = (\underset{\sim}{1} + 2\underset{\sim}{\chi}/c^2)_{ij} \end{cases} \tag{6-2}$$

where ϕ is called the Newtonian force function, ψ is called the non-linear part of the scalar force function, $\mathbf{g}$ is called the vector force function, and $\underset{\sim}{\chi}$ is called the tensor force function. Here the term 'force function' means the negative gravitational potential. The force functions are sometimes called the (gravitational) potentials loosely. It is well known that the expression of the tensor force function becomes simple in the PPN coordinate system as

$$\underset{\sim}{\chi} = \gamma\,\phi\,\underset{\sim}{1} \tag{6-3}$$

where γ is one of the universal constants named the PPN parameters.

The expression (6-2) is separated into g^{*} and g^{+} in the case of the Einstein-Infeld-Hoffmann(EIH) metric tensor as

$$\begin{cases} g^{*}_{00} = -1 + 2\phi^{*}/c^2 + 2\psi^{*}/c^4, \\ g^{*}_{0j} = (\mathbf{g}^{*}/c^3)_j, \qquad g^{*}_{ij} = (\underset{\sim}{1} + 2\underset{\sim}{\chi}^{*}/c^2)_{ij}, \end{cases} \tag{6-4}$$

$$\begin{cases} g^{+}_{00} = 2\phi^{+}/c^2 + 2\psi^{+}/c^4, \\ g^{+}_{0j} = (\mathbf{g}^{+}/c^3)_j, \qquad g^{+}_{ij} = 2\chi^{+}_{ij}/c^2 \end{cases} \tag{6-5}$$

where

$$\phi^{*} = \sum_{J\neq 0} GM_J/r_J, \tag{6-6}$$

$$\begin{aligned} \psi^{*} = &- \Big(\sum_{J\neq 0} GM_J/r_J\Big)^2 + 2\sum_{J\neq 0} GM_J v_J^2/r_J \\ &- \sum_{J\neq 0}\sum_{K\neq J} GM_J GM_K/(r_J r_{JK}) - \sum_{J\neq 0} GM_J(\mathbf{r}_J\cdot\mathbf{v}_J)^2/(2r_J^3) \\ &- \sum_{J\neq 0}\sum_{K\neq J} GM_J GM_K(\mathbf{r}_J\cdot\mathbf{r}_{JK})/(2r_J r_{JK}^3), \end{aligned} \tag{6-7}$$

$$\mathbf{g}^{*} = -4\sum_{J\neq 0} GM_J \mathbf{v}_J/r_J, \tag{6-8}$$

$$\phi^{+} = GM_0/r_0, \tag{6-9}$$

$$\psi^{+} = (GM_0/r_0)[\, - GM_0/r_0 + 2v_0^2 - (\mathbf{r}_0\cdot\mathbf{v}_0/r_0)^2/2 - \sum_{J\neq 0}\{\, 2GM_J/r_J + GM_J/r_{0J} - GM_J(\mathbf{r}_{0J}\cdot\mathbf{r}_0)/(2r_{0J}^3) \,\}\,], \tag{6-10}$$

$$\mathbf{g}^{+} = -\, 4\, GM_0\mathbf{v}_0/r_0 \tag{6-11}$$

Here

$$\mathbf{r}_J = \mathbf{x} - \mathbf{x}_J, \quad \mathbf{r}_{JK} = \mathbf{x}_J - \mathbf{x}_K, \quad \mathbf{v}_J = d\mathbf{x}_J/dt,$$

$$r_J = |\, \mathbf{r}_J \,|, \qquad r_{JK} = |\, \mathbf{r}_{JK} \,|$$

and G is the constant of gravitation, the suffix 0 denotes the central body, M_J is the mass of the body J, and $\mathbf{x}$ and $\mathbf{x}_J$ are the position of the considered point and the body J, respectively.

6.2. Time Coordinate

From the definition of the PRF (MTW, Section 13.6),

$$\tau = x^{\hat{0}}/c \tag{6-12}$$

is the proper time of the central body, namely the proper time of the clock comoving with the barycenter of the central body in the fictitious spacetime discussed in the previous section. The relation between τ and the coordinate time at the barycenter of the central body

$$t = x_0^0/c \tag{6-13}$$

in the fictitious spacetime is obtained by solving the following equation of proper time

$$d\tau/dt = 1 - (\, \phi_0^{*} + v_0^2/2 \,)/c^2 - [\, (\phi_0^{*})^2/2 + \phi_0^{*}v_0^2/2 + v_0^4/8 + \psi_0^{*} + \mathbf{g}_0^{*}\cdot\mathbf{v}_0 + \mathbf{v}_0\cdot\underset{\sim}{\chi}_0^{*}\mathbf{v}_0 \,]/c^4 \tag{6-14}$$

Here the gravitational potentials with the suffix 0 are those evaluated at the central body. In the EIH metric, they are written as

$$\phi_0^{*} = \sum_{J\neq 0} GM_J/r_{0J}, \tag{6-15}$$

$$\psi_0^{*} = -\,(\sum_{J\neq 0} GM_J/r_{0J})^2 + 2\sum_{J\neq 0} GM_J v_J^2/r_{0J} - \sum_{J\neq 0}\sum_{K\neq J} GM_J GM_K/(r_{0J}r_{JK}) - \sum_{J\neq 0} GM_J(\mathbf{r}_{0J}\cdot\mathbf{v}_J)^2/(2r_{0J}^3) - \sum_{J\neq 0}\sum_{K\neq J} GM_J GM_K(\mathbf{r}_{0J}\cdot\mathbf{r}_{JK})/(2r_{0J}r_{JK}^3), \tag{6-16}$$

$$g_0^* = -4 \sum_{J \neq 0} GM_J v_J / r_{0J}, \tag{6-17}$$

$$\underset{\sim}{\chi}_0^* = \phi_0^* \underset{\sim}{1} \tag{6-18}$$

Recall that the direct contribution of the self gravitational field of the central body g^+ is ignored in the equation (6-14).

