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RESEARCH ARTICLE

Local newforms for the general linear groups over a non-archimedean local field

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Abstract

In [14], Jacquet–Piatetskii-Shapiro–Shalika defined a family of compact open subgroups of p-adic general linear groups indexed by nonnegative integers and established the theory of local newforms for irreducible generic representations. In this paper, we extend their results to all irreducible representations. To do this, we define a new family of compact open subgroups indexed by certain tuples of nonnegative integers. For the proof, we introduce the Rankin–Selberg integrals for Speh representations.

Contents

1. Introduction

1.1. Background

The theory of newforms is fascinating and plays an important role in the theory of automorphic forms. It was first studied in the early 1970s by Atkin–Lehner [3] and Li [23] in terms of classical modular forms, and by Casselman [6] in terms of local newforms on GL_2. Their results become a bridge between classical modular forms and automorphic representations of GL_2. Casselman's result was generalised to GL_n by Jacquet–Piatetskii-Shapiro–Shalika [14] (see also Jacquet's erratum [13]) in the 1980s. Another proof was given by Matringe [26] in 2013.

After their works, the theory of local newforms was established

- for PGSp_4 and for $\widetilde{\mathrm{SL}}_2$, which is the double cover of SL_2, by Roberts–Schmidt [35, 36];
- for GSp_4 by Okazaki [33];
- for $\mathrm{U}(1,1)$ by Lansky–Raghuram [19];
- for unramified $\mathrm{U}(2,1)$ by Miyauchi [27, 28, 29, 30].

In 2010, Gross gave a conjecture on the local newforms for SO_{2n+1} in a letter to Serre (see the expansion [9] of this letter). It is a natural extension of the GL_2 case [6] and the PGSp_4 case [35]. This conjecture was proven for generic supercuspidal representations by Tsai [41].

One has to notice that in all previous works, representations are assumed to be *generic*. For GL_n, this assumption might be reasonable since all local components of an arbitrary irreducible cuspidal automorphic representation of GL_n are generic. However, for other groups, this assumption is too strong because there are many irreducible cuspidal automorphic representations whose local components are not generic (and not tempered), such as the Saito–Kurokawa lifting of PGSp_4.

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In this paper, we generalise the results in [14] to all the irreducible representations. Namely, we extend the theory of local newforms to not generic representations in the case of GL_n. By considering the endoscopic classification, our results would be useful for the study of local newforms for classical groups in the future.

1.2. Main results

Let us describe our results. Let F be a nonarchimedean local field of characteristic zero with the ring of integers $\mathfrak{o}$ and the maximal ideal $\mathfrak{p}$. Fix a nontrivial additive character ψ of F, which is trivial on $\mathfrak{o}$ but not on $\mathfrak{p}^{-1}$. We denote by q the order of $\mathfrak{o}/\mathfrak{p}$.

For an integer $n \geq 1$, set $\Lambda_n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid 0 \leq \lambda_1 \leq \cdots \leq \lambda_n\}$. We regard Λ_n as a totally ordered monoid with respect to the lexicographic order. For $\lambda = (\lambda_1, \ldots, \lambda_n)$, we set $|\lambda| = \lambda_1 + \cdots + \lambda_n$.

We set $G_n = \mathrm{GL}_n(F)$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$, we define a subgroup $\mathbb{K}_{n,\lambda}$ of $\mathrm{GL}_n(\mathfrak{o})$ by

$$\mathbb{K}_{n,\lambda} = \{(k_{i,j}) \in \mathrm{GL}_n(\mathfrak{o}) \mid k_{i,j} \equiv \delta_{i,j} \bmod \mathfrak{p}^{\lambda_i},\ 1 \leq i, j \leq n\},$$

where $\delta_{i,j}$ is the Kronecker delta.

Let π be an irreducible smooth complex representation of G_n. Godement–Jacquet [8] associated two local factors $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ with π. By [14, (5.1) Théorème (i)] and [8, Corollary 3.6] (or by the local Langlands correspondence [11], [12]), we have $\varepsilon(s, \pi, \psi) = \varepsilon(0, \pi, \psi)q^{-c_\pi s}$ for some nonnegative integer c_π. We call c_π the *conductor of* π.

Set $\pi^{(0)} = \pi$ and $\pi^{(i)}$ to be the highest derivative of $\pi^{(i-1)}$ in the sense of Bernstein–Zelevinsky [4] for $i = 1, \ldots, n$. (Note that our notation is different from the original in [4].) It is known that $\pi^{(i)}$ is irreducible so that one can consider its conductor $c_{\pi^{(i)}}$. We then define $\lambda_\pi = (\lambda_1, \ldots, \lambda_n)$ by

$$\lambda_k = c_{\pi^{(n-k)}} - c_{\pi^{(n-k+1)}}$$

for $1 \leq k \leq n$. In Section 2.3 (especially in Proposition 2.4) below, we will see that $\lambda_\pi \in \Lambda_n$. We note that $|\lambda_\pi| = c_\pi$.

We denote by $\pi^{\mathbb{K}_{n,\lambda}}$ the $\mathbb{K}_{n,\lambda}$-invariant subspace of π, which is finite-dimensional. Our main theorem is stated as follows:

Theorem 1.1 (Theorems 2.1, 2.2). *Let π be an irreducible representation of G_n.*

(1) *For $\lambda \in \Lambda_n$, we have*

$$\dim(\pi^{\mathbb{K}_{n,\lambda}}) = \begin{cases} 1 & \text{if } \lambda = \lambda_\pi, \\ 0 & \text{if } \lambda < \lambda_\pi. \end{cases}$$

(2) *If $\lambda \in \Lambda_n$ satisfies that $|\lambda| < |\lambda_\pi|$, then $\pi^{\mathbb{K}_{n,\lambda}} = 0$.*

We call any nonzero element in $\pi^{\mathbb{K}_{n,\lambda_\pi}}$ a *local newform* of π. Using Theorem 1.1, we can give a characterisation of the conductor in terms of the dimensions of fixed parts: that is,

$$c_\pi = \min\left\{|\lambda| \,\middle|\, \pi^{\mathbb{K}_{n,\lambda}} \neq 0\right\}.$$

Note that when π is generic, since $\pi^{(i)}$ is the trivial representation $\mathbf{1}_{G_0}$ for any $i \geq 1$, we have $\lambda_\pi = (0, \ldots, 0, c_\pi)$. In this case, $\mathbb{K}_{n,\lambda_\pi}$ is nothing but the compact group introduced by Jacquet–Piatetskii-Shapiro–Shalika [14]. Hence Theorem 1.1 (1) is an extension of a result in [14].

According to the Zelevinsky classification, the set of isomorphism classes of irreducible representations of G_n is classified by multisegments. We recall it in Section 2.1. When $\pi = Z(\mathfrak{m})$ is the irreducible representation associated with a multisegment $\mathfrak{m}$, we have another description of λ_π in terms of $\mathfrak{m}$ (Proposition 2.4), which allows us to compute λ_π in many important cases (Example 2.5). Moreover, Corollary 2.8 tells us how to compute λ_π inductively in general.

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The proof of Theorem 1.1 takes the following three steps:

Step 1: Reduce to two cases: the case where π is of type χ with an unramified character χ of $F^\times$ and the case where $L(s,\pi) = 1$. Here, we say that an irreducible representation π is *of type* χ if $\pi = Z(\Delta_1 + \cdots + \Delta_r)$ such that for $i = 1, \ldots, r$, the segment Δ_i is of the form $[a_i, b_i]_\chi$ for some integers a_i, b_i satisfying $a_i \le b_i$.
Step 2: Prove Theorem 1.1 for π of type χ with an unramified character χ of $F^\times$.
Step 3: Prove Theorem 1.1 for π such that $L(s,\pi) = 1$.

Let us give the details of each step.

1.3. Reduction

Using the Mackey theory, we study the $\mathbb{K}_{n,\lambda}$-invariant subspaces of parabolically induced representations in Section 5.1. To do this, in Section 4.1, we relate Λ_n with the set $|\mathcal{C}^n|$ of isomorphism classes $[M]$ of $\mathfrak{o}$-modules such that M is generated by at most n elements. In Section 5.1, we associate a compact open subgroup $\mathbb{K}_{n,[M]}$ of G_n with $[M] \in |\mathcal{C}^n|$. If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ and $M = \oplus_{i=1}^n \mathfrak{o}/\mathfrak{p}^{\lambda_i}$, then $\mathbb{K}_{n,[M]} = \mathbb{K}_{n,\lambda}$. Proposition 5.2 says that the $\mathbb{K}_{n,[M]}$-invariant subspace of a parabolically induced representation decomposes into a direct sum indexed by certain filtrations on M by $\mathfrak{o}$-modules. In particular, this proposition together with Corollary 4.7 reduces Theorem 1.1 to the following two types of irreducible representations:

- $\pi \in \mathrm{Irr}(G_n)$ of type χ with a fixed unramified character χ of $F^\times$
- $\pi \in \mathrm{Irr}(G_n)$ such that $L(s,\pi) = 1$

1.4. The case where π is of type χ

In Section 6, we prove Theorem 1.1 for irreducible representations $\pi \in \mathrm{Irr}(G_n)$ of type χ with a fixed unramified character χ of $F^\times$.

In the proof of Theorem 1.1 (1), we first consider the case where π is a ladder representation. The main ingredient in this case is Tadić's determinantal formula established by Lapid–Mínguez [21]. This formula describes π explicitly as an alternating sum of standard modules. The key point is that the standard modules appearing here are parabolically induced representations from one-dimensional representations. In particular, for $[M] \in |\mathcal{C}^n|$, the determinantal formula together with Proposition 5.2 expresses the dimension of $\pi^{\mathbb{K}_{n,[M]}}$ explicitly as an alternating sum of the numbers of certain filtrations on M by $\mathfrak{o}$-modules (Proposition 6.1). Surprisingly, there are many cancellations in this alternating sum (Lemma 6.3). From this lemma, we can deduce Theorem 1.1 (1) for a ladder representation π of type χ. For these miraculous cancellations, see Example 6.4.

The proof of Theorem 1.1 (1) for general π of type χ is by induction on a certain totally ordered set. The key is Proposition 2.7, whose proof relies on a highly nontrivial result of Knight–Zelevinsky [16] that describes the Zelevinsky dual of π (see also Proposition 3.7).

We reduce the proof of Theorem 1.1 (2) to the case where π is a Steinberg representation. In this case, by Tadić's determinantal formula (or by the definition of the Steinberg representations in Harish-Chandra [10]), we can express $\pi^{\mathbb{K}_{n,\lambda}}$ explicitly as an alternating sum of the numbers of certain filtrations on the $\mathfrak{o}$-module corresponding to λ. We realise this alternating sum as a coefficient of certain formal power series in one variable whose coefficients are in a graded ring. By giving another description of this formal power series, we deduce that $\pi^{\mathbb{K}_{n,\lambda}} = 0$.

1.5. The case where $L(s,\pi) = 1$

In Section 7, we firstly prove Theorem 1.1 (2) for π with $L(s,\pi) = 1$. We reduce the proof to the case where π is cuspidal. In this case, Lemma 7.1 says that certain Hecke operators depending on $\lambda \in \Lambda_n$

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act on π as nilpotent endomorphisms. We consider the Godement–Jacquet integral $Z(\Phi, s, f)$ defined in [8]. From this lemma, it follows that if $\pi^{\mathbb{K}_{n,\lambda}} \neq 0$, then we can obtain data Φ and f such that $Z(\Phi, s, f)$ is a nonzero constant, whereas $Z(\hat{\Phi}, s, \check{f}) \in q^{|\lambda|s}\mathbb{C}[[q^{-s}]]$. Since $\varepsilon(s, \pi, \psi) = \varepsilon(0, \pi, \psi)q^{-|\lambda_\pi|s}$, by the functional equation, we conclude that $|\lambda| \geq |\lambda_\pi|$.

By Proposition 5.2, we can reduce Theorem 1.1 (1) for π with $L(s, \pi) = 1$ to the case where $\pi = Z(\Delta)$ for a segment Δ (Lemma 7.2). The key point here is that the matrices defined by the multiplicities of irreducible representations appearing in standard modules are 'triangular' and unipotent ([42, 7.1 Theorem]).

Finally, we prove Theorem 1.1 (1) for $\pi = Z(\Delta)$ with $L(s, \pi) = 1$. Slightly generally, we do it in Section 9 for Speh representations $\mathrm{Sp}(\pi_{\mathrm{temp}}, m)$ with an irreducible tempered representation π_{temp} of G_n. For the notation of Speh representations, see Example 2.5 (4). The proof of this case is rather an analogue of the generic case in [14]. Namely, it is an application of the theory of Rankin–Selberg integrals. To carry out the proof, we establish this theory for Speh representations in Section 8.

1.6. Rankin–Selberg integrals for Speh representations

The theory of Rankin–Selberg integrals was developed by Jacquet–Piatetskii-Shapiro–Shalika [15]. These integrals are integrations of products of Whittaker functions of two irreducible representations of G_n and G_m, and they represent the Rankin–Selberg L-functions. Since representations are required to admit nontrivial Whittaker functions, they must be generic. As an application of Rankin–Selberg integrals for $G_n \times G_{n-1}$, the theory of local newforms for generic representations of G_n was established in [14].

