Glasgow Math. J. **58** (2016) 573–586. © Glasgow Mathematical Journal Trust 2015.
doi:10.1017/S0017089515000348.

CLASSES OF SEQUENTIALLY LIMITED OPERATORS

JAN H. FOURIE and ELROY D. ZEEKOEI
Unit for Business Mathematics and Informatics,
North-West University(NWU), Private Bag X6001,
Potchefstroom 2520, South Africa
e-mail: jan.fourie@nwu.ac.za; elroy.zeekoei@ nwu.ac.za

(Received 27 August 2014; revised 30 October 2014; accepted 15 December 2014;
first published online 22 July 2015)

Abstract. The purpose of this paper is to present a brief discussion of both the normed space of operator p-summable sequences in a Banach space and the normed space of sequentially p-limited operators. The focus is on proving that the vector space of all operator p-summable sequences in a Banach space is a Banach space itself and that the class of sequentially p-limited operators is a Banach operator ideal with respect to a suitable ideal norm - and to discuss some other properties and multiplication results of related classes of operators. These results are shown to fit into a general discussion of operator $[Y, p]$-summable sequences and relevant operator ideals.

2010 *Mathematics Subject Classification*. 47B10, 46A45.

1. Introduction and notation. Throughout the paper we work with Banach spaces X,Y,Z, etc. over the same scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and denote the space of bounded linear operators from X to Y by $B(X, Y)$. The continuous dual space $B(X, \mathbb{K})$ of X is denoted by X^*, whereas U_X denotes the closed unit ball of X. Other classical classes of linear operators from X to Y that will be encountered in this manuscript are $K(X, Y)$ (the space of compact linear operators), $\mathcal{W}(X, Y)$ (the space of weakly compact operators), $\Pi_p(X, Y)$ (the space of p-summing operators) and $\mathcal{N}_p(X, Y)$ (the space of p-nuclear operators). They are of course linear subspaces of the vector space $B(X, Y)$.

The scalar sequence $(\delta_{i,n})_i$ such that $\delta_{n,n} = 1$ and $\delta_{i,n} = 0$ if $i \neq n$ will be denoted by e_n. The set $\{e_n : n \in \mathbb{N}\}$ is a Schauder basis for the space ℓ_p of absolutely p-summable (scalar) sequences (with $1 \leq p < \infty$) as well as the space c_0 of scalar null sequences. For a Banach space Y we let

$$\ell_p^s(Y) = \{(y_n) \in Y^{\mathbb{N}} : (\|y_n\|) \in \ell_p\}, \text{ with norm } \|(y_n)\|_p = \left(\sum_{i=1}^{\infty} \|y_i\|^p\right)^{1/p}.$$

In the paper **[6]**, the authors introduce the operator p-summable sequences in a Banach space X and among some applications with respect to p-limited sets and the p-Dunford–Pettis-property, they introduce the sequentially p-limited operators which map weakly p-summable sequences to operator p-summable sequences. In a brief discussion of sequentially p-limited operators, they introduce a norm ℓt_p on each vector space $Lt_p(X, Y)$ of sequentially p-limited operators as X,Y run through the family of all Banach spaces and show that $(Lt_p, \ell t_p)$ is a normed operator ideal.

They could not verify the completeness of this normed ideal and therefore followed a completion procedure to obtain a Banach operator ideal.

Inspired by the paper [**6**], especially by the Banach ideal property of $(Lt_p, \ell t_p)$, we introduce in Section 1 the general concept of "operator $[Y, p]$-summable sequence" in a Banach space X, consider the vector space $Y_p(X)$ of all operator $[Y, p]$-summable sequences in X and introduce a norm on this space. We then prove that $Y_p(X)$ is a Banach space. The results of the general setting are then applied to the special setting of operator p-summable sequences in a Banach space X. Based on the discussion of Section 1, we consider the sequentially p-limited operators in Section 2. Following standard techniques for p-summing operators, we prove the main result of this section which states that given any pair X, Y of Banach spaces, then the normed space $(Lt_p(X, Y), \ell t_p(\cdot))$ is a Banach space. Thus, the pair $(Lt_p, \ell t_p)$ is a complete normed operator ideal. In Section 3, we study the normed operator ideal $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ of operators $T\colon X \to Y$ so that for the scalar Banach sequence space Λ we have $ST \in \mathcal{A}(X, \Lambda)$ for all $S \in B(Y, \Lambda)$, where X, Y run through the family of all Banach spaces and where $(\mathcal{A}, \alpha)$ is a given normed operator ideal. The corresponding normed operator ideal $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ is studied and it is shown that if $(\mathcal{A}, \alpha)$ is a Banach operator ideal, then so is $(\mathcal{A}_\Lambda, \alpha_\Lambda)$. The results of Section 2 may also be obtained by using the operator ideal approach of Section 3. In Section 4, we consider some other classes of sequentially limited operators, which are also special cases of the operator ideals discussed in Section 3, and prove some multiplication (or composition) results.

We now recall some definitions and notations in the literature. The space of all weakly p-summable sequences in a Banach space X is denoted by $\ell_p^{wk}(X)$; recall that it is a Banach space with norm

$$\|(x_i)\|_p^{wk} := \sup\left\{\left(\sum_{i=1}^{\infty} |\langle x_i, x^*\rangle|^p\right)^{1/p} : x^* \in X^*, \|x^*\| \le 1\right\}.$$

This space is isometrically isomorphic to $B(\ell_{p'}, X)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$). For $(x_i) \in \ell_p^{wk}(X)$ (taking $1 < p < \infty$), the linear operator

$$E_{(x_i)} : \ell_{p'} \to X : (\lambda_i) \mapsto \sum_{i=1}^{\infty} \lambda_i x_i,$$

is bounded, with $\|E_{(x_i)}\| = \|(x_i)\|_p^{wk}$. Conversely, it is also well-known that each $T \in B(\ell_{p'}, X)$ can be uniquely identified as an operator $E_{(x_i)}$ for some $(x_i) \in \ell_p^{wk}(X)$, so that $B(\ell_{p'}, X)$ is isometrically identified with $\ell_p^{wk}(X)$ by the mapping $(x_i) \mapsto E_{(x_i)}$. Similarly, $B(\ell_1, X) = \ell_\infty^{wk}(X)$ and $B(c_0, X) = \ell_1^{wk}(X)$.

The space of all weak* p-summable sequences in the dual space X^* of a Banach space X is denoted by $\ell_p^{wk^*}(X^*)$. Recall that it is a Banach space with norm

$$\|(x_i^*)\|_p^{wk^*} := \sup\left\{\left(\sum_{i=1}^{\infty} |\langle x, x_i^*\rangle|^p\right)^{1/p} : x \in X, \|x\| \le 1\right\}.$$

This space is isometrically isomorphic to $B(X, \ell_p)$. For a fixed $(x_i^*) \in \ell_p^{wk^*}(X^*)$ the operator $F_{(x_i^*)}\colon X \to \ell_p : x \mapsto (\langle x, x_i^*\rangle)_i$ is bounded and linear with $\|F_{(x_i^*)}\| = \|(x_i^*)\|_p^{wk^*}$.