The resulting function $\tau(t)$ or its inverse $t(\tau)$ is explicitly obtained in the case the central body is the Earth and the background coordinate system is the PPN coordinate system of the solar system mentioned in Section 4 (Moyer, 1981; Hirayama et al., 1984). In this case, t is called the solar system Barycentric Dynamical Time (TDB) and τ is the same as the Terrestrial Dynamical Time (TDT) after it is multiplied by a constant factor (Fukushima et al., 1986).

6.3. Space Origin

The expression of space coordinates $\mathbf{x}_0$ is obtained by solving the equation of translational motion of the central body such as the Einstein-Infeld-Hoffmann equation in the background spacetime (MTW, Section 39.6). The resulting $\mathbf{x}_0$ isre usually expressed as the function of not τ but t such as the solar system barycentric coordinates of the Earth in the astronomical ephemerides. Therefore the practical expression of $\mathbf{x}_0$ becomes $\mathbf{x}_0(t(\tau))$.

6.4. Tetrad

The post-Newtonian expression of the tetrad of the PRF of a massive body is given as

$$\begin{aligned} e^0{}_{\hat{0}} = 1 &+ (\phi_0^* + v_0^2/2)/c^2 + [3(\phi_0^*)^2/2 + 3\phi_0^* v_0^2/2 \\ &+ 3v_0^4/8 + \psi_0^* + \mathbf{v}_0 \cdot \mathbf{g}_0^* + \mathbf{v}_0 \cdot \underset{\sim}{\chi}_0^* \mathbf{v}_0]/c^4 \end{aligned} \tag{6-19}$$

$$e^i{}_{\hat{0}} = [\mathbf{v}_0/c + \{ (\phi_0^* + v_0^2/2)\mathbf{v}_0 \}/c^3]^i \tag{6-20}$$

$$\begin{aligned} e^0{}_{\hat{m}} = [\mathbf{v}_0/c + \{ &(2\phi_0^* + v_0^2/2)\mathbf{v}_0 + \underset{\sim}{\chi}_0^* \mathbf{v}_0 + \mathbf{g}_0^* \\ &+ \underset{\sim}{Q}\, \mathbf{v}_0 \}/c^3]^m \end{aligned} \tag{6-21}$$

$$e^i{}_{\hat{m}} = [\underset{\sim}{1} + \{ - \underset{\sim}{\chi}_0^* + \mathbf{v}_0 \otimes \mathbf{v}_0/2 - \underset{\sim}{Q} \}/c^2]^i{}_m \tag{6-22}$$

Here

$$\begin{aligned} \underset{\sim}{Q} &= \int \underset{\sim}{q}\, dt \\ &= \int [\mathbf{v}_0 \wedge (\nabla\phi_0^* - \mathbf{a}_0)/2 - \nabla \wedge (\underset{\sim}{\chi}_0^* \mathbf{v}_0 + \mathbf{g}_0^*/2)]dt \end{aligned} \tag{6-23}$$

where

$$(\mathbf{a} \wedge \mathbf{b})^{ij} = a^i b^j - b^i a^j \tag{6-24}$$

and ∇ is the 3-gradient operator in the background coordinate system, i.e., the partial derivative operator with respect to $\mathbf{x}$, $\mathbf{a}_0$ is the non-

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gravitational coordinate acceleration of the central body.

The matrix $\tilde{Q}$ shows the effect of the so-called dragging of inertial coordinate systems (MTW, Section 40.7). The above expressions of the teterad coincide with those of the Lorentz matrix (3-2) through (3-4) in the post-Newtonian approximation if the gravitational potentials are all ignored. This indicates that the coordinate transformation (6-1) is a sort of extension of the Poincaré transformation.

6.5. Deviation of Coordinate Grid from Straight Line

The expression (6-1) is obtained as the solution of the equation of space-like geodesics starting from the space coordinate origin of the PRF of the central body. If the space-like geodesic is likened to the orbit of a free tachyon, the first and the second terms in (6-1) show that the free tachyon runs along a straight line with a constant speed. The third term corresponds to the deviation of the tachyon's motion from a uniform, straight line motion. In the post-Newtonian framework, the deviation in coordinates δx^{μ} is obtained by solving the equations

$$d^2\delta t/(ds)^2 = [\mathbf{w}\cdot(\mathbf{w}\cdot\nabla)g^* - \mathbf{w}\cdot\dot{\tilde{\chi}}^*\mathbf{w} + 2(\mathbf{v}_0\cdot\mathbf{w})(\mathbf{w}\cdot\nabla)\phi^*]/c^4, \quad (6\text{-}25)$$

$$d^2\delta \mathbf{x}/(ds)^2 = [\nabla(\mathbf{w}\cdot\tilde{\chi}^*\mathbf{w}) - 2(\mathbf{w}\cdot\nabla)\tilde{\chi}^*\mathbf{w}]/c^2 \quad (6\text{-}26)$$

where $\delta t = \delta x^0/c$, $(\delta \mathbf{x})^j = \delta x^j$, and s and $\mathbf{w}$ are the proper length and the spatial tangential vector of the space-like geodesic approximated as

$$s \cong |\mathbf{X}| \cong r_0, \quad \mathbf{w} \cong \mathbf{X}/s, \quad |\mathbf{w}| \cong 1 \quad (6\text{-}27)$$