To prove Theorem 1.1 (1) for Speh representations, we need to extend the theory of Rankin–Selberg integrals to the case of Speh representations. In the equal rank case, this extension was done by Lapid–Mao [20]. In their paper, instead of Whittaker models, they used two models of a Speh representation that are called the *Zelevinsky model* and the *Shalika model*[1].

For our purpose, we need the Rankin–Selberg integrals in the 'almost equal rank case', which are easier than the equal rank case. The Zelevinsky model is a direct generalisation of the Whittaker model so that we can easily extend the theory of Rankin–Selberg integrals using this model (Theorem 8.5). On the other hand, the Shalika model has an important property of the Whittaker model (Theorem 8.2), which we need for the proof of Theorem 1.1 (1) for Speh representations. To transfer the Rankin–Selberg integrals in the Zelevinsky models to those in the Shalika models, we use the model transition established by Lapid–Mao (see Proposition 8.3).

After establishing the Rankin–Selberg integrals in the Shalika models, the proof of Theorem 1.1 (1) for Speh representations π with $L(s, \pi) = 1$ is exactly the same as in the generic case [14]. We do not compute the greatest common divisors of the Rankin–Selberg integrals in general (see Proposition 8.7). This is a main reason this method cannot be applied to Speh representations π with $L(s, \pi) \neq 1$. However, as an application of Theorem 1.1 (1) for all cases, we can specify the greatest common divisor when the Speh representation of the group of smaller rank is unramified (see Theorem 9.1).

1.7. Organisation

This paper is organised as follows. In Section 2, we state the main results (Theorems 2.1 and 2.2). We give two definitions of λ_π (Proposition 2.4) and explain how to compute it (Corollary 2.8). Some important examples of λ_π are given in Example 2.5. Propositions 2.4 and 2.7 are proven in Section 3. After preparing several facts on $\mathfrak{o}$-modules in Section 4, we prove the Mackey decomposition of the $\mathbb{K}_{n,[M]}$-invariant subspace of a parabolically induced representation (Proposition 5.2) in Section 5. It

[1]As mentioned in [20], this terminology does not coincide with the standard notion of the Shalika model in the literature. This model was also used in the theory of twisted doubling [5] established by Cai–Friedberg–Ginzburg–Kaplan, in which it is called the (k, c) *model*.

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reduces the proofs of the main results to two cases: $\pi \in \mathrm{Irr}(G_n)$ of type χ with a fixed unramified character χ of $F^\times$ and $\pi \in \mathrm{Irr}(G_n)$ such that $L(s,\pi) = 1$. For the former case, Theorems 2.1 and 2.2 are proven in Section 6. In Section 7, we treat the latter case. More precisely, for the latter case, we prove Theorem 2.2, but we reduce Theorem 2.1 to the case where π is a Speh representation. Theorem 2.1 for Speh representations π with $L(s,\pi) = 1$ is proven in Section 9 after establishing the theory of Rankin–Selberg integrals for Speh representations in Section 8.

Notation

Let F be a nonarchimedean local field of characteristic zero. Denote the ring of integers and its maximal ideal by $\mathfrak{o}$ and $\mathfrak{p}$, respectively. Fix a uniformiser ϖ of $\mathfrak{o}$, and normalise the absolute value $|\cdot|$ on F so that $|\varpi| = q^{-1}$, where $q = \#(\mathfrak{o}/\mathfrak{p})$. We fix a nontrivial additive character ψ of F such that ψ is trivial on $\mathfrak{o}$ but nontrivial on $\mathfrak{p}^{-1}$.

For an integer $n \geq 1$ and a commutative ring R, we let $M_n(R)$ denote the R-module of n-by-n matrices with entries in R.

In this paper, all representations are assumed to be smooth. For a representation π of $\mathrm{GL}_n(F)$, its contragredient representation is denoted by $\widetilde{\pi}$.

2. Statements of the main results

In this section, we fix notations and state the main results.

2.1. The Zelevinsky classification

We recall the Zelevinsky classification [42] of irreducible representations of $G_n = \mathrm{GL}_n(F)$. For a smooth representation π of G_n and a character χ of $F^\times$, the twisted representation $g \mapsto \pi(g)\chi(\det g)$ is denoted by $\pi\chi$. The set of equivalence classes of irreducible representations of G_n is denoted by $\mathrm{Irr}(G_n)$.

When $\pi_1, \dots, \pi_r$ are smooth representations of $G_{n_1}, \dots, G_{n_r}$, respectively, with $n_1 + \cdots + n_r = n$, we write $\pi_1 \times \cdots \times \pi_r$ for the parabolically induced representation of G_n from $\pi_1 \otimes \cdots \otimes \pi_r$ via the standard parabolic subgroup whose Levi subgroup is $G_{n_1} \times \cdots \times G_{n_r}$.

A *segment* Δ is a finite set of representations of the form

$$[x,y]_\rho = \{\rho|\cdot|^x, \rho|\cdot|^{x+1}, \dots, \rho|\cdot|^y\},$$

where ρ is an irreducible cuspidal representation of G_d for some $d \geq 1$, and $x, y \in \mathbb{R}$ with $x \equiv y \bmod \mathbb{Z}$ and $x \leq y$. We write $l(\Delta) = y - x + 1$ and call it the *length* of Δ.

Let $\Delta = [x,y]_\rho$ be a segment. Then the parabolically induced representation

$$\rho|\cdot|^x \times \rho|\cdot|^{x+1} \times \cdots \times \rho|\cdot|^y$$

of $G_{dl(\Delta)}$ has a unique irreducible subrepresentation. We denote it by $Z(\Delta)$. For example, if $\rho = \chi$ is a character of $F^\times$, then $Z([x,y]_\chi) = |\det|^{\frac{x+y}{2}}\chi(\det)$ is a one-dimensional representation of G_{y-x+1}.

Let $r \geq 1$. For $i = 1, \dots, r$, let $\Delta_i = [x_i, y_i]_{\rho_i}$ be a segment and $n_i \geq 1$ an integer such that ρ_i is a cuspidal representation of G_{n_i}. When ρ_i is unitary and the inequalities

$$x_1 + y_1 \geq \cdots \geq x_r + y_r$$

hold, the parabolically induced representation

$$Z(\Delta_1) \times \cdots \times Z(\Delta_r)$$

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has a unique irreducible subrepresentation. We denote it by $Z(\mathfrak{m})$, where $\mathfrak{m}$ denotes the multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$. The Zelevinsky classification says that for any irreducible representation π of G_n, there exists a unique multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ such that $\pi \cong Z(\mathfrak{m})$.

When $x_1 > \cdots > x_t$ and $y_1 > \cdots > y_t$, the irreducible representation

$$Z([x_1, y_1]_\rho, \ldots, [x_t, y_t]_\rho) = Z([x_1, y_1]_\rho + \cdots + [x_t, y_t]_\rho)$$

is called a *ladder representation*. A ladder representation of the form $Z([x, y]_\rho, [x-1, y-1]_\rho, \ldots, [x-t+1, y-t+1]_\rho)$ for some positive integer t is called a *Speh representation*.

2.2. *Main results*

Fix $n \geq 1$. Let Λ_n be the subset of $\mathbb{Z}^n$ consisting of $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$. Note that Λ_n is a submonoid of $\mathbb{Z}^n$. We endow Λ_n with the total order induced by the lexicographic order: that is, for $\lambda = (\lambda_1, \ldots, \lambda_n), \lambda' = (\lambda'_1, \ldots, \lambda'_n) \in \Lambda_n$, we write $\lambda < \lambda'$ if and only if there exists $1 \leq i \leq n$ such that $\lambda_j = \lambda'_j$ for $j < i$ and $\lambda_i < \lambda'_i$.

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$, define $\mathbb{K}_{n,\lambda}$ to be the subgroup of $G_n(\mathfrak{o}) = \mathrm{GL}_n(\mathfrak{o})$ consisting of matrices $k = (k_{i,j})_{1 \leq i,j \leq n}$ such that

$$k_{i,j} \equiv \delta_{i,j} \bmod \mathfrak{p}^{\lambda_i}$$

for any $1 \leq i, j \leq n$. For example, if $n = 4$ and $\lambda = (0, 0, 1, 2)$, then

$$\mathbb{K}_{4,(0,0,1,2)} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{pmatrix} \cap \mathrm{GL}_4(\mathfrak{o}).$$

In Section 1.2, we defined $\lambda_\pi \in \Lambda_n$ for any $\pi \in \mathrm{Irr}(G_n)$. The main results are as follows.

Theorem 2.1. *Let $\pi \in \mathrm{Irr}(G_n)$. Then the $\mathbb{K}_{n,\lambda_\pi}$-invariant subspace $\pi^{\mathbb{K}_{n,\lambda_\pi}}$ is one-dimensional. Moreover, if $\lambda \in \Lambda_n$ satisfies $\lambda < \lambda_\pi$, then $\pi^{\mathbb{K}_{n,\lambda}} = 0$.*

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$, we write $|\lambda|$ for $\lambda_1 + \cdots + \lambda_n$.

Theorem 2.2. *Let $\pi \in \mathrm{Irr}(G_n)$. If $\lambda \in \Lambda_n$ satisfies $|\lambda| < |\lambda_\pi|$, then $\pi^{\mathbb{K}_{n,\lambda}} = 0$.*

2.3. *Definition of $\lambda_\mathfrak{m}$*

For an irreducible representation π of G_n, we defined $\lambda_\pi \in \mathbb{Z}^n$ in Section 1.2. Here we describe it in terms of multisegments, which then implies that $\lambda_\pi \in \Lambda_n$.

A segment Δ is written as $\Delta = [a, b]_\rho$, where $a, b \in \mathbb{Z}$ with $a \leq b$ and ρ is a cuspidal representation of G_d for some $d \geq 0$. We write a multisegment as a sum $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ of segments, where r is a nonnegative integer. We call the integer r the *cardinality* of $\mathfrak{m}$ and denote it by $\mathrm{Card}(\mathfrak{m})$. Recall that we set $l(\Delta) = b - a + 1$. We write $l(\mathfrak{m})$ for the sum $l(\Delta_1) + \cdots + l(\Delta_r)$ and call $l(\mathfrak{m})$ the *length* of $\mathfrak{m}$.

If $a < b$, we write Δ^- for the segment $[a, b-1]_\rho$. When $a = b$, we understand Δ^- to be the empty multisegment. We set $\mathfrak{m}^- = \Delta_1^- + \cdots + \Delta_r^-$. By the fundamental result of Zelevinsky [42, 8.1 Theorem], the highest derivative of $Z(\mathfrak{m})$ is equivalent to $Z(\mathfrak{m}^-)$.

We call $\Delta = [a, b]_\rho$ *unipotent* if ρ is an unramified character of $F^\times$. Similarly, we say that $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ is *unipotent* if Δ_i is unipotent for $i = 1, \ldots, r$. Fix an unramified character χ of $F^\times$. We say that a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ is *of type* χ if for $i = 1, \ldots, r$, the segment Δ_i is of the form $[a_i, b_i]_\chi$ for some integers a_i, b_i satisfying $a_i \leq b_i$.

We denote by $\mathfrak{m}^\sharp$ the unique multisegment such that $Z(\mathfrak{m}^\sharp)$ is equivalent to the Zelevinsky dual of $Z(\mathfrak{m})$ (see, e.g., [34, Section 7]). We denote by $\mathfrak{m}^{\mathrm{ram}}$ the multisegment $((\mathfrak{m}^\sharp)^-)^\sharp$. When $\pi = Z(\mathfrak{m})$,

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we set $\pi^{\mathrm{ram}} = Z(\mathfrak{m}^{\mathrm{ram}})$. We use 'ram' only for unipotent multisegments. For an example of $\mathfrak{m}^{\mathrm{ram}}$, see Section 2.5 below.

When $n' < n$, we regard $\Lambda_{n'}$ as a submonoid of Λ_n via the inclusion $\Lambda_{n'} \hookrightarrow \Lambda_n$ given by $(\lambda_1, \ldots, \lambda_{n'}) \mapsto (0, \ldots, 0, \lambda_1, \ldots, \lambda_{n'})$.

Definition 2.3. Let $\mathfrak{m}$ be a multisegment.

(1) If $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ with $\Delta_i = [a_i, b_i]_{\rho_i}$ being not unipotent for all $i = 1, \ldots, r$, then we set

$$\lambda_{\mathfrak{m}} = \sum_{i=1}^{r} (0, \ldots, 0, \underbrace{c_{\rho_i}, \ldots, c_{\rho_i}}_{l(\Delta_i)}),$$

where c_{ρ_i} is the conductor of ρ_i. Note that $c_{\rho_i} > 0$ for $1 \leq i \leq r$ by [14, (5.1) Théorème].

(2) If $\mathfrak{m}$ is unipotent, and if we write $\mathfrak{m}^{\mathrm{ram}} = \Delta_1 + \cdots + \Delta_r$, then we set

$$\lambda_{\mathfrak{m}} = \sum_{i=1}^{r} (0, \ldots, 0, \underbrace{1, \ldots, 1}_{l(\Delta_i)}).$$

(3) In general, we decompose $\mathfrak{m}$ as $\mathfrak{m} = \mathfrak{m}' + \mathfrak{m}_{\mathrm{unip}}$, where $\mathfrak{m}_{\mathrm{unip}}$ is unipotent, and each segment in $\mathfrak{m}'$ is not unipotent. Then we set

$$\lambda_{\mathfrak{m}} = \lambda_{\mathfrak{m}'} + \lambda_{\mathfrak{m}_{\mathrm{unip}}}.$$

As seen in the next proposition, this is an alternative definition of λ_π.