Conversely, since each $T \in B(X, \ell_p)$ can be uniquely identified with an operator $F_{(x_i^*)}$ for some $(x_i^*) \in \ell_p^{wk^*}(X^*)$, the mapping $(x_i^*) \mapsto F_{(x_i^*)}$ identifies $B(X, \ell_p)$ isometrically with $\ell_p^{wk}(X^*)$. In the case of $p = \infty$ we consider the space $c_0^{wk^*}(X^*)$ of weak* null sequences in X^*. Note that $\ell_p^{wk^*}(X^*) = \ell_p^{wk}(X^*)$, but that $c_0^{wk}(X^*) \subseteq c_0^{wk^*}(X^*)$, whereby $c_0^{wk}(X^*)$ is isometrically isomorphic to the space $\mathcal{W}(X, c_0)$ of weakly compact operators.

In general it is not true that $\lim_{n\to\infty} \|(x_i) - (x_1, x_2, \dots x_n, 0, 0 \dots)\|_p^{wk} = 0$. The subspace $\ell_{p,c}^{wk}(X)$ of $\ell_p^{wk}(X)$ for which this is true, is a Banach space with respect to the norm $\|(\cdot)\|_p^{wk}$ and the identification of $(x_i) \in \ell_{p,c}^{wk}(X)$ with $E_{(x_i)}\colon \ell_{p'} \to X : (\lambda_i) \mapsto \sum_{i=1}^{\infty} \lambda_i x_i$ defines an isometric isomorphism between $\ell_{p,c}^{wk}(X)$ and the space $K(\ell_{p'}, X)$ of compact linear operators. Similarly, the subspace $\ell_{p,c}^{wk^*}(X^*)$ of $\ell_p^{wk^*}(X^*)$ consisting of all sequences $(x_i^*) \in \ell_p^{wk^*}(X^*)$ so that $\lim_{n\to\infty} \|(x_i^*) - (x_1^*, x_2^*, \dots x_n^*, 0, 0 \dots)\|_p^{wk^*} = 0$, is isometrically isomorphic to the space $K(X, \ell_p)$ (of compact operators) by the isometry $(x_i^*) \mapsto F_{(x_i^*)}$. Refer (for instance) to the paper **[5]** for these facts.

2. Operator p-summable sequences.

Definition 2.1. Let X, Y be given Banach spaces and let $1 \le p < \infty$. A sequence (x_n) in X is called operator $[Y, p]$-summable if $\sum_{n=1}^{\infty} \|Tx_n\|^p < \infty$ for all $T \in B(X, Y)$, i.e. if $(Tx_n) \in \ell_p^s(Y)$ for all $T \in B(X, Y)$.

We can extend the Definition 2.1 above to include the case when $p = \infty$ by adopting the convention that "operator $[Y, \infty]$-summable sequence (x_n) in X" will mean that for each $T \in B(X, Y)$ we have $\|Tx_n\| \xrightarrow{n} 0$ (i.e. $(Tx_n) \in c_0^s(Y)$, $\forall T \in B(X, Y)$).

Let

$$Y_p(X) := \{(x_i) \in X^{\mathbb{N}} : (x_i) \text{ is operator } [Y, p]\text{-summable}\}.$$

For a given $(x_i) \in Y_p(X)$, we define an operator

$$B(X, Y) \to \ell_p^s(Y) : T \mapsto (Tx_n),$$

which has closed graph. Therefore, we may define

$$\|(x_i)\|_{Y_p} := \sup\left\{\left(\sum_{n=1}^{\infty} \|Tx_n\|^p\right)^{1/p} : T \in B(X, Y), \|T\| \le 1\right\},$$

for each $(x_i) \in Y_p(X)$.

For $x^* \in U_{X^*}$, $y \in U_Y$, the rank one operator

$$x^* \otimes y : X \to Y : x \mapsto \langle x, x^* \rangle y,$$

has norm ≤ 1 and

$$|\langle x_n, x^* \rangle| \le \|(x^* \otimes y)x_n\| \le \left(\sum_{i=1}^{\infty} \|(x^* \otimes y)x_i\|^p\right)^{1/p} \le \|(x_i)\|_{Y_p}.$$

It is therefore clear that

$$\|(x_i)\|_{Y_p} = 0 \implies x_i = 0, \ \forall i \in \mathbb{N}.$$

https://doi.org/10.1017/S0017089515000348 Published online by Cambridge University Press

It is readily verified that $\|\cdot\|_{Y_p}$ defines a norm on the vector space $Y_p(X)$ and that for any $(x_i) \in Y_p(X)$,

$$(\dagger) \qquad \|x_j\| \leq \|(x_i)\|_{Y_p}, \ \forall j \in \mathbb{N},$$

showing that

LEMMA 2.2. *If $x_n \xrightarrow{n} x$ in $(Y_p(X), \|\cdot\|_{Y_p})$, where $x_n = (x_{n,j})_j$ and $x = (x_j)$, then for each $j \in \mathbb{N}$, we have $x_{n,j} \xrightarrow{n} x_j$ in X.*

Using Lemma 2.2, the completeness of the space X and the inequality $(\dagger)$ above, it is routine to verify that

THEOREM 2.3. *$(Y_p(X), \|\cdot\|_{Y_p})$ is a Banach space.*

In the paper **[6]** (and elsewhere in the literature), the well-known concept of "limited set" in a Banach space is generalized to introduce the so-called "p-limited" sets. A subset D of a Banach space X is said to be *p-limited* $(1 \leq p < \infty)$ if for each weak* p-summable sequence (x_n^*) in X^* there exists a sequence $(\lambda_i) \in \ell_p$ such $|\langle x, x_n^*\rangle| \leq \lambda_n$ for each $n \in \mathbb{N}$ and all $x \in D$, i.e if and only if for each weak* p-summable sequence (x_n^*) in X^*, we have $(\sup_{x\in D} |\langle x, x_n^*\rangle|)_n \in \ell_p$.

Replacing Y in Definition 2.1 by ℓ_p, we agree to use the phrase "operator p-summable" instead of "operator $[\ell_p, p]$-summable", thereby recalling a definition from the paper **[6]**

DEFINITION 2.4 (cf. **[6]**). Let $1 \leq p < \infty$. A sequence (x_n) is called operator p-summable if $(Tx_n) \in \ell_p^s(\ell_p)$ for all $T \in B(X, \ell_p)$.