Equations (6-25) and (6-26) are not explicitly solved in the general case. In the EIH metric, however, it is solved as (Fukushima, 1984)

$$\delta t = \sum_{J\neq 0}[\mathbf{v}_J - \{\mathbf{w}\cdot(4\mathbf{v}_J - 2\mathbf{v}_0)\}\mathbf{w}]\cdot\delta\mathbf{x}_J/c^2, \quad (6\text{-}28)$$

$$\delta\mathbf{x} = \sum_{J\neq 0}[\delta\mathbf{x}_J - 2(\mathbf{w}\cdot\delta\mathbf{x}_J)\mathbf{w}] \quad (6\text{-}29)$$

where

$$\delta\mathbf{x}_J = (GM_J/c^2)[\{\ln|(\mathbf{w}\cdot\mathbf{r}_J + r_J)/(\mathbf{w}\cdot\mathbf{r}_{0J} + r_{0J})| - s/r_{0J}\}\mathbf{w} + \{r_J - r_{0J} - s(\mathbf{w}\cdot\mathbf{r}_{0J})/r_{0J}\}\,\mathbf{w}\times(\mathbf{w}\times\mathbf{r}_{0J})/(\mathbf{w}\times\mathbf{r}_{0J})^2] \quad (6\text{-}30)$$

Thus the full expressions of the coordinate transformation (6-1) are explicitly obtained in the case the background metric is the EIH metric.

6.6. Effective Region

The PRF is not a global coordinate system. It is not well-defined far from the central body. This is because the transformation (6-1) ceases to be one-to-one where the distance from the coordinate origin is sufficiently large. The sub-spacetime where the inverse transformation of (6-1) is uniquely determined is named the effective region of the PRF. Roughly speaking, the effective region is a world tube with a spherical

(3-dimensional) cross section whose center is the central body.

Consider the case that the central body makes a uniform circular motion around one of the external bodies in the background coordinate system as the Earth approximately does in the solar system. Then the spatial size of the effective region is evaluated approximately as

$$R_{eff} \cong c^2 /(\omega v_0) \qquad (6\text{-}31)$$

where ω is the angular velocity of the orbital motion of the central body. In the Earth, R_{eff} amounts to about 0.5 kpc. This estimation is independent on the amount of the post-Newtonian corrections in (6-1).

7. KINEMATICS IN THE NON-ROTATING PROPER REFERENCE FRAME

The preceding section explicitly shows the coordinate transformation between the non-rotating PRF of the central body and the background coordinate system. Next the conversion formulas of various physical quantities such as the light direction, the metric tensor, etc., caused by this transformation must be investigated to obtain the equation of motion or the observational equation in the non-rotating PRF.

7.1. Transformation Law

The principle of the general relativity makes it very easy to obtain the conversion formulas of physical quantities caused by a coordinate transformation. The principle is rewritten as "The expressions of physical quantities and physical laws are covariant with any general but smooth coordinate transformation". Thus the following rules are established in the relativistic theory to obtain the conversion formulas:

1) Give the explicit formula of coordinate transformation

$$x^{\mu} = x^{\mu} (x^{\tilde{\alpha}}) \qquad (7\text{-}1)$$

where x^{μ} is the new coordinate and $x^{\tilde{\alpha}}$ is the old one.

2) Obtain the transformation matrix $E^{\mu}{}_{\tilde{\alpha}}$ (and its inverse $E^{\tilde{\alpha}}{}_{\mu}$) as

$$E^{\mu}{}_{\tilde{\alpha}} = \partial x^{\mu}/\partial x^{\tilde{\alpha}} \quad (E^{\tilde{\alpha}}{}_{\mu} = \partial x^{\tilde{\alpha}}/\partial x^{\mu}) \qquad (7\text{-}2)$$

3) The new expression of the physical quantity is obtained from its old one by multiplying $E^{\mu}{}_{\tilde{\alpha}}$ ($E^{\tilde{\alpha}}{}_{\mu}$) as many times as the number of superfices (suffices) of the quantity.

For example,

$$k^{\mu} = E^{\mu}{}_{\tilde{\alpha}} k^{\tilde{\alpha}}, \quad u^{\mu} = E^{\mu}{}_{\tilde{\alpha}} u^{\tilde{\alpha}}, \quad a^{\mu} = E^{\mu}{}_{\tilde{\alpha}} a^{\tilde{\alpha}}, \quad g_{\tilde{\alpha}\tilde{\beta}} = E^{\mu}{}_{\tilde{\alpha}} E^{\nu}{}_{\tilde{\beta}} g_{\mu\nu} \qquad (7\text{-}3)$$

where k^{μ} is the 4-wave vector, u^{μ} is the 4-velocity, and a^{μ} is the 4-acceleration.