Proposition 2.4. *Let $n \geq 1$, and let $\pi = Z(\mathfrak{m})$ be the irreducible representation of G_n corresponding to a multisegment $\mathfrak{m}$. Then we have $\lambda_\pi = \lambda_{\mathfrak{m}}$.*

This proposition will be proven in Section 3.2 below. We now give some examples.

Example 2.5. Let π be an irreducible representation of G_n.

(1) When $L(s, \pi) = 1$, then $\pi = Z(\Delta_1 + \cdots + \Delta_r)$ with Δ_i not unipotent. If $\pi = Z(\Delta)$ with a segment $\Delta = [x, y]_\rho$, then we have

$$\lambda_\pi = \lambda_\Delta = (\underbrace{0, \ldots, 0}_{n-l(\Delta)}, \underbrace{c_\rho, \ldots, c_\rho}_{l(\Delta)}) \in \Lambda_n.$$

Here, we note that $c_\rho > 0$. In general, if $\pi = Z(\Delta_1 + \cdots + \Delta_r)$, we have

$$\lambda_\pi = \lambda_{\Delta_1} + \cdots + \lambda_{\Delta_r} \in \Lambda_n.$$

(2) When $\pi = Z([x_1, y_1]_\chi, \ldots, [x_t, y_t]_\chi) \in \mathrm{Irr}(G_n)$ is a ladder representation of type χ, where χ is an unramified character of $F^\times$, we have

$$\lambda_\pi = \sum_{i=2}^{t} (0, \ldots, 0, \underbrace{1, \ldots, 1}_{\max\{y_i - x_{i-1} + 2, 0\}}) \in \Lambda_n.$$

Indeed, by the description of the Zelevinsky duals of ladder representations in [21, Section 3] (see also Section 2.5 below), we have

$$\pi^{\mathrm{ram}} = Z([x_1 - 1, y_2]_\chi, [x_2 - 1, y_3]_\chi, \ldots, [x_{t-1} - 1, y_t]_\chi).$$

Here, if $x_{i-1} - 1 > y_i$, we ignore $[x_{i-1} - 1, y_i]_\chi$.

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(3) Let $t \geq 1$, and let $\pi_i \in \mathrm{Irr}(G_{n_i})$ be as in either (1) or (2) above for $1 \leq i \leq t$. Assume $\pi = \pi_1 \times \cdots \times \pi_t$ is irreducible. Then we have $\lambda_\pi = \lambda_{\pi_1} + \cdots + \lambda_{\pi_t}$.

(4) Let π be an irreducible tempered representation of G_n. Then the parabolically induced representation

$$\pi|\cdot|^{-\frac{m-1}{2}} \times \pi|\cdot|^{-\frac{m-3}{2}} \times \cdots \times \pi|\cdot|^{\frac{m-1}{2}}$$

of G_{nm} has a unique irreducible subrepresentation σ, which is denoted by $\mathrm{Sp}(\pi, m)$. Note that σ is a (unitary) Speh representation. Combining the cases above, we obtain

$$\lambda_\sigma = (\underbrace{0, \ldots, 0}_{(n-1)m}, \underbrace{c_\pi, \ldots, c_\pi}_{m}) \in \Lambda_{nm}.$$

Remark 2.6. In the appendix of the paper [17] by the second and third authors, they introduce a notion of mirahoric representations (see Section A.1.6 of [17]). Let us recall the definition. Two segments Δ and Δ' are said to be *tightly linked* if they are linked and either Δ is not unipotent or $\Delta \cap \Delta'$ is nonempty. Let $\pi = L(\mathfrak{m})$ be an irreducible representation associated with a multisegment $\mathfrak{m}$ in the Langlands classification: that is, π is the Zelevinsky dual of $Z(\mathfrak{m})$. They defined π to be *mirahoric* if any two segments in $\mathfrak{m}$ are not tightly linked. In terms of the setup in this paper, the class of mirahoric representations is equal to the class of irreducible representations π such that $\lambda_\pi = (0, \ldots, 0, c)$ for some c. This can be seen from their proposition [17, Proposition A.15], which says that a representation π is mirahoric if and only if the conductor of the highest derivative of π is zero. Hence, a main result [17, Proposition A.3] in the appendix can be interpreted as a special case of Theorem 2.1 restricted to the mirahoric representations.

An irreducible representation $\pi = L(\mathfrak{m})$ is generic if and only if any two segments of $\mathfrak{m}$ are not linked. Therefore a generic representation is mirahoric. However, a simple multisegment such as $\mathfrak{m} = [0, 1]_\rho + [2, 3]_\rho$, where ρ is an unramified character, gives a mirahoric representation $L(\mathfrak{m})$, which is not generic. (This is one of the reasons for treating the unipotent case and the case $L(s, \pi) = 1$ separately.)

2.4. *Computation of* $\lambda_{\mathfrak{m}}$

When $\mathfrak{m}$ is a general unipotent multisegment, it is difficult to compute $\lambda_{\mathfrak{m}}$ directly from the definition. In this subsection, we explain how to compute $\lambda_{\mathfrak{m}}$ efficiently.

Let $\mathfrak{m}$ be a unipotent multisegment. We may assume that $\mathfrak{m}$ is of type χ for some unramified character χ of $F^\times$. We denote by $\mathfrak{m}_{\max}$ the set of segments Δ in $\mathfrak{m}$ such that Δ is maximal with respect to the inclusion among the segments in $\mathfrak{m}$. We regard $\mathfrak{m}_{\max}$ as a multisegment in which each segment has multiplicity at most 1. We set $\mathfrak{m}^{\max} = \mathfrak{m} - \mathfrak{m}_{\max}$. For example, if

$$\mathfrak{m} = [0, 0]_\chi + [1, 2]_\chi + [1, 2]_\chi + [2, 2]_\chi,$$

then we have

$$\mathfrak{m}_{\max} = [0, 0]_\chi + [1, 2]_\chi$$

and

$$\mathfrak{m}^{\max} = [1, 2]_\chi + [2, 2]_\chi.$$

Proposition 2.7. *We have* $\mathfrak{m}^{\mathrm{ram}} = (\mathfrak{m}_{\max})^{\mathrm{ram}} + (\mathfrak{m}^{\max})^{\mathrm{ram}}$.

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We will prove this proposition in Section 3.3 below.

Corollary 2.8. *We have* $\lambda_{\mathfrak{m}} = \lambda_{\mathfrak{m}_{\max}} + \lambda_{\mathfrak{m}^{\max}}$.

Proof. Write $(\mathfrak{m}_{\max})^{\mathrm{ram}} = \Delta_1 + \cdots + \Delta_r$ and $(\mathfrak{m}^{\max})^{\mathrm{ram}} = \Delta_{r+1} + \cdots + \Delta_t$. Then $\mathfrak{m}^{\mathrm{ram}} = \Delta_1 + \cdots + \Delta_t$ by Proposition 2.7. Hence we have

$$\begin{aligned}\lambda_{\mathfrak{m}} &= \sum_{i=1}^{t}(0,\ldots,0,\underbrace{1,\ldots,1}_{l(\Delta_i)})\\ &= \sum_{i=1}^{r}(0,\ldots,0,\underbrace{1,\ldots,1}_{l(\Delta_i)}) + \sum_{i=r+1}^{t}(0,\ldots,0,\underbrace{1,\ldots,1}_{l(\Delta_i)})\\ &= \lambda_{\mathfrak{m}_{\max}} + \lambda_{\mathfrak{m}^{\max}}.\end{aligned}$$

This completes the proof. □

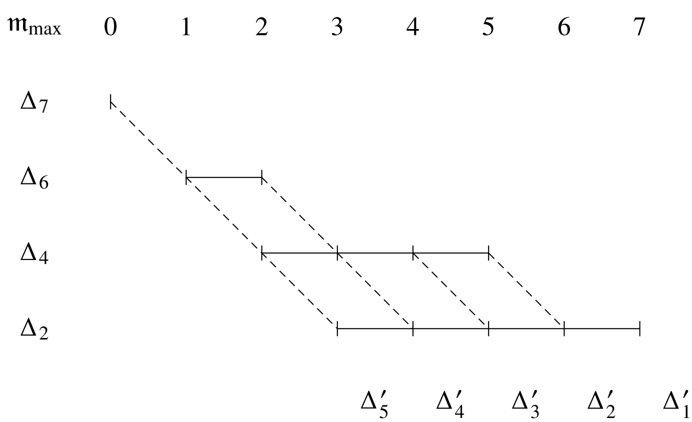
Since $\mathfrak{m}_{\max}$ is a ladder multisegment (i.e., the multisegment corresponding to a ladder representation), we can compute $\lambda_{\mathfrak{m}_{\max}}$ as in Example 2.5 (2). Hence, using this corollary, we can compute $\lambda_{\mathfrak{m}}$ inductively.

2.5. *An example of computation of* $\mathfrak{m}^{\mathrm{ram}}$

By using Proposition 2.7, one can compute $\mathfrak{m}^{\mathrm{ram}}$ for an arbitrary multisegment $\mathfrak{m}$ in a systematic way. Let us give an example.

Let $\mathfrak{m} = \sum_{i=1}^{7} \Delta_i$ be a multisegment where $\Delta_1 = [5,6]_\chi$, $\Delta_2 = [3,7]_\chi$, $\Delta_3 = [3,4]_\chi$, $\Delta_4 = [2,5]_\chi$, $\Delta_5 = [3,3]_\chi$, $\Delta_6 = [1,2]_\chi$, $\Delta_7 = [0,0]_\chi$. Then $\mathfrak{m}_{\max} = \Delta_2+\Delta_4+\Delta_6+\Delta_7$ and $\mathfrak{m}^{\max} = \Delta_1+\Delta_3+\Delta_5$. We also have $(\mathfrak{m}^{\max})_{\max} = \Delta_1 + \Delta_3$ and $(\mathfrak{m}^{\max})^{\max} = \Delta_5$. By Proposition 2.7, we are reduced to computing 'ram' of the three ladder multisegments.

As explained in Section 3 of [21], the Zelevinsky dual of a ladder multisegment can be calculated fairly easily. Let us compute the Zelevinsky of $\mathfrak{m}_{\max}$ by drawing pictures. In the xy-plane, we draw each segment of $\mathfrak{m}_{\max}$ so that each lies on the line $y = i$ for $i = 1,\ldots,4$. (See the following figure.) Whenever there exist points (e, f) and $(e+1, f-1)$ with $e, f \in \mathbb{Z}$, we draw a dotted line connecting them. Then the dotted lines form the multisegment of the Zelevinsky dual $(\mathfrak{m}_{\max})^\sharp$. One can use the algorithm of Mœglin–Waldspurger [32] to verify that the procedure above actually gives the Zelevinsky dual. We obtain $(\mathfrak{m}_{\max})^\sharp = \Delta_1'+\Delta_2'+\Delta_3'+\Delta_4'+\Delta_5'$, where $\Delta_1' = [7,7]_\chi$, $\Delta_2' = [5,6]_\chi$, $\Delta_3' = [4,5]_\chi$, $\Delta_4' = [2,4]_\chi$, $\Delta_5' = [0,3]_\chi$.



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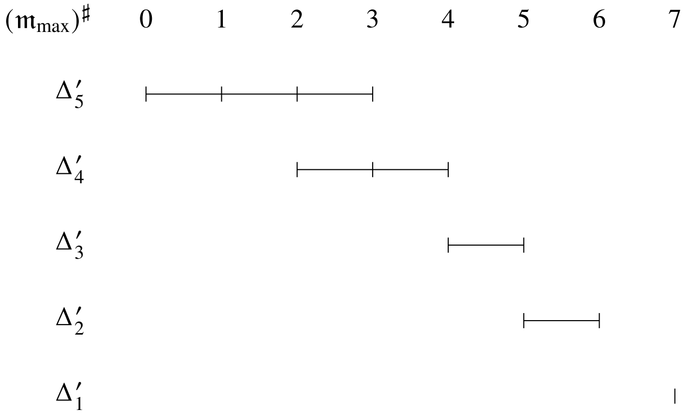
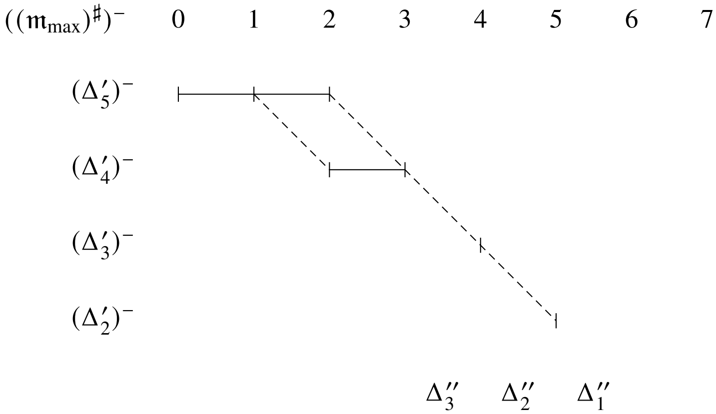
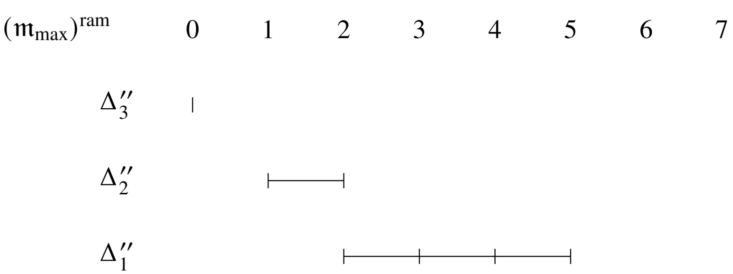


The multisegment of the highest derivative is obtained by shortening each segment by 1. Hence, we have $((\mathfrak{m}_{\max})^\sharp)^-$ as in the following figure. We obtain $((\mathfrak{m}_{\max})^\sharp)^- = (\Delta'_1)^- + (\Delta'_2)^- + (\Delta'_3)^- + (\Delta'_4)^-$ where $\Delta'_1 = [5,5]_\chi, \Delta'_2 = [4,4]_\chi, \Delta'_3 = [2,3]_\chi, \Delta'_4 = [0,2]_\chi$.