By Proposition 2.4 in **[6]** a sequence (x_n) in a Banach space X is operator p-summable if and only if $(x_n) \in \ell_p^{wk}(X)$ and $E_{(x_i)}(U_{\ell_{p'}})$ is a p-limited set. Let $\ell_p^o(X)$ denote the vector space of all operator p-summable sequences in the Banach space X (i.e. following the notation of Definition 2.1, we put $\ell_p^o(X) = (\ell_p)_p(X)$). If for $(x_i) \in \ell_p^o(X)$ we let

$$\|(x_i)\|_p^o := \sup\left\{\left(\sum_n \|Tx_n\|_p^p\right)^{1/p} : T \in B(X, \ell_p), \|T\| \leq 1\right\},$$

then it follows from Theorem 2.3 that:

THEOREM 2.5. *$(\ell_p^o(X), \|\cdot\|_p^o)$ is a Banach space.*

It is clear from Definition 2.4 above that $\ell_p^s(X)$ is a subspace of $\ell_p^o(X)$ and that the inclusion map has norm ≤ 1. Also, since the rank one operator

$$x^* \otimes e_j : X \to \ell_p : x \mapsto \langle x, x^*\rangle e_j,$$

has norm ≤ 1 for all $x^* \in U_{X^*}$ and all $j \in \mathbb{N}$, it is readily seen that for all $(x_i) \in \ell_p^o(X)$ we have $(x_i) \in \ell_p^{wk}(X)$ and $\|(x_i)\|_p^{wk} \leq \|(x_i)\|_p^o$. We thus conclude that

THEOREM 2.6. *Let $1 \leq p < \infty$. We have the following continuous (norm ≤ 1) inclusions:*

$$\ell_p^s(X) \subseteq \ell_p^o(X) \subseteq \ell_p^{wk}(X).$$

REMARK 2.7. Let $1 \leq p < \infty$. Recall that an operator $T \in B(X, Y)$ is said to be p-summing if $(Tx_n) \in \ell_p^s(Y)$ for all $(x_n) \in \ell_p^{wk}(X)$. The vector space $\Pi_p(X, Y)$ of all p-summing operators is a Banach space with respect to the norm

$$\pi_p(T) := \sup\{\|(Tx_n)\|_p : \|(x_n)\|_p^{wk} \leq 1\}.$$

For a detailed discussion of p-summing operators, the reader is referred to the book **[4]**. In **[6]** the authors introduce and study the so called *weak p-spaces*. A Banach space X is called a weak p-space (or X is said to have the p-Dunford–Pettis property) if $\ell_p^o(X) = \ell_p^{wk}(X)$. This is the case if and only if $\Pi_p(X, \ell_p) = B(X, \ell_p)$ (cf. **[6]**, Proposition 3.1). It is therefore immediately clear that ℓ_p itself is not a weak p-space. Moreover, it is shown in **[6]** that ℓ_p (for $1 < p < \infty$) is in fact not a weak r-space for any $r > 1$.

By Theorem 8.3.1 in **[1]** (page 213) every $T \in B(L_1(\mu), \ell_2)$ is absolutely summing and therefore also 2-summing. Thus, the space $L_1(\mu)$ is a weak 2-space. Since ℓ_1 is an $L_1(\mu)$-space for a suitable measure μ, the same theorem in **[1]** also holds for operators $T : \ell_1 \to \ell_2$. Thus, in contrast with the spaces ℓ_p for $1 < p < \infty$, the space ℓ_1 is a weak 2-space. If X is an infinite dimensional reflexive Banach space, then all p-summing operators on X are compact (cf. **[1]**, Corollary 8.2.15, page 211). Therefore, $K(X, \ell_p) = B(X, \ell_p)$ if X is a reflexive weak p-space. The equality $K(X, \ell_p) = B(X, \ell_p)$ holds if and only if $\ell_{p,c}^{wk^*}(X^*) = \ell_p^{wk^*}(X^*)$ and this is the case if and only if $\ell_p^{wk^*}(X^*) \subset c_0^s(X^*)$ (cf. for instance **[2]** and **[5]** for these facts).

3. Sequentially p-limited operators. In **[6]** (Definition 4.1), an operator $T \in B(X, Y)$ is said to be sequentially p-limited if it maps weakly p-summable sequences to operator p-summable sequences, i.e.

DEFINITION 3.1 (cf. **[6]**, Definition 4.1). Let $1 \leq p < \infty$. An operator $T \in B(X, Y)$ is called sequentially p-limited if $(Tx_n) \in \ell_p^o(Y)$ for all $(x_n) \in \ell_p^{wk}(X)$.

It is clear from the definition and Remark 2.7 that id_X is sequentially p-limited if and only if X is a weak p-space. An operator $T: X \to Y$ is sequentially p-limited if and only if RT is p-summing for all $R \in B(Y, \ell_p)$. Refer to **[6]** (Theorem 4.4, p.435) for this fact. Following **[6]** we let

$$Lt_p(X, Y) := \{T \in B(X, Y) : T \text{ is sequentially } p\text{-limited}\}.$$

The authors (in **[6]**) define a norm on $Lt_p(X, Y)$ by

$$\ell t_p(T) := \sup\{\pi_p(RT) : R \in B(Y, \ell_p) \text{ and } \|R\| \leq 1\},$$

and mention that it is routine to show that the pair $(Lt_p, \ell t_p)$ so defined is a normed operator ideal. However, the authors also note in **[6]** that they could not show that $(Lt_p, \ell t_p)$ is a Banach operator ideal. In order to obtain a Banach operator ideal, they settled for taking the completions of isometric copies of the components $Lt_p(X, Y)$ of the ideal $(Lt_p, \ell t_p)$ in the corresponding Banach spaces $B(B(Y, \ell_p), \Pi_p(X, \ell_p))$. Here is what happens in **[6]**: For each $T \in B(X, Y)$ the authors consider the operator $\varphi_T : B(Y, \ell_p) \to B(X, \ell_p)$, given by $\varphi_T(S) = ST$, and note that $T \mapsto \varphi_T$ is a linear isometry from $B(X, Y)$ into $B(B(Y, \ell_p), B(X, \ell_p))$ for $1 \leq p \leq \infty$ (where in the case of $p = \infty$, the space ℓ_∞ is replaced by c_0). Then $T \in Lt_p(X, Y)$ if and only if $\varphi_T(B(Y, \ell_p)) \subset \Pi_p(X, \ell_p)$ and $\ell t_p(T) = \|\varphi_T\|$; here $\|\varphi_T\|$ denotes the operator norm of φ_T considered as an element of $B(B(Y, \ell_p), \Pi_p(X, \ell_p))$. This can be done, since the Closed Graph Theorem implies that if $T \in Lt_p(X, Y)$, then

$$\sup_{R \in U_{B(Y,\ell_p)}} \pi_p(RT) < \infty.$$

A discussion of this fact in the general setting of operator ideals follows in Section 3. Finally, the authors consider the completion (closure) of the set $\{\varphi_T : T \in Lt_p(X, Y)\}$ in the complete space $B(B(Y, \ell_p), \Pi_p(X, \ell_p))$ and denote this completion (also) by $Lt_p(X, Y)$. In this way a Banach operator ideal is obtained (cf. [**6**], Proposition 4.7).