7.2. Transformation Matrix

The transformation matrix represents the reference frame accompanied with the coordinate transformation as is mentioned in Section 1. The expression of the coordinate transformation from the non-rotating PRF of the central body to the background coordinate system is already given as (6-1). Then the transformation matrix is obtained as

$$
\begin{aligned}
E^{0}{}_{\tilde{0}} = 1 &+ \{ \phi_0^* + v_0^2/2 + \mathfrak{x}\cdot(\nabla\phi_0^* + a_0) \}/c^2 + [\, 3(\phi_0^*)^2/2 \\
&+ 3\phi_0^* v_0^2/2 + 3v_0^4/8 + \psi_0^* + v_0\cdot g_0^* + v_0\cdot \underset{\sim}{\chi}{}_0^* v_0 \\
&+ \mathfrak{x}\cdot[(3\phi_0^* + v_0^2 - \underset{\sim}{\chi}{}_0^* + \underset{\sim}{Q})(\nabla\phi_0^* + a_0) \\
&+ \nabla(\psi_0^* + v_0\cdot g_0^* + v_0\cdot \underset{\sim}{\chi}{}_0^* v_0) + (v_0\cdot a_0)v_0 \\
&+ \{ \dot{\phi}_0^* + (v_0\cdot\nabla)\phi_0^* \} v_0 - \{ \dot{\underset{\sim}{\chi}}{}_0^* + (v_0\cdot\nabla)\underset{\sim}{\chi}{}_0^* \} v_0 \\
&+ \underset{\sim}{Q}\ (\nabla\phi_0^* + a_0) + \underset{\sim}{q}\, v_0 \,]\,]/c^4 + B^{0}{}_{\tilde{0}}
\end{aligned}
\tag{7-4}
$$

$$
\begin{aligned}
E^{i}{}_{\tilde{0}} = [\, v_0/c &+ [\, (\phi_0^* + v_0^2/2)v_0 - \{ \dot{\underset{\sim}{\chi}}{}_0^* + (v_0\cdot\nabla)\underset{\sim}{\chi}{}_0^* \}\mathfrak{x} \\
&+ \{ \mathfrak{x}\cdot(\nabla\phi_0^* + a_0)\} v_0/2 + (v_0\cdot\mathfrak{x})(\nabla\phi_0^* + a_0)/2 \\
&- \underset{\sim}{q}\,\mathfrak{x}\,]/c^3 \,]^{i} + B^{i}{}_{\tilde{0}},
\end{aligned}
\tag{7-5}
$$

$$
\begin{aligned}
E^{0}{}_{\tilde{m}} = [\, v_0/c &+ \{ (2\phi_0^* + v_0^2/2)v_0 + \underset{\sim}{\chi}{}_0^* v_0 + g_0^* \\
&+ \underset{\sim}{Q}\, v_0 \}/c^3 \,]^{m} + B^{0}{}_{\tilde{m}},
\end{aligned}
\tag{7-6}
$$

$$
E^{i}{}_{\tilde{m}} = [\, \underset{\sim}{1} + \{ - \underset{\sim}{\chi}{}_0^* + v_0 \otimes v_0/2 - \underset{\sim}{Q} \}/c^2 \,]^{i}{}_{m} + B^{i}{}_{\tilde{m}},
\tag{7-7}
$$

where

$$
B^{\mu}{}_{\tilde{\alpha}} = \partial\delta x^{\mu}/\partial x^{\tilde{\alpha}}
\tag{7-8}
$$

and $\dot{}$ is the partial derivative with respect to t, the time coordinate in the background coordinate system.

The partial derivatives of the gravitational potentials needed to evaluate E are obtained in the EIH metric as

$$
\nabla\phi_0^* = - \sum_{J\neq 0} (GM_J r_{0J}/r_{0J}^3)
\tag{7-9}
$$

$$
\begin{aligned}
\nabla\psi_0^* = \sum_{J\neq 0} [\, (GM_J/r_{0J}^3)[\, \{ & 2(\sum_{K\neq 0} GM_K/r_{0K}) - 2v_J^2 \\
&+ (\sum_{K\neq J} GM_K/r_{JK}) + 3(r_{0J}\cdot v_J/r_{0J})^2/2 \\
&+ r_{0J}\cdot\{ \sum_{K\neq J} GM_K r_{JK}/r_{JK}^3 \}/2 \,\}\, r_{0J} - (r_{0J}\cdot v_J)\, v_J \,] \\
&- (GM_J/r_{0J})\{ \sum_{K\neq J} GM_K r_{JK}/r_{JK}^3 \} \,]
\end{aligned}
\tag{7-10}
$$

$$
\nabla(v_0\cdot g_0^*) = 12 \sum_{J\neq 0} [\, GM_J\ (v_J\cdot v_0)\ r_{0J}/r_{0J}^3 \,]
\tag{7-11}
$$

$$\nabla(\mathbf{v}_0\cdot \underset{\sim}{\chi}{}_0^{*}\mathbf{v}_0) = (\mathbf{v}_0\cdot\nabla)\,\underset{\sim}{\chi}{}_0^{*}\mathbf{v}_0 = v_0^2\,\nabla\,\phi_0^{*} \tag{7-12}$$

$$\dot{\underset{\sim}{\chi}}{}_0^{*} = \dot{\phi}_0^{*}\,\underset{\sim}{1} = \sum_{J\neq 0}[\,GM_J\,(\mathbf{r}_{0J}\cdot\mathbf{v}_J)/r_{0J}^3\,]\,\underset{\sim}{1} \tag{7-13}$$

7.3. Transformation of Light Direction

The transformation formula of the light direction is obtained from that of the 4-wave vector as

$$\mathbf{n} = \hat{\mathbf{n}} + [\,\mathbf{v}_0 - (\hat{\mathbf{n}}\cdot\mathbf{v}_0)\,\hat{\mathbf{n}}\,]/c + [\,(\hat{\mathbf{n}}\cdot\mathbf{v}_0)^2\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\mathbf{v}_0)\mathbf{v}_0/2 - v_0^2\,\hat{\mathbf{n}}/2$$
$$- \underset{\sim}{\chi}{}_0^{*}\,\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\underset{\sim}{\chi}{}_0^{*}\hat{\mathbf{n}})\,\hat{\mathbf{n}} - \underset{\sim}{Q}\,\hat{\mathbf{n}}\,]/c^2 + \underset{\sim}{B}\,\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\underset{\sim}{B}\hat{\mathbf{n}})\,\hat{\mathbf{n}} \tag{7-14}$$

where $\mathbf{n}$ and $\hat{\mathbf{n}}$ are the unit vector of the light direction in the background coordinate system and in the PRF, respectively.