Taking the Zelevinsky dual again, we arrive at $(\mathfrak{m}_{\max})^{\mathrm{ram}}$ as in the following figure. We obtain $(\mathfrak{m}_{\max})^{\mathrm{ram}} = \Delta''_1 + \Delta''_2 + \Delta''_3$ where $\Delta''_1 = [2,5]_\chi, \Delta''_2 = [1,2]_\chi, \Delta''_3 = [0,0]_\chi$.



Similarly, we have $(\Delta_1 + \Delta_3)^{\mathrm{ram}} = [4,4]_\chi$ and $(\Delta_5)^{\mathrm{ram}} = \emptyset$. Thus $\mathfrak{m}^{\mathrm{ram}} = [4,4]_\chi + [2,5]_\chi + [1,2]_\chi + [0,0]_\chi$.

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2.6. The Weil–Deligne representations

In this subsection, we give some justification for the use of the term ‘ram’ in the notation π^{ram}. This comes from the Galois side of the local Langlands correspondence ([11], [12]). For several materials in this subsection, see [40].

Let us fix an algebraic closure $\overline{F}$ of F. Let $W_F \subset \mathrm{Gal}(\overline{F}/F)$ denote the Weil group of F. By definition, W_F is a locally profinite topological group. If we denote by W_F^{ab} the quotient of W_F by the closure of $[W_F, W_F]$, then there exists an isomorphism $r_F : W_F^{\mathrm{ab}} \xrightarrow{\cong} F^\times$ that sends any lift of geometric Frobenius to a uniformiser of F.

A *Weil–Deligne representation* is a triple (τ, V, N), where (τ, V) is a finite-dimensional complex representation of W_F and N is a linear endomorphism of V such that the kernel of τ is open in W_F, and we have $\tau(\sigma)N = |r_F(\sigma)|N\tau(\sigma)$ for any $\sigma \in W_F$. Let $I_F \subset W_F$ denote the inertia subgroup. A Weil–Deligne representation (τ, V, N) is called *unramified* if I_F acts trivially and N acts as 0 on V. Any Weil–Deligne representation $V = (\tau, N, V)$ has a unique maximal unramified Weil–Deligne subrepresentation V_{ur}. Explicitly, we have $V_{\mathrm{ur}} = V^{I_F} \cap \mathrm{Ker}\, N$. We denote by V^{ram} the quotient V/V_{ur}, and we call it the *ramified quotient of* V.

The local Langlands correspondence gives a one-to-one correspondence between the isomorphism classes of irreducible complex representations of G_n and the isomorphism classes of Frobenius semisimple n-dimensional Weil–Deligne representations over the complex numbers.

Lemma 2.9. *Let π be a unipotent irreducible admissible representation of G_n, and let V denote the Weil–Deligne representation corresponding to π via the local Langlands correspondence. Then V^{ram} corresponds to π^{ram}.*

Proof. For a segment $[a, b]_\rho$, we denote by $\Delta[a, b]_\rho$ the generalised Steinberg representation: that is, the unique irreducible quotient of

$$\rho|\cdot|^a \times \rho|\cdot|^{a+1} \times \cdots \times \rho|\cdot|^b.$$

As in the Langlands classification, we write $\pi = L([a_1, b_1]_{\rho_1} + \cdots + [a_r, b_r]_{\rho_r})$ if π is the unique irreducible subrepresentation of

$$\Delta[a_1, b_1]_{\rho_1} \times \cdots \times \Delta[a_r, b_r]_{\rho_r}$$

with ρ_i unitary and $a_1 + b_1 \leq \cdots \leq a_r + b_r$. Then the Zelevinsky dual $\pi^\sharp$ of π is given by

$$\pi^\sharp = Z([a_1, b_1]_{\rho_1} + \cdots + [a_r, b_r]_{\rho_r}).$$

By [42, 8.1 Theorem], the highest derivative $(\pi^\sharp)^-$ of $\pi^\sharp$ is

$$(\pi^\sharp)^- = Z([a_1, b_1 - 1]_{\rho_1} + \cdots + [a_r, b_r - 1]_{\rho_r}).$$

Here, if $a_i = b_i$, we ignore $[a_i, b_i - 1]_{\rho_i}$. Hence

$$\pi^{\mathrm{ram}} = ((\pi^\sharp)^-)^\sharp = L([a_1, b_1 - 1]_{\rho_1} + \cdots + [a_r, b_r - 1]_{\rho_r}).$$

Therefore, the map $\pi \mapsto \pi^{\mathrm{ram}}$ corresponds to $V \mapsto V/\mathrm{Ker}\, N$ (see, e.g., [37]). Since π is unipotent, the corresponding V satisfies that $V = V^{I_F}$ so that $V^{\mathrm{ram}} = V/\mathrm{Ker}\, N$. □

3. Proofs of Propositions 2.4 and 2.7

The purpose of this section is to prove Propositions 2.4 and 2.7. To do these, we introduce the notions of VN-*pair*s and WL-*pair*s.

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3.1. VN-pairs and WL-pairs

A *VN-pair* (over $\mathbb{C}$) is a pair (V, N) of a finite-dimensional $\mathbb{Z}$-graded vector space V over $\mathbb{C}$ and a $\mathbb{C}$-linear endomorphism $N : V \to V$ of degree 1. Similarly, a *WL-pair* (over $\mathbb{C}$) is a pair (W, L) of a finite-dimensional $\mathbb{Z}$-graded vector space W over $\mathbb{C}$ and a $\mathbb{C}$-linear endomorphism $L : W \to W$ of degree -1.

Let (V, N) and (V', N') be two VN-pairs. A morphism $f : (V, N) \to (V', N')$ is a $\mathbb{C}$-linear map $V \to V'$ preserving the degrees such that $f \circ N = N' \circ f$.

Lemma 3.1. *Let (V, N) and (V', N') be two VN-pairs. Then $(V, N) \cong (V', N')$ if and only if $V \cong V'$ as graded vector spaces and $(\mathrm{Image}\, N, N|_{\mathrm{Image}\, N}) \cong (\mathrm{Image}\, N', N'|_{\mathrm{Image}\, N'})$.*

Proof. The 'only if' part is trivial. We prove the 'if' part. Assume the two conditions. Let us choose an isomorphism

$$f_1 : (\mathrm{Image}\, N, N|_{\mathrm{Image}\, N}) \xrightarrow{\cong} (\mathrm{Image}\, N', N'|_{\mathrm{Image}\, N'})$$

of VN-pairs. Let us also choose homogeneous elements $v_1, \ldots, v_r \in \mathrm{Image}\, N$ whose images in $\mathrm{Image}\, N/\mathrm{Image}\, N^2$ form a basis of this space. For $i = 1, \ldots, r$, let us choose homogeneous elements $e_1, \ldots, e_r \in V$ and $e'_1, \ldots, e'_r \in V'$ in such a way that we have $N(e_i) = v_i$ and $N'(e'_i) = f_1(v_i)$ for $i = 1, \ldots, r$. Let W (respectively, W') denote the graded vector subspace of V (respectively, V') generated by $\mathrm{Image}\, N$ and $e_1, \ldots, e_r$ (respectively, $\mathrm{Image}\, N'$ and $e'_1, \ldots, e'_r$).

Let $\overline{N} : V/\mathrm{Image}\, N \to \mathrm{Image}\, N/\mathrm{Image}\, N^2$ denote the homomorphism induced by N. It follows from the construction of W that the restriction of $\overline{N}$ to $W/\mathrm{Image}\, N$ gives an isomorphism $W/\mathrm{Image}\, N \xrightarrow{\cong} \mathrm{Image}\, N/\mathrm{Image}\, N^2$. Hence we have $V/\mathrm{Image}\, N = \mathrm{Ker}\, \overline{N} \oplus (W/\mathrm{Image}\, N)$. By applying the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Image}\, N & \longrightarrow & V & \longrightarrow & V/\mathrm{Image}\, N & \longrightarrow & 0 \\ & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle \overline{N}} & & \\ 0 & \longrightarrow & \mathrm{Image}\, N^2 & \longrightarrow & \mathrm{Image}\, N & \longrightarrow & \mathrm{Image}\, N/\mathrm{Image}\, N^2 & \longrightarrow & 0, \end{array}$$

we see that the homomorphism $\alpha : \mathrm{Ker}\, N \to \mathrm{Ker}\, \overline{N}$ induced by the quotient map $V \twoheadrightarrow V/\mathrm{Image}\, N$ is surjective. Let us choose a graded vector subspace $U \subset \mathrm{Ker}\, N$ such that the restriction of α to U gives an isomorphism $U \xrightarrow{\cong} \mathrm{Ker}\, \overline{N}$. Since $V/\mathrm{Image}\, N = \mathrm{Ker}\, \overline{N} \oplus (W/\mathrm{Image}\, N)$, we have $V = U \oplus W$.

A similar argument shows that there exists a graded vector subspace $U' \subset \mathrm{Ker}\, N'$ such that $V' = U' \oplus W'$. Since V and V' are isomorphic as graded vector spaces, U and U' are isomorphic as graded vector spaces. Let us choose an isomorphism $f_2 : U \to U'$ of graded vector spaces.

Let $f : V \to V'$ denote the homomorphism defined as follows: $f(v) = f_1(v)$ for $v \in \mathrm{Image}\, N$, $f(e_i) = e'_i$ for $i = 1, \ldots, r$ and $f(u) = f_2(u)$ for $u \in U$. Then f is an isomorphism of VN-pairs from (V, N) to (V', N'). This completes the proof. □

Let (V, N) be a VN-pair. For an integer $c \in \mathbb{Z}$, we let $(V, N)(c)$ denote the cth degree-shift of (V, N). By definition, $(V, N)(c) = (V(c), N(c))$, where $V(c)$ is the $\mathbb{Z}$-graded vector space over $\mathbb{C}$ whose degree-a-part is equal to the degree-$(a - c)$-part of V for any $a \in \mathbb{Z}$, and $N(c) : V(c) \to V(c)$ is the endomorphism induced by N. (This notation of degree-shift corresponds to the Tate twist on the Galois side of the local Langlands correspondence.)

For a segment $\Delta = [a, b]_\chi$ with $a, b \in \mathbb{Z}$, we let (V_Δ, N_Δ) denote the VN-pair such that V_Δ is the graded complex vector space with basis $e_a, e_{a+1}, \ldots, e_b$, where for $i = a, \ldots, b$, the vector e_i is homogeneous of degree i and $N_\Delta : V_\Delta \to V_\Delta$ is the endomorphism that sends e_i to e_{i+1} for $i = a, \ldots, b-1$ and sends e_b to 0. Similarly, we denote by (W_Δ, L_Δ) the WL-pair such that $W_\Delta = V_\Delta$ and $L_\Delta : W_\Delta \to W_\Delta$ is the endomorphism that sends e_i to e_{i-1} for $i = a + 1, \ldots, b$ and sends e_a to 0.

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Let χ be an unramified character of $F^\times$. For a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ of type χ, we define the VN-pair $(V_\mathfrak{m}, N_\mathfrak{m})$ and the WL-pair $(W_\mathfrak{m}, L_\mathfrak{m})$ as the direct sums

$$(V_\mathfrak{m}, N_\mathfrak{m}) = \left(\bigoplus_{i=1}^{r} V_{\Delta_i}, \bigoplus_{i=1}^{r} N_{\Delta_i} \right)$$

and

$$(W_\mathfrak{m}, L_\mathfrak{m}) = \left(\bigoplus_{i=1}^{r} W_{\Delta_i}, \bigoplus_{i=1}^{r} L_{\Delta_i} \right).$$

It follows from the Gabriel theory [7], or from the theory of Jordan normal forms and some elementary argument (compare to [16]), that these give one-to-one correspondence among the multisegments of type χ, the isomorphism classes of VN-pairs and the isomorphism classes of WL-pairs.