The presence of the Banach operator ideal (Π_p, π_p) in the definition of $(Lt_p, \ell t_p)$ suggests a different approach in the study of sequentially p-limited operators via the theory of p-summing operators. Based on our discussion of the sequence space $\ell_p^o(X)$ in Section 1, we are now ready to discuss the completeness of the normed space $(Lt_p(X, Y), \ell t_p(\cdot))$ in the following theorem.

THEOREM 3.2. *Let $1 \le p < \infty$ and let X, Y be Banach spaces. The space $(Lt_p(X, Y), \ell t_p(\cdot))$ of sequentially p-limited operators is a Banach space. Thus, $(Lt_p, \ell t_p)$ is a Banach operator ideal.*

Proof. We associate with each $T \in Lt_p(X, Y)$ the operator

$$\widehat{T} : \ell_p^{wk}(X) \to \ell_p^o(Y) : (x_i) \mapsto (Tx_i).$$

A routine argument, involving the operators $y^* \otimes e_1 \in U_{B(Y,\ell_p)}$ where y^* runs through the unit ball U_{Y^*} of Y^*, shows that $\widehat{T}$ has closed graph and hence is bounded. Note that

$$\|\widehat{T}\| = \sup_{\|(x_i)\|_p^{wk} \le 1} \|(Tx_i)\|_p^o.$$

Also

$$\sup_{R \in U_{B(Y,\ell_p)}} \pi_p(RT) = \sup_{\|(x_i)\|_p^{wk} \le 1} \sup\{\|(RTx_i)\|_p : R \in U_{B(Y,\ell_p)}\} = \|\widehat{T}\|,$$

showing that $\ell t_p(T) = \|\widehat{T}\|$. From this discussion it is clear that the mapping

$$\Phi : Lt_p(X, Y) \to B(\ell_p^{wk}(X), \ell_p^o(Y)) : T \mapsto \widehat{T},$$

is an isometry which associates the space $(Lt_p(X, Y), \ell t_p(\cdot))$ isometrically with a subspace of the Banach space $B(\ell_p^{wk}(X), \ell_p^o(Y))$. We denote the range space $\Phi(Lt_p(X, Y))$ by $\widehat{Lt}_p(X, Y)$ and prove that it is a closed subspace of the complete normed space $B(\ell_p^{wk}(X), \ell_p^o(Y))$: Consider any sequence $(\widehat{T}_n)$ in $\widehat{Lt}_p(X, Y)$ which converges in operator norm to some operator $S \in B(\ell_p^{wk}(X), \ell_p^o(Y))$. If $(x_i) \in \ell_p^{wk}(X)$ and $S((x_i)) = (y_i) \in \ell_p^o(Y)$, then

$$\|(T_n x_i) - (y_i)\|_p^o = \|\widehat{T}_n((x_i)) - (y_i)\|_p^o \xrightarrow{n} 0,$$

where $T_n \in Lt_p(X, Y)$ such that $\Phi(T_n) = \widehat{T}_n$ for all $n \in \mathbb{N}$. By Lemma 2.2, it thus follows that $T_n x_j \xrightarrow{n} y_j$ for all $j \in \mathbb{N}$. This implies that if we put $Tx = \lim_n T_n x$ for each $x \in X$, then T is a bounded linear operator and given any $(x_i) \in \ell_p^{wk}(X)$ and $S((x_i)) = (y_i)$, we have $Tx_i = y_i$ for all $i \in \mathbb{N}$. Thus $T \in Lt_p(X, Y)$ and $S = \widehat{T} \in \widehat{L}t_p(X, Y)$, showing that $\widehat{L}t_p(X, Y)$ is a closed subspace of the Banach space $B(\ell_p^{wk}(X), \ell_p^o(Y))$, i.e. $Lt_p(X, Y)$ is a Banach space for all Banach spaces X, Y. □

Let $1 < p < \infty$. In this case, using Proposition 2.19 in **[4]** (page 50), it is easy to see that if the second dual operator $T^{**}: X^{**} \to Y^{**}$ of the operator $T \in B(X, Y)$ is sequentially p-limited, then so is T. Moreover, we also have:

PROPOSITION 3.3. *Let $1 < p < \infty$. If an operator $T: X \to Y$ is sequentially p-limited and weakly compact, then so is its second dual T^{**}.*

Proof. Assume that $T: X \to Y$ is sequentially p-limited and weakly compact. Since T is weakly compact, it follows from Theorem 5.5 in **[3]** (page 185) that T^{**} is weakly compact and that $T^{**}(X^{**}) \subseteq Y$. Let $S \in B(Y^{**}, \ell_p)$ and denote the canonical embedding (evaluation) from Y into Y^{**} by C_Y. Recall that C_Y^* defines a canonical projection from Y^{***} to Y^*. Using the above information we get

$$\langle ST^{**}x^{**}, \gamma\rangle = \langle S^{**}C_Y^{**}T^{**}x^{**}, \gamma\rangle,$$

for all $x^{**} \in X^{**}$, $\gamma \in \ell_{p'}$ and thus that

$$ST^{**} = S^{**}C_Y^{**}T^{**} = (SC_YT)^{**}.$$

Since SC_YT is p-summing, it follows from Proposition 2.19 in **[4]** that ST^{**} is p-summing. This completes the proof. □

There are sequentially p-limited operators which are not weakly compact. Refer to the discussion in Remark 2.7 above. Each bounded linear operator from ℓ_1 to ℓ_2 is absolutely summing, hence by the Inclusion Theorem (see **[4]**, Theorem 2.8, page 39) each bounded linear operator from ℓ_1 to ℓ_2 is p-summing for all $1 \le p < \infty$, in particular, each $S \in B(\ell_1, \ell_2)$ is 2-summing. Therefore, the identity $id_{\ell_1} : \ell_1 \to \ell_1$ is sequentially 2-limited, but not weakly compact. This argument, of course, will also imply that for all nonreflexive Banach spaces X such that $B(X, \ell_p) = \Pi_p(X, \ell_p)$ (i.e. for all nonreflexive weak-p spaces) the identity operator id_X on X is sequentially p-limited but not weakly compact. The discussion in Remark 2.7 shows that $L_1(\mu)$ and ℓ_1 are (nonreflexive) weak 2-spaces. Also, c_0 is a (nonreflexive) weak 2-space (see **[6]**). In general, if X, Y are nonreflexive Banach spaces, where X is also a weak p-space, then each $T \in B(X, Y)$ such that $T \notin \mathcal{W}(X, Y)$ is an example of a sequentially p-limited operator which is not weakly compact. By an application of Grothendieck's Inequality, it follows that for each compact Hausdorff space K and any measure μ, we have $B(C(K), L_p(\mu)) = \Pi_2(C(K), L_p(\mu))$ if $1 \le p \le 2$. In particular, this yields $B(C(K), \ell_2) = \Pi_2(C(K), \ell_2)$, i.e. that $C(K)$ is a nonreflexive weak 2-space (cf. **[4]**, Theorem 3.5 for the details). This is also true for all Banach spaces X such that X^{**} is a $C(K)$ space.