7.4. Transformation of Velocity

The transformation formula of the coordinate velocity is obtained from that of the 4-velocity as

$$\mathbf{v} = \mathbf{v}_0 + \hat{\mathbf{v}} - [\,\{\,\phi_0^{*} + v_0^2/2 + \mathbf{v}_0\cdot\hat{\mathbf{v}} + \hat{\mathbf{x}}\cdot(\nabla\phi_0^{*} + \mathbf{a}_0)\,\}\,\hat{\mathbf{v}}$$
$$+ \underset{\sim}{\chi}{}_0^{*}\hat{\mathbf{v}} + (\mathbf{v}_0\cdot\hat{\mathbf{v}})\mathbf{v}_0/2 - \{\,\mathbf{v}_0 \times (\nabla\,\phi_0^{*} + \mathbf{a}_0)\,\} \times \hat{\mathbf{x}}/2$$
$$+ \{\,\dot{\underset{\sim}{\chi}}{}_0^{*} + (\mathbf{v}_0\cdot\nabla)\,\underset{\sim}{\chi}{}_0^{*}\,\}\hat{\mathbf{x}} + \underset{\sim}{q}\,\hat{\mathbf{x}} + \underset{\sim}{Q}\,\hat{\mathbf{v}}\,]/c^2 + \mathbf{B}_{\delta} + \underset{\sim}{B}\,\hat{\mathbf{v}} \tag{7-15}$$

where $\mathbf{v}$ and $\hat{\mathbf{v}}$ are the coordinate velocity in the background coordinate system and in the PRF, respectively.

7.5. Transformation of Non-gravitational Acceleration

The transformation formula of the non-gravitational coordinate acceleration is obtained from that of the 4-acceleration as

$$\mathbf{a} = \hat{\mathbf{a}} - [\,\{\,2\phi_0^{*} + v_0^2 + 2\mathbf{v}_0\cdot\hat{\mathbf{v}} + 2\hat{\mathbf{x}}\cdot(\nabla\phi_0^{*} + \mathbf{a}_0)\,\}\,\hat{\mathbf{a}}$$
$$+ \underset{\sim}{\chi}{}_0^{*}\,\hat{\mathbf{a}} + (\mathbf{v}_0\cdot\hat{\mathbf{a}})\,\hat{\mathbf{v}} + (\hat{\mathbf{v}}\cdot\hat{\mathbf{a}})\,\mathbf{v}_0/2 + \underset{\sim}{Q}\,\hat{\mathbf{a}}\,]/c^2 + \underset{\sim}{B}\,\hat{\mathbf{a}} \tag{7-16}$$

where $\mathbf{a}$ and $\hat{\mathbf{a}}$ are the non-gravitational coordinate acceleration in the background coordinate system and in the PRF, respectively.

7.6. Transformation of Metric Tensor

The transformation formulas of the force functions are obtained from that of the metric tensor as

$$\tilde{\phi} = \phi - \phi_0^{*} - \hat{\mathbf{x}}\cdot(\,\nabla\,\phi_0^{*} + \mathbf{a}_0\,), \tag{7-17}$$

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$$
\begin{aligned}
\tilde{\psi} = & \{ \psi - \psi_0^* - (\mathfrak{X}\cdot\nabla)\psi_0^* \} \\
& + v_0 \cdot \{ g - g_0^* - (\mathfrak{X}\cdot\nabla) g_0^* \} \\
& + v_0 \cdot \{ \underset{\sim}{\chi} - \underset{\sim}{\chi}_0^* - (\mathfrak{X}\cdot\nabla) \underset{\sim}{\chi}_0^* \} v_0 \\
& + (2\phi_0^* + v_0^2)\{ \phi - \phi_0^* - \mathfrak{X}\cdot(\nabla\phi_0^* + a_0) \} \\
& + \mathfrak{X}\cdot(\nabla\phi_0^* + a_0) \{ 2(\phi - \phi_0^*) + \mathfrak{X}\cdot(\nabla\phi_0^* + a_0)/2 \} \\
& - (\mathfrak{X}\cdot v_0) \{ v_0\cdot(\nabla\phi_0^* + a_0)/2 + \dot{\phi}_0^* \} \\
& + \mathfrak{X}\cdot\{ (\underset{\sim}{\chi}_0^* + \underset{\sim}{Q})(\nabla\phi_0^* + a_0) \} + v_0\cdot B_{\mathfrak{o}} - B^0{}_{\mathfrak{o}}
\end{aligned} \qquad (7\text{-}18)
$$