For a VN-pair (V, N) (respectively, a WL-pair (W, L)), let us consider the set $S(V, N)$ (respectively, $S(W, L)$) of $\mathbb{C}$-linear endomorphisms $L\colon V \to V$ (respectively, $N\colon V \to V$) of degree -1 (respectively, degree 1) satisfying $L \circ N = N \circ L$. We sometimes regard $S(V, N)$ and $S(W, L)$ as algebraic varieties over $\mathbb{C}$. Since $S(V, N)$ and $S(W, L)$ are finite-dimensional complex vector spaces, $S(V, N)$ and $S(W, L)$ are, as algebraic varieties over $\mathbb{C}$, isomorphic to affine spaces over $\mathbb{C}$.

Lemma 3.2. *Let (V, N) be a VN-pair and (W, L) be a WL-pair:*

(1) *The map $S(V, N) \to S(\mathrm{Image}\, N, N|_{\mathrm{Image}\, N})$ that sends L to $L|_{\mathrm{Image}\, N}$ is surjective.*
(2) *The map $S(W, L) \to S(\mathrm{Image}\, L, L|_{\mathrm{Image}\, L})$ that sends N to $N|_{\mathrm{Image}\, L}$ is surjective.*

Proof. We only give a proof of assertion (1). We can prove assertion (2) in a similar manner.

Let us choose homogeneous, linearly independent elements $v_1, \ldots, v_m \in V$ such that V is a direct sum of $\mathrm{Image}\, N$ and the subspace of V generated by $v_1, \ldots, v_m$. For $i = 1, \ldots, m$, we let d_i denote the degree of v_i. Given $L' \in S(\mathrm{Image}\, N, N|_{\mathrm{Image}\, N})$, choose a homogeneous element $w_i \in V$ of degree $d_i - 1$ that satisfies $L'(N(v_i)) = N(w_i)$ for each $i = 1, \ldots, m$. Let L denote the unique $\mathbb{C}$-linear map $V \to V$ such that $L(v) = L'(v)$ for $v \in \mathrm{Image}\, N$ and that $L(v_i) = w_i$ for $i = 1, \ldots, m$. It is then straightforward to check that $L \in S(V, N)$. It follows from the construction of L that $L|_{\mathrm{Image}\, N} = L'$. Hence the claim follows. □

Let (V, N) be a VN-pair, and let $\mathfrak{m}$ be the multisegment (of type χ) corresponding to (V, N). It follows from [43] and [31] that there exists a Zariski open dense subset $S^o(V, N) \subset S(V, N)$ such that, for $L \in S(V, N)$, the multisegment (of type χ) corresponding to (V, L) is equal to $\mathfrak{m}^\sharp$ if and only if $L \in S^o(V, N)$.

Let V be a finite-dimensional $\mathbb{Z}$-graded vector space over $\mathbb{C}$ and $N, L\colon V \to V$ be $\mathbb{C}$-linear endomorphisms of degree $1, -1$, respectively. We say that the triple (V, N, L) is *admissible* if $N \circ L = L \circ N$ and the multisegment corresponding to the WL-pair (V, L) is the Zelevinsky dual of the one corresponding to the VN-pair (V, N).

Lemma 3.3. *Let (V, N) (respectively, (W, L)) be a VN-pair (respectively, a WL-pair), and let $\mathfrak{m}$ denote the multisegment corresponding to (V, N) (respectively, (W, L)). Then the multisegment $\mathfrak{m}^-$ corresponds to* $(\mathrm{Image}\, N, N|_{\mathrm{Image}\, N})(-1)$ *(respectively,* $(\mathrm{Image}\, L, L|_{\mathrm{Image}\, L})$*).*

Proof. Easy. □

Let us give an example. Let $\mathfrak{m} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$, where $\Delta_1 = [3, 7]_\chi$, $\Delta_2 = [2, 5]_\chi$, $\Delta_3 = [1, 2]_\chi$ and $\Delta_4 = [0, 0]_\chi$.

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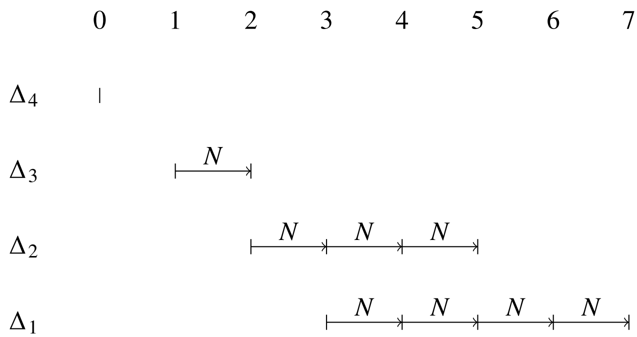
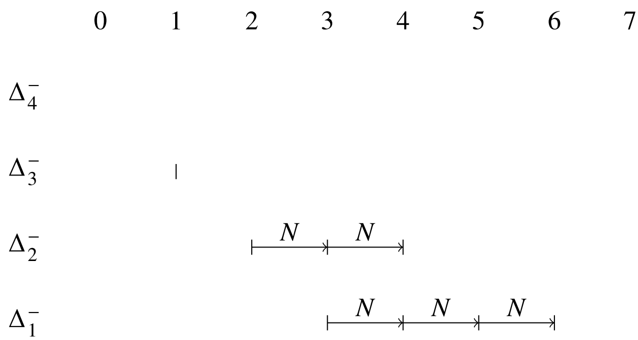


The picture of $(\mathrm{Image}\ N(-1), N|_{\mathrm{Image}\ N}(-1))$ is as follows:



We see that this corresponds to $\mathfrak{m}^-$.

Lemma 3.4. *Let (V, N) be a VN-pair, and let $\mathfrak{m}$ be the multisegment corresponding to (V, N). Then there exists a Zariski open dense subset $S^\theta(V, N) \subset S(V, N)$ such that for $L \in S(V, N)$, both (V, N, L) and $(\mathrm{Image}\, L, N|_{\mathrm{Image}\, L}, L|_{\mathrm{Image}\, L})$ are admissible triples if and only if $L \in S^\theta(V, N)$.*

Proof. It is easy to see that there exists a Zariski open subset $S^\theta(V, N) \subset S(V, N)$ such that for $L \in S(V, N)$, both (V, N, L) and $(\mathrm{Image}\, L, N|_{\mathrm{Image}\, L}, L|_{\mathrm{Image}\, L})$ are admissible triples if and only if $L \in S^\theta(V, N)$.

It remains to show that $S^\theta(V, N)$ is dense in $S(V, N)$. Since $S(V, N)$ is irreducible as an algebraic variety over $\mathbb{C}$, it suffices to show that $S^\theta(V, N)$ is nonempty. Let us choose $L \in S^o(V, N)$. Since the morphism $S(V, L) \longrightarrow S(\mathrm{Image}\, L, L|_{\mathrm{Image}\, L})$ is surjective by Lemma 3.2, there exists $N' \in S(V, L)$ such that both (V, N', L) and$(\mathrm{Image}\, L, N'|_{\mathrm{Image}\, L}, L|_{\mathrm{Image}\, L})$ are admissible triples. Then (V, N) and (V, N') are isomorphic since both correspond to the same multisegment. Hence (V, N', L) is isomorphic to (V, N, L') for some $L' \in S(V, N)$. Since L' belongs to $S^\theta(V, N)$, it follows that $S^\theta(V, N)$ is nonempty, as desired. □

3.2. *Proof of Proposition 2.4*

Now we prove Proposition 2.4.

Proof of Proposition 2.4. Let $\pi = Z(\mathfrak{m})$ be an irreducible representation of G_n. We decompose $\mathfrak{m}$ as

$$\mathfrak{m} = \mathfrak{m}' + \mathfrak{m}_1 + \cdots + \mathfrak{m}_t,$$

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where

- each segment in $\mathfrak{m}'$ is not unipotent;
- each $\mathfrak{m}_i$ is of type χ_i for some unramified character χ_i of $F^\times$ for $1 \leq i \leq t$;
- if $i \neq j$, then $\chi_i \chi_j^{-1}$ is not of the form $|\cdot|^a$ for any $a \in \mathbb{Z}$.

Set $\pi' = Z(\mathfrak{m}')$ and $\pi_i = Z(\mathfrak{m}_i)$. Then π is isomorphic to the parabolic induction $\pi' \times \pi_1 \times \cdots \times \pi_t$.

For $\Pi = \pi, \pi', \pi_1, \ldots, \pi_t$, let $\Pi^{(0)} = \Pi$ and $\Pi^{(i)}$ denote the highest derivative of $\Pi^{(i-1)}$ for $i \geq 1$. Then we have $\pi^{(i)} = \pi'^{(i)} \times \pi_1^{(i)} \times \cdots \times \pi_t^{(i)}$ for any integer $i \geq 0$. Thus, to prove the claim, we may assume that $\mathfrak{m} = \mathfrak{m}'$ or $\mathfrak{m} = \mathfrak{m}_1$.

First, we consider the case where $\mathfrak{m} = \mathfrak{m}'$. Let us write $\pi = Z(\mathfrak{m})$ and $\mathfrak{m} = [a_1, b_1]_{\rho_1} + \cdots + [a_r, b_r]_{\rho_r}$. Then $\rho_1, \ldots, \rho_r$ are ramified cuspidal representations. For $i = 1, \ldots, r$, let $c_i = c_{\rho_i}$ denote the conductor of ρ_i. Then for $j \geq 0$, we have $\pi^{(j)} = Z(\mathfrak{m}^{(j)})$, where

$$\mathfrak{m}^{(j)} = \sum_{\substack{1 \leq i \leq r \\ b_i - a_i \geq j}} [a_i, b_i - j]_{\rho_i}.$$

This shows that the conductor of $\pi^{(j)}$ is equal to

$$c^{(j)} = \sum_{\substack{1 \leq i \leq r \\ b_i - a_i \geq j}} (b_i - a_i + 1 - j) c_i.$$

Hence we have

$$c^{(j)} - c^{(j+1)} = \sum_{\substack{1 \leq i \leq r \\ b_i - a_i \geq j}} c_i.$$

From this, one can easily see that

$$\lambda_\pi = \sum_{i=1}^{r} (0, \ldots, 0, \underbrace{c_i, \ldots, c_i}_{b_i - a_i + 1}) = \lambda_{\mathfrak{m}},$$

as desired.

Now we consider the case where $\pi = Z(\mathfrak{m})$ is of type χ for an unramified character χ of $F^\times$. Let us consider the VN-pair (V, N) corresponding to $\mathfrak{m}$. For $i \geq 0$, let us write $\pi^{(i)} = Z(\mathfrak{m}^{(i)})$. As we remarked at the beginning of Section 2.3, we have $\pi^{(1)} = Z(\mathfrak{m}^-)$. Hence $\mathfrak{m}^{(i)}$ is obtained from $\mathfrak{m}$ by the i-fold iteration of the operation $(\ )^-$. Therefore, it follows from Lemma 3.3 that $\mathfrak{m}^{(i)}$ corresponds to the VN-pair $(\operatorname{Image} N^i, N|_{\operatorname{Image} N^i})(-i)$. Let us choose $L \in S^\theta(V, N)$ such that $L|_{\operatorname{Image} N^i}$ belongs to $S^o(\operatorname{Image} N^i, N|_{\operatorname{Image} N^i})$ for any integer $i \geq 0$. By Lemma 3.2, such an L exists. Then the conductor of $\pi^{(i)}$ is equal to the dimension of $\operatorname{Image} L \circ N^i$. Hence if we write $\lambda_\pi = (\lambda_1, \ldots, \lambda_n)$ and $d_i = \dim \operatorname{Image} L \circ N^i$ for $i \geq 0$, then we have

$$\lambda_k = d_{n-k} - d_{n-k+1}$$

for $k = 1, \ldots, n$. Let us write $\pi^{\mathrm{ram}} = Z(\mathfrak{m}^{\mathrm{ram}})$ with $\mathfrak{m}^{\mathrm{ram}} = \Delta_1 + \cdots + \Delta_r$ and $\Delta_i = [a_i, b_i]_\chi$ for $1 \leq i \leq r$. Then $\lambda_{\mathfrak{m}} = (\lambda'_1, \ldots, \lambda'_n)$, with

$$\lambda'_k = \sum_{\substack{1 \leq i \leq r \\ b_i - a_i \geq n-k}} 1$$

for $k = 1, \ldots, n$. By Lemmas 3.3 and 3.4, $\mathfrak{m}^{\mathrm{ram}}$ corresponds to the VN-pair $(\mathrm{Image}\, L, N|_{\mathrm{Image}\, L})$. Since L and N commute, we have $\dim \mathrm{Image}\, N^i \circ L = d_i$ for $i \geq 0$. Hence we have

$$d_i - d_{i+1} = \sum_{\substack{1 \leq i \leq r \\ b_i - a_i \geq i}} 1$$

for $i = 0, \ldots, n-1$. Therefore, we have

$$\lambda'_k = d_{n-k} - d_{n-k+1} = \lambda_k$$

for $k = 1, \ldots, n$. This completes the proof. □

We do not use the following proposition, but it might be interesting.