The immediate question arising from Proposition 3.3 is when a sequentially p-limited operator $T : X \to Y$ will be weakly compact. In the following lemma we list some conditions (by no means all possible conditions) from the literature which imply that each bounded linear operator is weakly compact.

LEMMA 3.4. *For two Banach spaces X and Y, each $T \in B(X, Y)$ is weakly compact if*

(i) *Either X or Y is reflexive.*
(ii) *$X = C(K)$ for some compact Hausdorff space K and no closed subspace of Y is isomorphic to c_0 (cf. [**1**], Corollary 5.5.4, page 120).*
(iii) *X does not contain a copy of ℓ_1 and $Y = L_1$, where L_1 denotes the space $L_1([0, 1], \lambda)$ and λ is Lebesgue measure on $[0, 1]$ (cf. [**1**], page 125).*
(iv) *If X has type $r > 1$ and $Y = L_1(\mu)$ (for some σ-finite measure μ), for in this case each T factors through the reflexive space $L_q(\mu)$ for all $1 < q < r$ (cf. [**1**], Theorem 7.1.8, page 172).*
(v) *X has type 2 and Y has cotype 2, for in this case each T factors through a Hilbert space by the well-known Kwapień–Maurey Theorem (cf. [**1**], Theorem 7.4.2, page 187).*
(vi) *X^* has cotype 2 and $Y = L_1$, for in this case each T factors through a Hilbert space (cf. [**1**], Theorem 8.1.7, page 203).*
(vii) *X^* has cotype 2, Y has cotype 2 and either X or Y has the approximation property, for in this case each T factors through a Hilbert space by Pisier's Abstract Grothendieck Theorem (cf. [**1**], Theorem 8.1.8, page 204).*

From Proposition 3.3 and Lemma 3.4 we conclude that

COROLLARY 3.5. *Let $1 < p < \infty$. If the Banach spaces X and Y satisfy any one of the conditions* (i) *to* (vii) *in Lemma 3.4, then if $T : X \to Y$ is sequentially p-limited, so is T^{**}.*

4. An operator ideal approach. The reader is referred to [**7**] for information on operator ideals. Consider a Banach operator ideal $(\mathcal{A}, \alpha)$. Fix a Banach sequence space $(\Lambda, \|\cdot\|_\Lambda)$ which contains the set ϕ of all sequences having only a finite number of non-zero terms and for which $\|e_n\|_\Lambda = 1$ for all $n \in \mathbb{N}$. Clearly, $\Lambda = \ell_p$ (for $1 \le p \le \infty$) and $\Lambda = c_0$ satisfy these properties. However, there are more Banach sequence spaces with these properties (see for instance Remark 4.6 at the end of this section). With the vector space $\mathcal{A}(X, Y)$ we associate

$$\mathcal{A}_\Lambda(X, Y) := \{T \in B(X, Y) : ST \in \mathcal{A}(X, \Lambda),\ \forall S \in B(Y, \Lambda)\}.$$

From the operator ideal properties of $\mathcal{A}$ it is easily verified that $\mathcal{A}_\Lambda$ also defines an operator ideal.

For $T \in B(X, Y)$ we let

$$\phi_T : B(Y, \Lambda) \to B(X, \Lambda) : S \mapsto ST.$$

Then ϕ_T is a bounded linear operator for which $\|\phi_T\| \le \|T\|$ is clear from its definition. On the other hand

$$\|\phi_T\| \ge \sup_{\|y^*\|\le 1} \|(y^* \otimes e_1) \circ T\| = \sup_{\|y^*\|\le 1} \sup_{\|x\|\le 1} \|\langle Tx, y^*\rangle e_1\|_\Lambda = \|T\|.$$

Thus, we define an isometry $T \mapsto \phi_T$ from $B(X, Y)$ into $B(B(Y, \Lambda), B(X, \Lambda))$. It is then clear that $T \in \mathcal{A}_\Lambda(X, Y)$ if and only if $\phi_T(B(Y, \Lambda)) \subseteq \mathcal{A}(X, \Lambda)$.

Now let $T \in \mathcal{A}_\Lambda(X, Y)$ be given. By the above discussion ϕ_T is a linear operator from $B(Y, \Lambda)$ into $\mathcal{A}(X, \Lambda)$. Using that $\|R\| \le \alpha(R)$ for all $R \in \mathcal{A}(X, \Lambda)$, a routine argument shows that $\phi_T : B(Y, \Lambda) \to (\mathcal{A}(X, \Lambda), \alpha)$ has closed graph. Thus, we may define

$$\alpha_\Lambda(T) := \sup\{\alpha(ST) : S \in B(Y, \Lambda), \|S\| \leq 1\}.$$

Then $\alpha_\Lambda(\cdot)$ defines a norm on $\mathcal{A}_\Lambda(X, Y)$ and $\|T\| \leq \alpha_\Lambda(T)$ for all $T \in \mathcal{A}_\Lambda(X, Y)$. Since this is true for all Banach spaces X and Y, it therefore follows that:

PROPOSITION 4.1. $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ *is a normed operator ideal.*

Clearly, $\alpha_\Lambda(T)$ is the operator norm of ϕ_T considered as an element of $B(B(Y, \Lambda), \mathcal{A}(X, Y))$. Therefore, $T \mapsto \phi_T$ defines an isometry from $\mathcal{A}_\Lambda(X, Y)$ into $B(B(Y, \Lambda), \mathcal{A}(X, Y))$, associating $\mathcal{A}_\Lambda(X, Y)$ isometrically with a subspace $A_\Lambda := \{\phi_T : T \in \mathcal{A}_\Lambda(X, Y)\}$ of the Banach space $B(B(Y, \Lambda), \mathcal{A}(X, Y))$.

Taking $R \in B(B(Y, \Lambda), \mathcal{A}(X, Y))$ from the closure of the subspace A_Λ, let $(\phi_{T_n}) \subset A_\Lambda$ so that $\phi_{T_n} \xrightarrow{n} R$ in the operator norm of $B(B(Y, \Lambda), \mathcal{A}(X, Y))$. Then, since

$$\|ST_n - R(S)\| \leq \alpha(ST_n - R(S)) \xrightarrow{n} 0, \ \forall S \in B(Y, \Lambda),$$

$ST_nx \xrightarrow{n} R(S)x$ for each $x \in X$ and each $S \in B(Y, \Lambda)$. Denote the restriction of the operator norm of $B(B(Y, \Lambda), \mathcal{A}(X, Y))$ to the subspace A_Λ by $\|\cdot\|_{A_\Lambda}$. From the isometry $T \mapsto \phi_T$ discussed above, we then conclude that

$$\|T_n - T_m\| \leq \alpha_\Lambda(T_n - T_m) = \|\phi_{T_n} - \phi_{T_m}\|_{A_\Lambda} \to 0 \text{ as } m, n \to \infty.$$