$$
\begin{aligned}
\mathfrak{g} = & g - g_0^* + 2 (\underset{\sim}{\chi} - \underset{\sim}{\chi}_0^*) v_0 + 2 (\phi - \phi_0^*) v_0 \\
& + \{ v_0 \times (\nabla\phi_0^* + a_0) \} \times \mathfrak{X}/2 - \{ \dot{\underset{\sim}{\chi}}_0^* + (v_0\cdot\nabla) \underset{\sim}{\chi}_0^* \} \mathfrak{X} \\
& - \underset{\sim}{q}\, \mathfrak{X} + {}^t\underset{\sim}{B}\, v_0 + B_{\mathfrak{o}} - B^0
\end{aligned} \qquad (7\text{-}19)
$$

$$
\tilde{\underset{\sim}{\chi}} = \underset{\sim}{\chi} - \underset{\sim}{\chi}_0^* + ({}^t\underset{\sim}{B} + \underset{\sim}{B})/2 \qquad (7\text{-}20)
$$

where

$$
(B^0)_m = B^0{}_{\mathfrak{m}}, \quad (B_{\mathfrak{o}})^i = B^i{}_{\mathfrak{o}}, \quad (\underset{\sim}{B})^i{}_m = B^i{}_{\mathfrak{m}} \qquad (7\text{-}21)
$$

and the symbol ~ denotes the quantity in the non-rotating PRF.

If there is no non-gravitational accelerations, then

$$
\tilde{\phi} = \phi^+ + [\phi^* - \phi_0^* - (\mathfrak{X}\cdot\nabla)\phi_0^*] \qquad (7\text{-}22)
$$

In this case, the Newtonian potential in the non-rotating PRF of the central body is that of the central body itself plus the tidal potential of the external bodies. And if the relation

$$
\tilde{\underset{\sim}{\chi}} = \chi \underset{\sim}{1} \qquad (7\text{-}23)
$$

holds such as in the PPN coordinate system, then it is shown that

$$
\tilde{\chi} = \chi - \chi^* = \chi^+ \qquad (7\text{-}24)
$$

Namely the spatial curvature of the non-rotating PRF includes no tidal effects if the background spacetime has an isotropic spatial curvature. This is because to take the new space coordinate axes as the space-like geodesics makes the new space Euclidean as long as the spatial metric of the background spacetime is isotropic and diagonal.

8. RIGIDLY ROTATING COORDINATE SYSTEM

Besides the non-rotating coordinate systems, the rotating coordinate systems are of much use to construct the dynamical theories such as the meteorological dynamics on the Earth. Above all, the rigidly rotating coordinate system is important since the Earth as well as other solid planets is rotating almost rigidly. Here we discuss the rigidly rotating coordinate systems in the post-Newtonian framework.

Just as same as in the Newtonian mechanics, a rotating coordinate system is obtained from a non-rotating one by the following coordinate transformation in the general relativistic theory

$$x^{\tilde{0}} = x^{\hat{0}}, \qquad x^{\tilde{m}} = x^{\hat{p}} R^{\tilde{m}}{}_{\hat{p}} \tag{8-1}$$

where $x^{\tilde{\alpha}}$ and $x^{\hat{\xi}}$ are the coordinates in the non-rotating and the rotating coordinate systems, respectively, and R is the matrix of rigid rotation. The accompanied 4-dimensionl transformation matrix is evaluated as

$$\begin{aligned} &E^{\tilde{0}}{}_{\hat{0}} = 1, \qquad E^{\tilde{0}}{}_{\hat{m}} = 0, \\ &E^{\tilde{m}}{}_{\hat{0}} = -R^{\tilde{m}}{}_{\hat{n}} \Omega^{\hat{n}}{}_{\hat{p}} x^{\hat{p}}/c = (\mathbf{v}_{rot}/c)^{m}, \qquad E^{\tilde{m}}{}_{\hat{p}} = R^{\tilde{m}}{}_{\hat{p}} \end{aligned} \tag{8-2}$$

where $\mathbf{v}_{rot}$ is the rotational velocity of the coordinate system. Clearly the conversion formulas of physical quantities except that the metric tensor are just the same as those in the Newtonian framework. The conversion formulas of the gravitational potentials are as follows

$$\hat{\phi} = \tilde{\phi} + \mathbf{v}_{rot}{}^{2}/2, \tag{8-3}$$

$$\hat{\psi} = \tilde{\psi} + \mathbf{v}_{rot} \cdot (\mathbf{g} + \underset{\sim}{\tilde{\chi}}\, \mathbf{v}_{rot}), \tag{8-4}$$

$$\hat{\mathbf{h}} = \underset{\sim}{R}\, \mathbf{v}_{rot}, \tag{8-5}$$

$$\mathbf{g} = \underset{\sim}{R}\, (\mathbf{g} + 2\, \underset{\sim}{\tilde{\chi}}\, \mathbf{v}_{rot}), \tag{8-6}$$

$$\underset{\sim}{\hat{\chi}} = \underset{\sim}{R}\, \underset{\sim}{\tilde{\chi}}\, \underset{\sim}{R}^{-1} \tag{8-7}$$

where the quantities with ^ and ~ are those in the rigidly rotating and the non-rotating PRFs, respectively.

Here the term $\hat{\mathbf{h}}$ is the gravitational vector potential of order of v, which is named the Coriolis potential. The Coriolis potential $\hat{\mathbf{h}}$ corresponds to the electromagnetic vector potential if the Newtonian gravitational potential is likened to the electromagnetic scalar potential. Namely the rotation produces the Coriolis force field in the gravitational theory almost in the same way as the boost does the magnetic field in the electromagnetic theory. Similarly the choice of the rotating coordinate system in the gravitational theory corresponds to the choice of the gauge field in the electromagnetic theory.