Proposition 3.5. *For any multisegment $\mathfrak{m}$, we have $(\mathfrak{m}^-)^{\mathrm{ram}} = (\mathfrak{m}^{\mathrm{ram}})^-$.*

Proof. Let (V, N) be the VN-pair corresponding to the multisegment $\mathfrak{m}$. If we choose a sufficiently general $L \in S(V, N)$, then $(\mathfrak{m}^-)^{\mathrm{ram}}$ and $(\mathfrak{m}^{\mathrm{ram}})^-$ correspond to the pairs $(\mathrm{Image}\, L \circ N, N|_{\mathrm{Image}\, L \circ N})(-1)$ and $(\mathrm{Image}\, N \circ L, N|_{\mathrm{Image}\, N \circ L})(-1)$, respectively. Since $L \circ N = N \circ L$, the claim follows. □

3.3. *Proof of Proposition 2.7*

The following statement is easy to check. However, we record it as a lemma for later use. A proof is omitted.

Lemma 3.6. *For any multisegment $\mathfrak{m}$, we have $(\mathfrak{m}^-)_{\max} = (\mathfrak{m}_{\max})^-$ and $(\mathfrak{m}^-)^{\max} = (\mathfrak{m}^{\max})^-$.*

For a multisegment $\mathfrak{m}$, a *full-sub-multisegment* of $\mathfrak{m}$ is a multisegment $\mathfrak{m}'$ such that for any segment Δ in $\mathfrak{m}'$, its multiplicity in $\mathfrak{m}'$ is equal to that in $\mathfrak{m}$.

We say that a multisegment $\mathfrak{m}$ is *totally ordered* if for any two segments Δ, Δ' in $\mathfrak{m}$, we have either $\Delta \subset \Delta'$ or $\Delta' \subset \Delta$.

Proposition 3.7. *Let $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ be a multisegment of type χ and $a \in \mathbb{Z}$ an integer. Let us write $\delta_a = [a, a+1]_\chi$. Let $\mathfrak{m}_a$ denote the full-sub-multisegment of $\mathfrak{m}$ that consists of segments that intersect δ_a, and let $\mathfrak{m}^\sharp_{(a)}$ denote the full-sub-multisegment of $\mathfrak{m}^\sharp = \Delta'_1 + \cdots + \Delta'_s$ that consists of segments that contain δ_a. Namely,*

$$\mathfrak{m}_a := \sum_{\substack{1 \leq i \leq r \\ \Delta_i \cap \delta_a \neq \emptyset}} \Delta_i, \quad \mathfrak{m}^\sharp_{(a)} := \sum_{\substack{1 \leq i \leq s \\ \Delta'_i \supset \delta_a}} \Delta'_i.$$

Then we have the equality

$$\mathrm{Card}(\mathfrak{m}^\sharp_{(a)}) = \mathrm{Card}(\mathfrak{m}_a) - \max_{\mathfrak{m}'} \mathrm{Card}(\mathfrak{m}'),$$

where $\mathfrak{m}'$ runs over the set of totally ordered full-sub-multisegments of $\mathfrak{m}_a$.

Proof. By replacing χ with $\chi|\cdot|^c$ for some integer c, we may and will assume that there exists an integer r such that any segment in $\mathfrak{m}$ is contained in $[1, r]_\chi$.

For two integers a, b with $1 \leq a \leq b \leq r$, let $d_{a,b} = d_{a,b}(\mathfrak{m})$ denote the multiplicity of the segment $[a, b]_\chi$ in $\mathfrak{m}$. When $a > b$, we set $d_{a,b} = 0$. Then it follows from the result of Knight–Zelevinsky [16, Theorem 1.2] that $\mathrm{Card}(\mathfrak{m}^\sharp_{(a)})$ is equal to the right-hand side of the equality (1.6) in [16] for $(i, j) = (a, a+1)$.

For two integers x, y with $x \leq y$, let $[x, y]$ denote the set of integers c satisfying $x \leq c \leq y$. Let $a \in [1, r-1]$. We rewrite the right-hand side of (1.6) in [16] for $(i, j) = (a, a+1)$. Let us first recall

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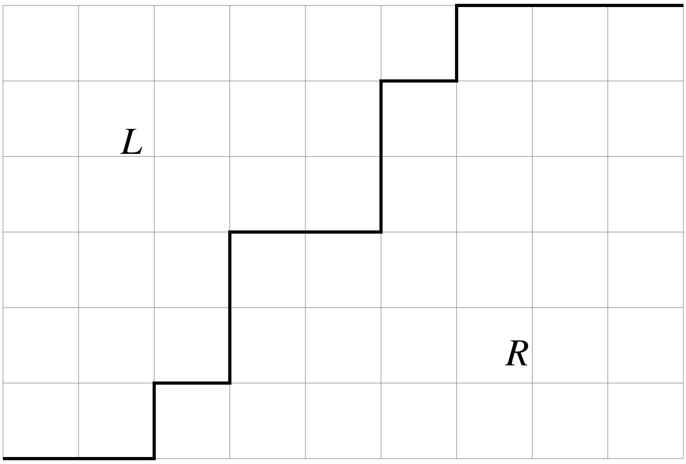

Figure 1.

some notation in [16]. They fix a positive integer r and consider the set S of pairs of integers (i, j) such that $1 \leq i \leq j \leq r$. For $1 \leq i \leq j \leq r$, they consider the set $T_{i,j}$ of functions $\nu\colon [1, i] \times [j, r] \to [i, j]$ such that $\nu(k, l) \leq \nu(k', l')$ whenever $k \leq k', l \leq l'$.

Let $1 \leq a \leq r$. We only use the case $i = a$, $j = a + 1$ and consider $T_{a,a+1}$. In this case, any function $\nu \in T_{a,a+1}$ takes one of two values $a, a+1$. We express this using Figure 1. The rectangle depicts the set $[1, a] \times [a + 1, r]$. The upper-left corner is $(1, a + 1)$, the lower-left corner is $(a, a + 1)$, the upper-right corner is $(1, r)$ and the lower-right corner is (a, r). Because of the condition on ν, there exists a bold line as in the picture such that ν takes the value a on the left (call the region L) and the value $a + 1$ on the right (call the region R).

We look at the sum from (1.6) [16]:

$$\sum_{(k,l)\in[1,a]\times[a+1,r]} d_{\nu(k,l)+k-a,\nu(k,l)+l-a-1}.$$

This equals

$$\sum_{(k,l)\in L} d_{k,l-1} + \sum_{(k,l)\in R} d_{k+1,l}.$$

Now consider Figure 2. The rectangle depicts the set $U = [1, a+1]\times[a, r]$. Let L' be the region L moved to the left by 1 and R' be the region R moved down by 1. These are subsets of U, and the complement $V_\nu = U \setminus (L' \cup R')$ is shown in blue in the picture.

A path from $(a + 1, a)$ to $(1, r)$ is a map $p\colon [0, r] \to \mathbb{Z}\times\mathbb{Z}$ satisfying the following conditions:

(1) $p(0) = (a + 1, a)$.
(2) For $i = 1, \ldots, r$, the element $p(i) \in \mathbb{Z}\times\mathbb{Z}$ is equal to $p(i - 1) - (1, 0)$ or $p(i - 1) + (0, 1)$.
(3) $p(r) = (1, r)$.

Then V_ν is equal to the image of a path from $[a + 1, a]$ to $[1, r]$. By sending ν to this path, we obtain a bijection from $T_{a,a+1}$ to the set A_a of paths from $(a + 1, a)$ to $(1, r)$.

Notice now that the sum above is equal to

$$\sum_{(k,l)\in L'} d_{k,l} + \sum_{(k,l)\in R'} d_{k,l} = \sum_{(k,l)\in U} d_{k,l} - \sum_{(k,l)\in V_\nu} d_{k,l}.$$

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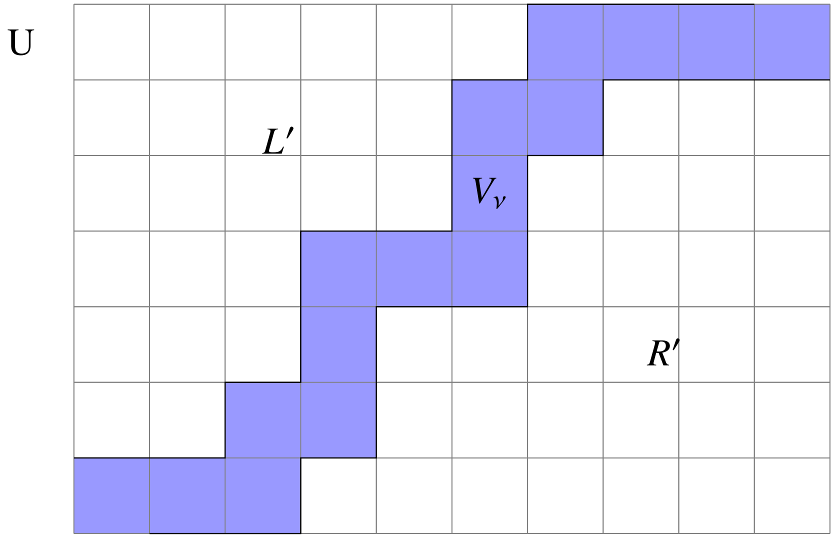

Figure 2.

Conversely, given a path from $[a+1, a]$ to $[a, 1]$, we obtain a function $\nu \in T_{a,a+1}$ such that V_ν is the image of the given path. Thus, the right-hand side of (1.6) of [16] is equal to

$$\sum_{(k,l)\in U} d_{k,l} - \max_{p\in A_a} \sum_{i=0}^{r} d_{p(i)}.$$

Notice that $\mathrm{Card}(\mathfrak{m}_a) = \sum_{(k,l)\in U} d_{k,l}$. From this, we see that $\mathrm{Card}(\mathfrak{m}^{\sharp}_{(a)})$ is equal to

$$\mathrm{Card}(\mathfrak{m}_a) - E_a(\mathfrak{m}),$$

where

$$E_a(\mathfrak{m}) = \max_{p\in A_a} \sum_{i=0}^{r} d_{p(i)}.$$

For $p \in A_a$, let $\mathfrak{m}_{a,p}$ denote the full-sub-multisegment of $\mathfrak{m}_a$ that consists of the segments $[a', b']_\chi$ in $\mathfrak{m}_a$ of the form $(a', b') = p(i)$ for some integer $i \in [0, r]$. Then $\mathfrak{m}_{a,p}$ is totally ordered, and we have $\sum_{i=0}^{r} d_{p(i)} = \mathrm{Card}(\mathfrak{m}_{a,p})$. By sending p to $\mathfrak{m}_{a,p}$, we obtain a map from A_a to the set T_a of totally ordered full-sub-multisegments of $\mathfrak{m}_a$. In general, this map is neither injective nor surjective. However, for any totally ordered full-sub-multisegment $\mathfrak{m}'$ of $\mathfrak{m}_a$, there exists a path $p \in A_a$ such that $\mathfrak{m}'$ is a full-sub-multisegment of $\mathfrak{m}_{a,p}$. In particular $\mathrm{Card}(\mathfrak{m}') \leq \mathrm{Card}(\mathfrak{m}_{a,p})$ for this p.

Thus, we obtain an equality

$$\max_{\mathfrak{m}'\in T_a} \mathrm{Card}(\mathfrak{m}') = \max_{p\in A_a} \mathrm{Card}(\mathfrak{m}_{a,p}) = E_a(\mathfrak{m}),$$

which completes the proof. □

Now we can prove Proposition 2.7.

Proof of Proposition 2.7. We prove the claim by induction on $l(\mathfrak{m})$.

Let (V, N), (V_1, N_1) and (V_2, N_2) be the VN-pairs corresponding to the multisegments $\mathfrak{m}$, $\mathfrak{m}_{\max}$ and $\mathfrak{m}^{\max}$, respectively. Let us consider the sets $S(V, N)$, $S(V_1, N_1)$ and $S(V_2, N_2)$ introduced in Section 3.1. Let us choose sufficiently general $L \in S(V, N)$, $L_1 \in S(V_1, N_1)$ and $L_2 \in S(V_2, N_2)$. Then the six multisegments $\mathfrak{m}^{\mathrm{ram}}$, $(\mathfrak{m}^-)^{\mathrm{ram}}$, $(\mathfrak{m}_{\max})^{\mathrm{ram}}$, $((\mathfrak{m}_{\max})^-)^{\mathrm{ram}}$, $(\mathfrak{m}^{\max})^{\mathrm{ram}}$ and $((\mathfrak{m}^{\max})^-)^{\mathrm{ram}}$ correspond to the pairs $(\mathrm{Image}\, L, N|_{\mathrm{Image}\, L})$, $(\mathrm{Image}\, L \circ N, N|_{\mathrm{Image}\, L\circ N})(-1)$, $(\mathrm{Image}\, L_1, N_1|_{\mathrm{Image}\, L_1})$, $(\mathrm{Image}\, L_1 \circ N_1, N_1|_{\mathrm{Image}\, L_1\circ N_1})(-1)$, $(\mathrm{Image}\, L_2, N_2|_{\mathrm{Image}\, L_2})$ and $(\mathrm{Image}\, L_2 \circ N_2, N_2|_{\mathrm{Image}\, L_2\circ N_2})(-1)$, respectively.