Thus, there exists $T \in B(X, Y)$ so that $T_n \xrightarrow{n} T$ in the operator norm of $B(X, Y)$ and $R(S)x = STx$ for each $x \in X$ and each $S \in B(Y, \Lambda)$, i.e. $R = \phi_T$. We have thus shown that for each pair of Banach spaces X and Y, the vector space A_Λ is a closed subspace of the complete space $B(B(Y, \Lambda), \mathcal{A}(X, Y))$. Therefore, we may conclude that

THEOREM 4.2. $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ *is a Banach operator ideal.*

It is clear from the definition that $\mathcal{A}_\Lambda(X, \Lambda) = \mathcal{A}(X, \Lambda)$ for all Banach spaces X. Recall that an operator ideal $\mathcal{A}$ is said to be *surjective* if for all Banach spaces X, Y, Z, and each $T \in B(X, Y)$ for which there exists a surjective operator (quotient map) $Q \in B(Z, X)$ so that $TQ \in \mathcal{A}(Z, Y)$, it follows that $T \in \mathcal{A}(X, Y)$. It is easily seen that if $\mathcal{A}$ is a surjective ideal, then so is $\mathcal{A}_\Lambda$.

For a given Banach operator ideal $(\mathcal{A}, \alpha)$, consider the associated operator ideal $(\mathcal{A}_\Lambda^\diamond, \alpha_\Lambda^\diamond)$, whereby we let

$$\begin{aligned} \mathcal{A}_\Lambda^\diamond(X, Y) &= \{T \in B(X, Y) : TS \in \mathcal{A}(\Lambda, Y), \forall S \in B(\Lambda, X)\} \\ \alpha_\Lambda^\diamond(T) &= \sup\{\alpha(TS) : S \in B(\Lambda, X), \|S\| \leq 1\}. \end{aligned}$$

To verify that $\alpha_\Lambda^\diamond$ is an ideal norm, note that

$$\|T\| \leq \sup_{\|x\| \leq 1} \|T \circ (e_1 \otimes x)\| \leq \sup\{\alpha(TS) : S \in B(\Lambda, X), \|S\| \leq 1\} = \alpha_\Lambda^\diamond(T).$$

From the definition we have $\mathcal{A}_\Lambda^\diamond(\Lambda, Y) = \mathcal{A}(\Lambda, Y)$ for all Banach spaces Y. Recall that an operator ideal $\mathcal{A}$ is said to be *injective* if for all Banach spaces X, Y, Y_0 such that Y is isometrically embedded into Y_0 by $J \in B(Y, Y_0)$, it follows from $T \in B(X, Y)$ and $JT \in \mathcal{A}(X, Y_0)$ that $T \in \mathcal{A}(X, Y)$. It is easily seen that if $\mathcal{A}$ is an injective ideal, then so is $\mathcal{A}_\Lambda^\diamond$.

https://doi.org/10.1017/S0017089515000348 Published online by Cambridge University Press

We conclude this section with a brief discussion of dual ideals in the context of this manuscript. Given a Banach operator ideal $(\mathcal{A}, \alpha)$, we recall that a Banach operator ideal $(\mathcal{A}^d, \alpha^d)$, called the *dual ideal* of $(\mathcal{A}, \alpha)$, is defined by the components

$$\mathcal{A}^d(X, Y) := \{T \in B(X, Y) : T^* \in \mathcal{A}(Y^*, X^*)\},$$

where X, Y run through the family of all Banach spaces. Here

$$\alpha^d(T) = \alpha(T^*).$$

Let $1 < p < \infty$. Since each $T \in B(\ell_{p'}, X^*)$ is weak-to-weak continuous and $\ell_{p'}$ is reflexive, we have

LEMMA 4.3. *Let $1 < p < \infty$. Each $T \in B(\ell_{p'}, X^*)$ is weak*-to-weak* continuous.*

Consider the case when $\Lambda = \ell_p$ with $1 < p < \infty$. In this case denote the Banach operator ideal $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ (respectively, $(\mathcal{A}_\Lambda^\diamond, \alpha_\Lambda^\diamond)$) by $(\mathcal{A}_p, \alpha_p)$ (respectively, $(\mathcal{A}_p^\diamond, \alpha_p^\diamond)$). Using Lemma 4.3 to realize that $S \in B(\ell_{p'}, Y^*)$ if and only if there exists $R \in B(Y, \ell_p)$ such that $R^* = S$, one verifies easily that

PROPOSITION 4.4. *Let $1 < p < \infty$. Then $T \in (\mathcal{A}_{p'}^\diamond)^d(X, Y)$ if and only if $T \in (\mathcal{A}^d)_p(X, Y)$; in this case $(\alpha_{p'}^\diamond)^d(T) = (\alpha^d)_p(T)$.*

It follows from Proposition 4.4 that:

COROLLARY 4.5. *For $1 < p < \infty$, we have*

$$(\Pi_p^d)_p(X, Y) := \{T \in B(X, Y) : T^*S \in \Pi_p(\ell_{p'}, X^*), \forall S \in B(\ell_{p'}, Y^*)\}.$$

REMARK 4.6. In our discussion above it is clear that to obtain the necessary isometric embedding $T \mapsto \phi_T$, we need to assume the properties on Λ stated at the beginning of this section (in particular, that $\|e_n\|_\Lambda = 1$ for all n). These properties are for instance also shared by some Orlicz sequence spaces. For example, if for $1 \le p < \infty$ we let

$$N_p(t) = \begin{cases} t^p(1 + |\ln t|), & \text{for } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

then N_p defines an Orlicz function. The norm on the corresponding Orlicz sequence space

$$h_{N_p} := \left\{(\alpha_i) : \sum_{n=1}^{\infty} N_p\left(\frac{|\alpha_n|}{\rho}\right) < \infty, \ \forall \rho > 0\right\},$$

is given by

$$\|(\alpha_i)\|_{N_p} := \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} N_p\left(\frac{|\alpha_n|}{\rho}\right) \le 1\right\}.$$

The set $\{e_n\}_{n=1}^{\infty}$ of unit vectors is a symmetric basis for h_{N_p} and since $N_p(t) > 1$ if $t > 1$, $N_p(1) = 1$ and $\lim_{t \downarrow 1} N_p(t) = 1$, it follows that $\|e_n\|_{N_p} = 1$ for all $n \in \mathbb{N}$.

Our result Theorem 3.2 also follows from the (general) operator ideal approach discussed in this section (in particular from Theorem 4.2): Take $(\mathcal{A}, \alpha) = (\Pi_p, \pi_p)$ and $\Lambda = \ell_p$ (where $1 \leq p < \infty$) to verify that in this case $(\mathcal{A}_\Lambda, \alpha_\Lambda) = (Lt_p, \ell t_p)$ and hence that the space $(Lt_p(X, Y), \ell t_p(\cdot))$ of sequentially p-limited operators is a Banach space.