9. PRACTICAL CHOICE OF COORDINATE SYSTEMS FOR THE ASTROMETRY

Although the non-rotating PRF has a number of merits from the viewpoint to construct the post-Newtonian dynamics, it has some defects in the kinematical sense. Above all, it is a clear demerit that the coordinate transformation formula (6-1) has the term $\underset{\sim}{Q}$, which is not given explicitly until the history of the central body are known. Also this term breaks the symmetry of spatial part of the transformation matrix E. It is also a demerit of the PRF that the existence of δx^{μ}, the deviation of space-like geodesics from straight lines and its partial derivative matrix B makes the formulas of transformation in Section 7 too complicated. Furthermore the fully explicit expression of the coordinate transformation of the PRF is obtained only if the background metric is approximated to be the EIH metric.

Recalling that in the relativistic theory we can choose any coordinate system whether it is realistic or fictitious, we should choose the coordinate system so that it requires us the least effort. There are two types of effort we must do when we analyze the observations in the astrometry. The one is the effort to give a dynamical theory, i.e. to establish the appropriate equation of motion and solve it. The other is the effort to give an observational theory, namely to obtain the observational equation and reduce the observed quantities with use of it. We think that the latter effort must be reduced as largely as possible because the post-Newtonian dynamical theory is complicated very much even if there is no additional complexity caused by the choice of coordinate system. In fact, the amount of effort saved by adopting the PRF as the basic coordinate system is relatively small in constructing a dynamical theory practically.

Then we propose another coordinate system as the basic coordinate system which is defined by the coordinate transformation

$$t = t_0(\bar{t}) + \bar{\mathbf{x}}\cdot\mathbf{v}_0/c^2 + \bar{\mathbf{x}}\cdot[\ (\ 2\phi_0^* + v_0^2/2\ )\ \mathbf{v}_0 + \underset{\sim}{\chi}_0^*\ \mathbf{v}_0 + \mathbf{g}_0^*\ ]/c^4 \qquad (9\text{-}1)$$

$$\mathbf{x} = \mathbf{x}_0(\bar{t}) + \bar{\mathbf{x}} + [\ (\bar{\mathbf{x}}\cdot\mathbf{v}_0)\ \mathbf{v}_0/2 - \underset{\sim}{\chi}_0^*\ \bar{\mathbf{x}}\ ]/c^2 \qquad (9\text{-}2)$$

where the quantities with $\bar{}$ are those in the proposed coordinate system. This coordinate transformation is obtained from the formula (6-1) by dropping $\underset{\sim}{Q}$ and δx^{μ}. The transformation matrix and other transformation formulas from the proposed to the background coordinate systems are obtained by dropping $\underset{\sim}{Q}$, $\underset{\sim}{q}$, and B from those in Section 7.

The new coordinate system has the following features :

1) The coordinate transformation is explicitly defined as long as the background metric has no Coriolis potential.
2) The spatial part of the transformation matrix is symmetric.
3) The transformation formulas of the physical quantities are not complicated too much.

Among them, the second feature is important since the direction of the incident light from the distant stars or the quasars does not change in a secular way. This character of the new coordinate system seems very suitable in constructing the practical coordinate system by the VLBI observations or the precise optical observations by the orbital telescopes. In this sense the new coordinate system is an extension of the natural frame of Murray. Then we name the new coordinate system the non-rotating natural coordinate system (NCS) comoving with a massive body.

It is noted that the meaning of 'non-rotating' is different between in the PRF and in the NCS. In the PRF, it means that the tetrad of the PRF suffers the Fermi-Walker transportation. In the NCS, it means that the space-space component of the tetrad is symmetric. In other words, the spatial direction in the non-rotating PRF is referred to a gyroscope at the space origin, while the spatial direction in the non-rotating NCS coincides with that in the background coordinate system.

As is stated in the preceding section, a rigidly rotating coordinate system can be constructed from any non-rotating coordinate system. Then we define the rigidly rotating NCS of a massive body in the same manner we did for the rigidly rotating PRF of the body in the previous section.

Now we are ready to define the six basic coordinate systems stated in Section 1 within the general relativistic framework. We propose to define these coordinate systems as follows:

1) the extragalactic coordinate system as the local quasi-Cartesian coordinate system defined by Will,
2) the galactic coordinate system as the non-rotating NCS of the galaxy where its background coordinate system is the extragalactic coordinate system defined above,
3) the solar system barycentric coordinate system as the non-rotating NCS of the solar system where its background coordinate system is the galactic coordinate system defined above,
4) the terrestrial coordinate system as the non-rotating NCS of the Earth where its background coordinate system is the solar system coordinate system defined above,
5) the terrestrial rotating coordinate system as the rigidly rotating NCS of the Earth obtained from the non-rotating natural coordinate system of the Earth defined above via a rigid spatial rotation, and
6) the coordinate system of the observer on the Earth as the non-rotating NCS of the observer where the background coordinate system is the terrestrial coordinate system defined above.

We note that these definitions of the terrestrial coordinate system and the terrestrial rotating coordinate system agrees with the present convention that the geodesic precession is included into the general precession obtained from the observation.