To prove the claim, it suffices to show that the pair $(\mathrm{Image}\, L, N|_{\mathrm{Image}\, L})$ is isomorphic to the pair $(\mathrm{Image}\, L_1 \oplus \mathrm{Image}\, L_2, N_1|_{\mathrm{Image}\, L_1} \oplus N_2|_{\mathrm{Image}\, L_2})$. By the inductive hypothesis, the claim is true for the multisegment $\mathfrak{m}^-$. Hence it follows from Lemma 3.6 that

$$(\mathrm{Image}\, L \circ N, N|_{\mathrm{Image}\, L\circ N}) \cong (\mathrm{Image}\, L_1 \circ N_1 \oplus \mathrm{Image}\, L_2 \circ N_2, N_1|_{\mathrm{Image}\, L_1\circ N_1} \oplus N_2|_{\mathrm{Image}\, L_2\circ N_2}).$$

Hence by Lemma 3.1, it suffices to show that the graded vector space $\mathrm{Image}\, L$ is isomorphic to the graded vector space $\mathrm{Image}\, L_1 \oplus \mathrm{Image}\, L_2$.

Let $\mathfrak{m}_a$ and $\mathfrak{m}^{\sharp}_{(a)}$ be as in Proposition 3.7. Note that the dimension of the degree-a-part of $\mathrm{Image}\, L$ is equal to $\mathrm{Card}(\mathfrak{m}^{\sharp}_{(a)})$. Let $\mathfrak{m}'$ be a totally ordered full-sub-multisegment of $\mathfrak{m}_a$ with the maximum cardinality. When $\mathfrak{m}_a$ is nonempty, the maximal segment Δ_1 of $\mathfrak{m}'$ must belong to $\mathfrak{m}_{\max}$, since otherwise one can find a totally ordered full-sub-multisegment of $\mathfrak{m}_a$ that strictly contains $\mathfrak{m}'$ by adding to $\mathfrak{m}'$ a segment of $\mathfrak{m}_{\max}$ that contains Δ_1, which is a contradiction. It is then easy to see that

- when $\mathfrak{m}_a$ is nonempty, $\mathfrak{m}' - \Delta_1$ is a totally ordered full-sub-multisegment of $(\mathfrak{m}^{\max})_a$ with the maximum cardinality; and
- Δ_1, which is regarded as a multisegment with $\mathrm{Card}(\Delta_1) = 1$, is a totally ordered full-sub-multisegment of $(\mathfrak{m}_{\max})_a$ with the maximum cardinality.

Thus, it follows from Proposition 3.7 that the dimension of the degree-a-part of $\mathrm{Image}\, L$ is equal to the sum of those of $\mathrm{Image}\, L_1$ and $\mathrm{Image}\, L_2$, as desired. □

4. Preliminaries on $\mathfrak{o}$-modules

To prove our main theorems, we prepare some results on $\mathfrak{o}$-modules in this section.

4.1. *On $\mathfrak{o}$-modules of finite length*

In this subsection, we introduce some terminologies on $\mathfrak{o}$-modules and give two basic results (Propositions 4.4, 4.6), which we call convexity and uniqueness, respectively. The authors suspect that these two results are well-known to some experts. In fact, one can deduce them from the description of Hall polynomials given in [25, II, (4.3)] in terms of sequences of partitions related with the Littlewood–Richardson rule. However, for the sake of completeness, we do not omit the proof of these results, which the authors believe to be helpful for most readers.

Let $|\mathcal{C}|$ denote the set of isomorphism classes of $\mathfrak{o}$-modules of finite length. For an $\mathfrak{o}$-module M of finite length, we denote by $[M] \in |\mathcal{C}|$ its isomorphism class.

For an integer $n \geq 1$, let $|\mathcal{C}^n| \subset |\mathcal{C}|$ denote the subset of isomorphism classes $[M]$ such that M is generated by at most n elements. We denote by $\iota_n : |\mathcal{C}^n| \hookrightarrow |\mathcal{C}^{n+1}|$ the inclusion map.

Recall that Λ_n is the set of n-tuples $(\lambda_1, \ldots, \lambda_n)$ of integers satisfying $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$. For $[M] \in |\mathcal{C}^n|$, there exists a unique element $(\lambda_1, \ldots, \lambda_n)$ of Λ_n such that the $\mathfrak{o}$-module M is isomorphic to

$$\mathfrak{o}/\mathfrak{p}^{\lambda_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\lambda_n}.$$

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By sending $[M]$ to the n-tuple $(\lambda_1, \ldots, \lambda_n)$, we obtain a bijective map $\mathrm{seq}_n \colon |\mathcal{C}^n| \to \Lambda_n$. We denote by $\jmath_n \colon \Lambda_n \to \Lambda_{n+1}$ the injective map that sends $(\lambda_1, \ldots, \lambda_n)$ to $(0, \lambda_1, \ldots, \lambda_n)$. Then the diagram

$$\begin{tikzcd} |\mathcal{C}^n| \arrow[r, "\mathrm{seq}_n"] \arrow[d, "\iota_n"'] & \Lambda_n \arrow[d, "\jmath_n"] \\ |\mathcal{C}^{n+1}| \arrow[r, "\mathrm{seq}_{n+1}"] & \Lambda_{n+1} \end{tikzcd}$$

is commutative.

For two elements $[M], [M'] \in |\mathcal{C}^n|$, we write $[M] \le [M']$ if $\mathrm{seq}_n([M]) \le \mathrm{seq}_n([M'])$ with respect to the lexicographic order on Λ_n. This gives a total order on the set $|\mathcal{C}^n|$. The map ι_n is compatible with the total orders on $|\mathcal{C}^n|$ and on $|\mathcal{C}^{n+1}|$ since the map $\jmath_n$ is compatible with the lexicographic orders on Λ_n and on Λ_{n+1}. Hence the total orders on $|\mathcal{C}^n|$ for all n induce a total order $\le$ on the set $|\mathcal{C}|$.

We regard Λ_n as a subset of $\mathbb{Z}^n$. Then Λ_n is closed under the addition $+$ on $\mathbb{Z}^n$ and becomes a commutative submonoid of $\mathbb{Z}^n$ with the addition $+$. For two elements $[M], [M'] \in |\mathcal{C}^n|$, we denote by $[M] \vee [M']$ the unique element of $|\mathcal{C}^n|$ whose image under seq_n is equal to $\mathrm{seq}_n([M]) + \mathrm{seq}_n([M'])$. Then the set $|\mathcal{C}^n|$ becomes a commutative monoid with the operation $\vee$ and the diagram

$$\begin{tikzcd} |\mathcal{C}^n| \times |\mathcal{C}^n| \arrow[r, "\vee"] \arrow[d, "\mathrm{seq}_n \times \mathrm{seq}_n"'] & |\mathcal{C}^n| \arrow[d, "\mathrm{seq}_n"] \\ \Lambda_n \times \Lambda_n \arrow[r, "+"] & \Lambda_n \end{tikzcd}$$

is commutative. The map ι_n is compatible with the monoid structures on $|\mathcal{C}^n|$ and $|\mathcal{C}^{n+1}|$ since the map $\jmath_n$ is compatible with the addition $+$. Hence the binary operations $\vee$ on $|\mathcal{C}^n|$ for all n induce a binary operation on the set $|\mathcal{C}|$, also denoted by $\vee$. This gives a structure of a commutative monoid on the set $|\mathcal{C}|$.

The following lemma says that the total order $\le$ on $|\mathcal{C}|$ is compatible with the monoid structure on $|\mathcal{C}|$.

Lemma 4.1. *Let $[M], [M'], [N], [N'] \in |\mathcal{C}|$ and suppose that $[M] \le [N]$ and $[M'] \le [N']$. Then we have $[M] \vee [M'] \le [N] \vee [N']$.*

Proof. We can easily see that the lexicographic order on Λ_n is compatible with the monoid structure on Λ_n given by $+$. Hence the claim follows. □

Recall that F is the field of fractions of $\mathfrak{o}$. For an $\mathfrak{o}$-module M of finite length, we let $M^\vee$ denote the $\mathfrak{o}$-module $\mathrm{Hom}_{\mathfrak{o}}(M, F/\mathfrak{o})$.

Lemma 4.2. *For any $\mathfrak{o}$-module M of finite length, we have $[M] = [M^\vee]$.*

Proof. We may assume $M = \mathfrak{o}/\mathfrak{p}^{\lambda_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\lambda_n}$. Since $(\ )^\vee$ commutes with finite direct sums, we may further assume that $M = \mathfrak{o}/\mathfrak{p}^{\lambda}$. Then we have $M^\vee \cong \mathfrak{p}^{-\lambda}/\mathfrak{o}$. Hence by choosing a uniformiser $\varpi \in \mathfrak{p}$, we obtain a desired isomorphism $M \cong M^\vee$. □

Lemma 4.3. *Let M, M' be $\mathfrak{o}$-modules of finite length.*

(1) *If there exists an injective homomorphism $M' \hookrightarrow M$, then we have $[M'] \le [M]$.*
(2) *If there exists a surjective homomorphism $M \twoheadrightarrow M'$, then we have $[M'] \le [M]$.*

Proof. Since an injective homomorphism $M' \hookrightarrow M$ induces a surjective homomorphism $M^\vee \twoheadrightarrow M'^\vee$, claim (1) follows from claim (2) and Lemma 4.2. Let us prove claim (2) below.

Let M, M' be $\mathfrak{o}$-modules of finite length, and suppose that there exists a surjective homomorphism $M \twoheadrightarrow M'$. Let us take an integer $n \ge 1$ such that both $[M]$ and $[M']$ belong to $|\mathcal{C}^n|$. We prove the claim by induction on n. If $n = 1$, then the claim is clear. We assume $n > 1$. Let us write $\mathrm{seq}_n([M]) = (\lambda_1, \ldots, \lambda_n)$ and $\mathrm{seq}_n([M']) = (\lambda'_1, \ldots, \lambda'_n)$.

First, suppose that $\lambda_1 > \lambda'_1$. Then we have $[M'] < [M]$ as claimed. Next, suppose that $\lambda_1 = \lambda'_1$. Then both $M/\mathfrak{p}^{\lambda_1}M$ and $M'/\mathfrak{p}^{\lambda_1}M'$ are isomorphic to $(\mathfrak{o}/\mathfrak{p}^{\lambda_1})^{\oplus n}$, and we have $[M] = [\mathfrak{p}^{\lambda_1}M] \vee [(\mathfrak{o}/\mathfrak{p}^{\lambda_1})^{\oplus n}]$ and $[M'] = [\mathfrak{p}^{\lambda_1}M'] \vee [(\mathfrak{o}/\mathfrak{p}^{\lambda_1})^{\oplus n}]$. Note that the surjective homomorphism $M \twoheadrightarrow M'$ induces a surjective homomorphism $\mathfrak{p}^{\lambda_1}M \twoheadrightarrow \mathfrak{p}^{\lambda_1}M'$. Hence by Lemma 4.1, we are reduced to proving claim (2) for $\mathfrak{p}^{\lambda_1}M$ and $\mathfrak{p}^{\lambda_1}M'$. Since both $[\mathfrak{p}^{\lambda_1}M]$ and $[\mathfrak{p}^{\lambda_1}M']$ belong to $|\mathcal{C}^{n-1}|$, the inductive hypothesis proves the claim in the case where $\lambda_1 = \lambda'_1$.

Finally, suppose that $\lambda_1 < \lambda'_1$. Again in this case, the surjective homomorphism $M \twoheadrightarrow M'$ induces a surjective homomorphism $\mathfrak{p}^{\lambda_1}M \twoheadrightarrow \mathfrak{p}^{\lambda_1}M'$. Note that $\mathfrak{p}^{\lambda_1}M$ is generated by fewer than n elements, whereas the minimum number of generators of $\mathfrak{p}^{\lambda_1}M'$ is equal to n. This leads to a contradiction. □

Proposition 4.4 (Convexity). *Let*

$$0 \to M' \to M \to M'' \to 0 \tag{4.1}$$

be a short exact sequence of $\mathfrak{o}$*-modules of finite length. Then we have the inequality*

$$[M] \geq [M'] \vee [M''].$$

Proof. Let n, n' and n'' denote the minimal numbers of generators of the $\mathfrak{o}$-modules M, M' and M'', respectively. We prove the claim by induction on $n' + n''$.

If $n' + n'' = 0$, then we have $M' = M'' = 0$, and the claim is clear. Since $M \to M''$ and $M^\vee \to M'^\vee$ are surjective, we have $n \geq n''$ and $n \geq n'$. If $n > \max\{n', n''\}$, then the claim is obvious. Hence we may assume that $n = \max\{n', n''\}$. By considering the short exact sequence

$$0 \to M''^\vee \to M^\vee \to M'^\vee \to 0$$

instead of equation (4.1), if necessary, we may further assume that $n = n''$. Let us write $\mathrm{seq}_n(M'') = (\lambda''_1, \ldots, \lambda''_n)$ and $I = \mathfrak{p}^{\lambda''_1}$. Then both M/IM and M''/IM'' are isomorphic to $(\mathfrak{o}/I)^{\oplus n}$, and we have $[M] = [IM] \vee [(\mathfrak{o}/I)^{\oplus n}]$ and $[M''] = [IM''] \vee [(\mathfrak{o}/I)^{\oplus n}]$. Moreover, equation (4.1) induces a short exact sequence

$$0 \to M' \to IM \to IM'' \to 0.$$

Since $\mathrm{seq}_n(IM'') = (0, \lambda''_2 - \lambda''_1, \ldots, \lambda''_n - \lambda''_1)$, the minimal number of generators of IM'' is strictly smaller than n''. Hence, by induction, we have $[IM] \geq [M'] \vee [IM'']$. By adding $[(\mathfrak{o}/I)^{\oplus n}]$ to both sides and using Lemma 4.1, we obtain the desired inequality. □

Lemma 4.5. *Let M be an* $\mathfrak{o}$*-module of finite length. Then for any nonzero ideal $I \subset \mathfrak{o}$, we have* $[M] = [IM] \vee [M/IM]$.