5. More classes of operators. Recall from [**4**] (Chapter 10, page 197) that an operator $T : X \to Y$ is called (q, p)-summing (with $1 \leq p, q < \infty$) if there is an induced operator

$$\widehat{T} : \ell_p^{wk}(X) \to \ell_q^s(Y) : (x_n) \mapsto (Tx_n).$$

The vector space of (q, p)-summing operators is denoted by $\Pi_{q,p}(X, Y)$; it is normed by the norm

$$\pi_{q,p}(T) = \|\widehat{T}\|,$$

where $\|\widehat{T}\|$ denotes the operator norm of $\widehat{T}$. A bounded linear operator $T \in B(X, Y)$ is (q, p)-summing if and only if there is some $C \geq 0$ for which

$$(*) \qquad \left(\sum_{i=1}^{n} \|Tx_i\|^q\right)^{1/q} \leq C \sup_{x^* \in U_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, x^*\rangle|^p\right)^{1/p},$$

no matter how the finite set $\{x_1, \ldots, x_n\}$ of vectors from X is chosen. Moreover, $\pi_{q,p}(T)$ is the least such constant C. Using the version $(*)$ of the definition of a (q, p)-summing operator, one soon verifies that only the zero operator can be (q, p)-summing if $q < p$. It will therefore be natural to assume that $p \leq q$. Under the assumption $1 \leq p \leq q < \infty$, the pair $(\Pi_{q,p}, \pi_{q,p})$ is an injective Banach ideal (cf. [**4**], Proposition 10.2). Observe that an operator $T \in B(X, Y)$ is (p, p)-summing if and only if it is p-summing (refer to [**4**], page 31 for the definition of p-summing operator in terms of the corresponding version of the inequality $(*)$ for the case $p = q$).

Let $1 \leq p \leq q < \infty$ and let $1 \leq r < \infty$. If we let $(\mathcal{A}, \alpha) = (\Pi_{q,p}, \pi_{q,p})$ in our general discussion of Section 3, then

(1) We denote $(\mathcal{A}_{\ell_r}, \alpha_{\ell_r})$ by $(Lt_{q,p,r}, \ell t_{q,p,r})$. In this case we have $T \in Lt_{q,p,r}(X, Y)$ if and only if $ST \in \Pi_{q,p}(X, \ell_r)$ for all $S \in B(Y, \ell_r)$, i.e. if and only if

$$\sum_{n=1}^{\infty} \|STx_n\|_r^q < \infty, \quad \forall\, S \in B(Y, \ell_r),\ \forall\, (x_n) \in \ell_p^{wk}(X).$$

Also, for $T \in Lt_{q,p,r}(X, Y)$, we have

$$\ell t_{q,p,r}(T) = \sup_{S \in U_{B(Y,\ell_r)}} \pi_{q,p}(ST).$$

(2) In case of $p = q = r$, we clearly have $(Lt_{p,p,p}, \ell t_{p,p,p}) = (Lt_p, \ell t_p)$.
(3) In case of $p = q$, we denote $(Lt_{q,p,r}, \ell t_{q,p,r})$ by $(Lt_{p,r}, \ell t_{p,r})$. In this case we have $T \in Lt_{p,r}(X, Y)$ if and only if $ST \in \Pi_p(X, \ell_r)$ for all $S \in B(Y, \ell_r)$. The operators $T \in Lt_{p,r}(X, Y)$ will be called sequentially (p, r)-limited. In case of $p = r$, we again have $(Lt_{p,p}, \ell t_{p,p}) = (Lt_p, \ell t_p)$.

By Theorem 4.2, the pairs $(Lt_{q,p,r}, \ell t_{q,p,r})$ and $(Lt_{p,r}, \ell t_{p,r})$ are Banach operator ideals.

Using Theorem 2.8 in **[4]**, we have the following inclusion result:

THEOREM 5.1. *Let $1 \leq p \leq q < \infty$ and $1 \leq r < \infty$. Then $Lt_{p,r}(X, Y) \subseteq Lt_{q,r}(X, Y)$. Moreover, for $T \in Lt_{p,r}(X, Y)$ we have $\ell t_{q,r}(T) \leq \ell t_{p,r}(T)$.*

Proof. Given $T \in Lt_{p,r}(X, Y)$ and $S \in B(Y, \ell_r)$, it follows that $ST \in \Pi_p(X, \ell_r)$ and $\pi_p(ST) \leq \|S\| \ell t_{p,r}(T)$. By Theorem 2.8 in **[4]** we therefore have $ST \in \Pi_q(X, \ell_r)$ and

$$\pi_q(ST) \leq \pi_p(ST) \leq \|S\| \ell t_{p,r}(T).$$

Since S was arbitrary, it follows that $T \in Lt_{q,r}(X, Y)$ and

$$\ell t_{q,r}(T) = \sup_{S \in U_{B(Y,\ell_r)}} \pi_q(ST) \leq \sup_{S \in U_{B(Y,\ell_r)}} \|S\| \ell t_{p,r}(T) = \ell t_{p,r}(T).$$

□

Generalizing Theorem 5.1, we may use Theorem 10.4 in **[4]** in a similar fashion to prove that

THEOREM 5.2. *Let $1 \leq t < \infty$ and suppose that $1 \leq p_j \leq q_j < \infty$ $(j = 1, 2)$ satisfy $p_1 \leq p_2$, $q_1 \leq q_2$ and*

$$\frac{1}{p_1} - \frac{1}{q_1} \leq \frac{1}{p_2} - \frac{1}{q_2}.$$

Then

$$Lt_{q_1,p_1,r}(X, Y) \subseteq Lt_{q_2,p_2,r}(X, Y),$$

and for each $T \in Lt_{q_1,p_1,r}(X, Y)$ we have

$$\ell t_{q_2,p_2,r}(T) \leq \ell t_{q_1,p_1,r}(T).$$

Using two more results from **[4]** (namely, Lemma 2.23 and Theorem 2.22) we obtain the following multiplication theorem:

THEOREM 5.3. *Let $1 \leq p, q, r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $T \in \Pi_q(X, Y)$ and $S \in Lt_{p,r}(Y, Z)$. Then $ST \in Lt_r(X, Z)$ and*

$$\ell t_r(ST) \leq \ell t_{p,r}(S)\, \pi_q(T).$$

Proof. Let $(x_i) \in \ell_r^{wk}(X)$ and put $\gamma := \|(x_m)\|_r^{wk}$. By Lemma 2.23 of **[4]**, let $Tx_n = \sigma_n y_n$, for all $n \in \mathbb{N}$, where $(\sigma_n) \in \ell_q$ and $(y_n) \in \ell_p^{wk}(Y)$ such that

$$\|(\sigma_n)\|_q \leq \gamma^{r/q} \text{ and } \|(y_n)\|_p^{wk} \leq \gamma^{r/p}\, \pi_q(T).$$

https://doi.org/10.1017/S0017089515000348 Published online by Cambridge University Press