10. BARYCENTRIC COORDINATE SYSTEM AND TERRESTRIAL COORDINATE SYSTEM

Let the solar system Barycentric Coordinate System (BCS) and the Terrestrial Coordinate System (TCS) be defined as proposed in the previous section. The deviation of the metric tensor expressed in the BCS from the EIH metric tensor is negligiblly small in the neighbourhood of the Earth as of order of 10^{-25} in the case of the tidal gravitational potential of the galaxy or of order of 10^{-30} in the case of the cosmological curvature (See Table 3). Also the non-gravitational acceleration of the Earth in the BCS is too small to be taken into account. Then the

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System	Size	Mass	Newtonian	Post-Newtonian	Tidal		
					Cosmological	Galactic	Luni-Solar
	R (AU)	M ($M_\odot$)	$\frac{GM}{c^2R}$	$\left(\frac{GM}{c^2R}\right)^2$	$\left(\frac{R}{R_U}\right)^2$	$\frac{GM_GR^2}{c^2R_G^3}$	$\frac{GM_\odot R^2}{c^2R_\odot^3}$
Universe	6×10^{14}	—	—	—	—	—	—
Galaxy	2×10^{9}	10^{11}	5×10^{-7}	3×10^{-13}	10^{-11}	—	—
Solar System	1	1	10^{-8}	10^{-16}	3×10^{-30}	10^{-25}	—
Terrestrial	6×10^{-5}	3×10^{-6}	5×10^{-10}	3×10^{-19}	10^{-38}	5×10^{-34}	10^{-18}
Observer	10^{-11}	—	—	—	3×10^{-52}	10^{-47}	10^{-30}

Table 3 Magnitudes of Effects in Coordinate Systems

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coordinate transformation between the BCS and the TCS is given as

$$t_B = t_T + \mathbf{x}_T \cdot \mathbf{v}_E / c^2 + \Delta t_E(t_T) + [\ (v_E{}^2)(\mathbf{x}_T \cdot \mathbf{v}_E)/2$$
$$+ \mathbf{x}_T \cdot \{ \sum_{J \neq E} GM_J (3\mathbf{v}_E - 4\mathbf{v}_J)/r_{EJ} \}\ ]/c^4 \qquad (10\text{-}1)$$

$$\mathbf{x}_B = \mathbf{x}_E + \mathbf{x}_T + [\ (\mathbf{x}_T \cdot \mathbf{v}_E)\ \mathbf{v}_E/2 - (\sum_{J \neq E} GM_J/r_{EJ})\ \mathbf{x}_T\ ]/c^2 \qquad (10\text{-}2)$$

and the inverse coordinate transformation is given as

$$t_T = t_B - (\mathbf{x}_B - \mathbf{x}_E) \cdot \mathbf{v}_E/c^2 - \Delta t_E(t_T)$$
$$- 4\ (\mathbf{x}_B - \mathbf{x}_E) \cdot [\ \sum_{J \neq E} GM_J\ (\mathbf{v}_E - \mathbf{v}_J)/r_{EJ}\ ]/c^4 \qquad (10\text{-}3)$$

$$\mathbf{x}_T = \mathbf{x}_B - \mathbf{x}_E - [\ \{\ (\mathbf{x}_B - \mathbf{x}_E) \cdot \mathbf{v}_E\ \}\ \mathbf{v}_E/2$$
$$- (\sum_{J \neq E} GM_J/r_{EJ})(\mathbf{x}_B - \mathbf{x}_E)\ ]/c^2 \qquad (10\text{-}4)$$

where

t_T : the TDT of the desired event

t_B : the TDB of the desired event

Δt_E : the correction to t_T so that $t_E = t_T + \Delta t_E$ gives the TDB of the geocenter at the same TDT (See Moyer or Hirayama et al.)

$\mathbf{x}_T$: the position of the desired event in the TCS

$\mathbf{x}_B$: the position of the desired event in the BCS

$\mathbf{x}_E$: the position of the geocenter in the BCS when the TDB is t_E

$\mathbf{v}_E$: the velocity of the geocenter in the BCS when the TDB is t_E

$\mathbf{v}_J$: the velocity of the body J in the BCS when the TDB is t_E

r_{EJ} : the distance between the geocenter and the body J in the BCS when the TDB is t_E

GM_J : the gravitational constant of the body J

The transformation formula of the light direction is given as

$$\mathbf{n}_B = \mathbf{n}_T + [\ \mathbf{v}_E - (\mathbf{n}_T \cdot \mathbf{v}_E)\ \mathbf{n}_T\ ]/c$$
$$+ [\ (\mathbf{n}_T \cdot \mathbf{v}_E)^2\ \mathbf{n}_T - (\mathbf{n}_T \cdot \mathbf{v}_E)\ \mathbf{v}_E/2 - v_E^2\ \mathbf{n}_T/2\ ]/c^2 \qquad (10\text{-}5)$$

and its inverse formula is given as

$$\mathbf{n}_T = \mathbf{n}_B - [\ \mathbf{v}_E - (\mathbf{n}_B \cdot \mathbf{v}_E)\ \mathbf{n}_B\ ]/c$$
$$+ [\ (\mathbf{n}_B \cdot \mathbf{v}_E)^2\ \mathbf{n}_B - (\mathbf{n}_B \cdot \mathbf{v}_E)\ \mathbf{v}_E/2 - v_E^2\ \mathbf{n}_B/2\ ]/c^2 \qquad (10\text{-}6)$$

where

$\mathbf{n}_T$: the unit vector of the light direction in the TCS

$\mathbf{n}_B$: the unit vector of the light direction in the BCS

Note that the formulas (10-5) and (10-6) have the same form as those in the special theory of relativity though the boost velocity is not constant but a function of the time coordinate in the TCS. It is one of the merits of adopting the NCS as the basic coordinate system that the transformation formula of the light direction is independent on the spatial coordinates in the NCS.

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