For $R \in B(Z, \ell_r)$, with $\|R\| \leq 1$, we have $(RSy_n) \in \ell_p^s(\ell_r)$ and

$$\begin{aligned}\left(\sum_n \|RSTx_n\|_r^r\right)^{1/r} &= \left(\sum_n |\sigma_n|^r \|RSy_n\|_r^r\right)^{1/r} \\ &\leq \left(\sum_n |\sigma_n|^q\right)^{1/q} \left(\sum_n \|RSy_n\|_r^p\right)^{1/p} \\ &\leq \gamma^{r/q} \ell t_{p,r}(S) \|(y_n)\|_p^{wk} \\ &\leq \ell t_{p,r}(S)\, \pi_q(T)\, \|(x_n)\|_r^{wk}.\end{aligned}$$

This shows that $ST \in Lt_r(X, Z)$. Taking firstly the supremum over all $\|(x_n)\|_r^{wk} \leq 1$ and then the supremum over all $R \in B(Z, \ell_r)$, with $\|R\| \leq 1$, it also follows that $\ell t_r(ST) \leq \ell t_{p,r}(S)\, \pi_q(T)$. □

Corollary 5.4. *Let $1 \leq p, q < \infty$ be such that $1 \leq \frac{1}{p} + \frac{1}{q}$. If $S \in Lt_{p,1}(Y, Z)$ and $T \in \Pi_q(X, Y)$, then $ST \in Lt_1(X, Z)$ and*

$$\ell t_1(ST) \leq \ell t_{p,1}(S)\, \pi_q(T).$$

Proof. For $p = 1$ we have $S \in Lt_1(Y, Z)$ and so by the operator ideal properties we also have $ST \in Lt_1(X, Z)$ and

$$\begin{aligned}\ell t_1(ST) &= \sup_{R \in U_{B(Z,\ell_1)}} \pi_1(RST) \\ &\leq \sup_{R \in U_{B(Z,\ell_1)}} \pi_1(RS)\|T\| \leq \ell t_{1,1}(S)\pi_q(T).\end{aligned}$$

Now, assume $p > 1$. Then $1 \leq q \leq p' < \infty$, hence $T \in \Pi_{p'}(X, Y)$ and $\pi_{p'}(T) \leq \pi_q(T)$ by Theorem 2.8 in [**4**]. The result follows by application of Theorem 5.3. □

Let $1 \leq q < \infty$. Recall from Proposition 5.23 in [**4**] (page 112) that $T \in B(X, Y)$ is q-nuclear if and only if it has a representation $T = \sum_{i=1}^{\infty} x_i^* \otimes y_i$, where $(x_i^*) \in \ell_q^s(X^*)$ and $(y_i) \in \ell_{q'}^{wk}(Y)$. The norm on the vector space $\mathcal{N}_q(X, Y)$ of q-nuclear operators is then given by

$$\nu_q(T) := \inf \left\{ \|(x_i^*)\|_q^s \|(y_i)\|_{q'}^{wk} : T = \sum_{i=1}^{\infty} x_i^* \otimes y_i \right\}.$$

From the notation of our earlier discussion in this paper it follows that if $\mathcal{A} = \mathcal{N}_q$ (and $\alpha = \nu_q$) and $\Lambda = \ell_r$, then $\mathcal{A}_\Lambda$ becomes $(\mathcal{N}_q)_r$ and $\alpha_\Lambda = (\nu_q)_r$. With this notation in mind, we have the following composition result:

Theorem 5.5. *Let $1 \leq p, q, r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $T \in \mathcal{N}_q(X, Y)$ and $S \in Lt_{p,r}(Y, Z)$. Then $ST \in (\mathcal{N}_r)_r(X, Z)$ and*

$$(\nu_r)_r(ST) \leq \ell t_{p,r}(S)\, \nu_q(T).$$

Proof. Let $T \in \mathcal{N}_q(X, Y)$. For $\delta > 0$ being arbitrarily given, let $T = \sum_{i=1}^{\infty} x_i^* \otimes y_i$, with $(x_i^*) \in \ell_q^s(X^*)$ and $(y_i) \in \ell_{q'}^{wk}(Y)$ so chosen that

$$\|(x_i^*)\|_q^s \leq 1 \text{ and } \|(y_i)\|_{q'}^{wk} \leq \nu_q(T) + \delta\,.$$

Let $R \in B(Z, \ell_r)$ and apply Lemma 2.23 in [**4**] to obtain $(\sigma_n) \in \ell_p$ and $(z_n) \in \ell_{r'}^{wk}(\ell_r)$ so that if $\gamma = \|(y_i)\|_{q'}^{wk}$, then

$$\|(\sigma_n)\|_p \leq \gamma^{q'/p},\ \|(z_n)\|_{r'}^{wk} \leq \gamma^{q'/r'} \pi_p(RS) \text{ and } RSy_n = \sigma_n z_n,\ \forall n.$$

Then, $(\sigma_i x_i^*) \in \ell_r^s(X^*)$ and $RST = \sum_{i=1}^{\infty} \sigma_i x_i^* \otimes z_i$, thus $RST \in \mathcal{N}_r(X, \ell_r)$. From $R \in B(Z, \ell_r)$ being arbitrary, it follows that $ST \in (\mathcal{N}_r)_r(X, Z)$. Moreover,

$$\begin{aligned}
\nu_r(RST) &\leq \|(\sigma_i x_i^*)\|_r^s \,\|(z_i)\|_{r'}^{wk} \\
&\leq \|(\sigma_i)\|_p \,\gamma^{q'/r'} \pi_p(RS) \\
&\leq \|(y_i)\|_{q'}^{wk} \pi_p(RS) \\
&\leq (\nu_q(T) + \delta)\pi_p(RS),
\end{aligned}$$

from which $(\nu_r)_r(ST) \leq \nu_q(T)\ell t_{p,r}(S)$ follows. □

ACKNOWLEDGEMENT. Financial support from the National Research Foundation (NRF) in South Africa is acknowledged. Any opinion, findings and conclusions or recommendations expressed in this material are those of the authors and therefore the NRF does not accept any liability in regard thereto.

REFERENCES

1. F. Albiac and N. J. Kalton, *Topics in Banach spaces*, Graduate Texts in Mathematics (Springer Inc., USA, 2006).

2. S. Aywa and J. H. Fourie, On convergence of sections of sequences in Banach spaces, *Rendiconti del Circolo Matematico di Palermo*, Serie II, **XLIX** (2000), 141–150.

3. J. B. Conway, *A course in Functional Analysis*, Graduate Texts in Mathematics, 2nd Edition (Springer-Verlag, New York, Inc., 1990).

4. J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators* (Cambridge University Press, Cambridge, 1995).

5. J. H. Fourie and J. Swart, Banach ideals of p-compact operators, *Manuscr. Math.*, **26**(4) (1979), 349–362.

6. A. K. Karn and D. P. Sinha, An operator summability of sequences in Banach spaces, *Glasgow Math. J.* **56**(2) (2014), 427–437.

7. A. Pietsch, *Operator ideals* (North-Holland, North-Holland Publishing Company, Amsterdam, New York, Oxford. 1980).