Proof. Let us write $I = \mathfrak{p}^\lambda$. Let us choose a positive integer n such that $[M] \in |\mathcal{C}^n|$. Let us write $\mathrm{seq}_n([M]) = (\lambda_1, \ldots, \lambda_n)$. For $i = 1, \ldots, n$, set $\lambda''_i = \min\{\lambda, \lambda_i\}$. Then M/IM is isomorphic to $\bigoplus_{i=1}^n \mathfrak{o}/\mathfrak{p}^{\lambda''_i}$ and IM is isomorphic to $\bigoplus_{i=1}^n \mathfrak{p}^{\lambda''_i}/\mathfrak{p}^{\lambda_i}$. Thus, we have $[M] = [IM] \vee [M/IM]$, as desired. □

Proposition 4.6 (Uniqueness). *Suppose that $[M], [M'], [M''] \in |\mathcal{C}|$ satisfy $[M] = [M'] \vee [M'']$. Then there exists a unique* $\mathfrak{o}$*-submodule $N \subset M$ satisfying $[N] = [M']$ and $[M/N] = [M'']$. Moreover, for any* $\mathfrak{o}$*-submodule $N' \subset M$ other than N, we have either $[N'] < [M']$ or $[M/N'] < [M'']$.*

Proof. First, we prove the existence and uniqueness of N. Let n, n' and n'' denote the minimal numbers of generators of the $\mathfrak{o}$-modules M, M' and M'', respectively. We prove the claim by induction on $n' + n''$.

If $n' + n'' = 0$, then we have $M = M' = M'' = 0$, and the claim is obvious. The relation $[M] = [M'] \vee [M'']$ implies that $n = \max\{n', n''\}$. By considering $M^\vee$ instead of M if necessary, we may assume that $n = n''$. Let us write $\mathrm{seq}_n(M'') = (\lambda''_1, \ldots, \lambda''_n)$ and $I = \mathfrak{p}^{\lambda''_1}$. Then for any $\mathfrak{o}$-submodule $N \subset M$ satisfying $[M/N] = [M'']$, we have $N \subset IM$. Since $\mathrm{seq}_n(IM'') = (0, \lambda''_2 - \lambda''_1, \ldots, \lambda''_n - \lambda''_1)$, the minimal number of generators of IM'' is strictly smaller than n''. Hence, by induction, there exists a unique $\mathfrak{o}$-submodule $N \subset IM$ satisfying $[N] = [M']$ and $[IM/N] = [IM'']$. Since N is contained in IM, the $\mathfrak{o}$-module $I(M/N)$ is isomorphic to IM/N, and hence $(M/N)/I(M/N)$ is isomorphic to M/IM. It follows from Lemma 4.5 that we have $[M/N] = [IM/N] \vee [M/IM] = [M'']$. Hence the claim follows.

Finally, let us prove the last assertion of the proposition. Let $N' \subset M$ be an $\mathfrak{o}$-submodule other than N. Suppose that $[N'] \geq [M']$ and $[M/N'] \geq [M'']$. Since $N' \neq N$, we have either $[N'] > [M']$ or $[M/N'] > [M'']$. Hence it follows from Lemma 4.1 and Proposition 4.4 that

$$[M] \geq [N'] \vee [M/N'] > [M'] + [M''] = [M],$$

which is a contradiction. Hence we have either $[N'] < [M']$ or $[M/N'] < [M'']$. This completes the proof. $\square$

Corollary 4.7. *Suppose that* $[M], [M_1], \ldots, [M_r] \in |\mathcal{C}|$ *satisfy* $[M] = [M_1] \vee \cdots \vee [M_r]$. *Then there exists a unique increasing filtration*

$$0 = \mathrm{F}^0_0 M \subset \cdots \subset \mathrm{F}^0_r M = M$$

of M *by* $\mathfrak{o}$*-submodules satisfying* $[M_i] = [\mathrm{Gr}^{\mathrm{F}^0}_i M]$ *for* $i = 1, \ldots, r$, *where* $\mathrm{Gr}^{\mathrm{F}^0}_i M = \mathrm{F}^0_i M/\mathrm{F}^0_{i-1} M$. *Moreover, for any filtration*

$$0 = \mathrm{F}_0 M \subset \cdots \subset \mathrm{F}_r M = M$$

of M *by* $\mathfrak{o}$*-submodules other than* $\mathrm{F}^0_\bullet M$, *we have* $[\mathrm{Gr}^{\mathrm{F}}_i M] < [M_i]$ *for some* $i \in \{1, \ldots, r\}$.

Proof. We prove the existence and uniqueness of $\mathrm{F}^0_\bullet M$ by induction on r. If $r = 1$, it is obvious. If $r > 1$, set $[M'] = [M_1]$ and $[M''] = [M_2] \vee \cdots \vee [M_r]$. By Proposition 4.6, there exists a unique $\mathfrak{o}$-submodule N of M such that $[N] = [M_1]$ and $[M/N] = [M_2] \vee \cdots \vee [M_r]$. By the inductive hypothesis, we have a unique filtration $\mathrm{F}^0_\bullet(M/N)$ satisfying the conditions with respect to $[M/N] = [M_2] \vee \cdots \vee [M_r]$. By setting $\mathrm{F}^0_{i+1} M$ to be the inverse image of $\mathrm{F}^0_i(M/N)$ for $1 \leq i \leq r-1$, and $\mathrm{F}^0_1 M = N$, we obtain $\mathrm{F}^0_\bullet M$.

The last assertion follows from the same argument as in the proof of Proposition 4.6. $\square$

4.2. Generators of $\mathfrak{o}$*-modules*

Lemma 4.8. *Let* $f : M \twoheadrightarrow N$ *be a surjective homomorphism of* $\mathfrak{o}$*-modules, and* $M' \subset M$ *an* $\mathfrak{o}$*-submodule. Let* $x \in N$ *and* $y \in M/M'$ *be elements whose images in* $N/f(M')$ *coincide. Then there exists a lift* $\widetilde{y} \in M$ *of* y *satisfying* $f(\widetilde{y}) = x$.

Proof. Let us take an arbitrary lift $\widetilde{y}' \in M$ of y and set $x' = f(\widetilde{y}')$. Since the images of x and x' coincide in $N/f(M')$, there exists $z \in M'$ satisfying $x - x' = f(z)$. Then the element $\widetilde{y} = \widetilde{y'} + z \in M$ has the desired property. $\square$

Lemma 4.9. *Let* N *be an* $\mathfrak{o}$*-module of finite length. Let* L *and* L' *be finitely generated free* $\mathfrak{o}$*-modules of the same rank, and let* $f : L \twoheadrightarrow N$ *and* $f' : L' \twoheadrightarrow N$ *be surjective homomorphisms of* $\mathfrak{o}$*-modules. Then there exists an isomorphism* $\alpha : L \xrightarrow{\cong} L'$ *of* $\mathfrak{o}$*-modules satisfying* $f = f' \circ \alpha$.

Proof. Since N is of finite length over a noetherian local ring $\mathfrak{o}$, one can take a projective cover $\beta : P \to N$ of N (see [1, 17.16 Examples (3)]). Then there exist homomorphisms $\gamma : L \to P$ and $\gamma' : L' \to P$ satisfying $f = \beta \circ \gamma$ and $f' = \beta \circ \gamma'$. Since projective covers are essential surjections, the homomorphisms γ and γ' are surjective. Hence by the projectivity of P, one can choose a right inverse s and s' of γ and γ', respectively. Since $\mathrm{Ker}\,\gamma$ and $\mathrm{Ker}\,\gamma'$ are free $\mathfrak{o}$-modules of the same rank, there exists an isomorphism $\alpha' : \mathrm{Ker}\,\gamma \xrightarrow{\cong} \mathrm{Ker}\,\gamma'$ of $\mathfrak{o}$-modules. By taking the direct sum of α' and the isomorphism $s(P) \xrightarrow{\cong} s'(P)$ given by $s' \circ \gamma$, we obtain a desired isomorphism $\alpha : L \to L'$. $\square$

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Corollary 4.10. *Let N be an $\mathfrak{o}$-module of finite length generated by n elements $x_1, \dots, x_n$. Then for any free $\mathfrak{o}$-module L of rank n and for any surjective homomorphism $f\colon L \to N$, there exists an $\mathfrak{o}$-basis $y_1, \dots, y_n$ of L satisfying $f(y_i) = x_i$ for $i = 1, \dots, n$.*

Proof. Let $L' = \mathfrak{o}^{\oplus n}$, and let $f'\colon L' \twoheadrightarrow N$ denote the surjection that sends the standard basis of L' to the elements $x_1, \dots, x_n$. By applying Lemma 4.9, we obtain an isomorphism $\alpha\colon L \xrightarrow{\cong} L'$ satisfying $f = f' \circ \alpha$. Then the image under α^{-1} of the standard basis of L' gives a desired basis of L. □

From now until the end of this section, we fix an integer $n \geq 1$ and a partition

$$\mathbf{n} = (n_1, \dots, n_r),\ n = n_1 + \cdots + n_r,\ n_i \geq 1$$

of n. For $i = 1, \dots, r$, we set

$$a_i = n_1 + \cdots + n_{i-1} + 1,\ b_i = n_1 + \cdots + n_i.$$

We use the following terminology.

Definition 4.11. Let M be an $\mathfrak{o}$-module generated by at most n elements.

(1) We say that an increasing filtration $\mathrm{F}_\bullet M$ of M by $\mathfrak{o}$-submodules is $\mathbf{n}$-*admissible* if the following conditions are satisfied:
- $\mathrm{F}_0 M = 0$ and $\mathrm{F}_r M = M$.
- For $i = 1, \dots, r$, the graded quotient $\mathrm{Gr}^{\mathrm{F}}_i M = \mathrm{F}_i M / \mathrm{F}_{i-1} M$ is generated by at most n_i elements.

(2) Let $\mathrm{F}_\bullet M$ be an $\mathbf{n}$-admissible filtration of M. We say that a sequence $y_1, \dots, y_n$ of elements of M is *compatible with* $\mathrm{F}_\bullet M$ if, for $i = 1, \dots, r$, the b_i elements $y_1, \dots, y_{b_i}$ generate the $\mathfrak{o}$-module $\mathrm{F}_i M$.

Lemma 4.12. *Let M be an $\mathfrak{o}$-module of finite length. Let L be a free $\mathfrak{o}$-module of rank n, and let $f\colon L \twoheadrightarrow M$ be a surjective homomorphism of $\mathfrak{o}$-modules. Suppose that an $\mathbf{n}$-admissible filtration $\mathrm{F}_\bullet L$ of L is given. Let $\mathrm{F}_\bullet M$ denote the filtration on M induced from $\mathrm{F}_\bullet L$ via f: that is, $\mathrm{F}_i M = f(\mathrm{F}_i L)$. For $i = 1, \dots, r$, let $f_i\colon \mathrm{Gr}^{\mathrm{F}}_i L \twoheadrightarrow \mathrm{Gr}^{\mathrm{F}}_i M$ denote the surjective homomorphism induced by f. Then we have the following.*

(1) *$\mathrm{F}_\bullet M$ is an $\mathbf{n}$-admissible filtration of M.*

(2) *Let $x_1, \dots, x_n$ be a sequence of elements of M compatible with $\mathrm{F}_\bullet M$. Then there exists a sequence $y_1, \dots, y_n$ of elements of L compatible with $\mathrm{F}_\bullet L$ such that $x_j = f(y_j)$ for $j = 1, \dots, n$.*

(3) *Let $x_1, \dots, x_n$ be a sequence of elements of M compatible with $\mathrm{F}_\bullet M$. Suppose that, for $i = 1, \dots, r$, an $\mathfrak{o}$-basis $z_{a_i}, \dots, z_{b_i}$ of $\mathrm{Gr}^{\mathrm{F}}_i L$ is given in such a way that for $j = a_i, \dots, b_i$, the image $f_i(z_j)$ is equal to the class of x_j in $\mathrm{Gr}^{\mathrm{F}}_i M$. Then there exists a sequence $y_1, \dots, y_n$ of elements of L compatible with $\mathrm{F}_\bullet L$ such that $x_j = f(y_j)$ for $j = 1, \dots, n$ and such that the class of y_j in $\mathrm{Gr}^{\mathrm{F}}_i L$ is equal to z_j for $j = a_i, \dots, b_i$.*

Proof. Assertion (1) is clear. We can deduce assertion (2) from assertion (3), since in the situation of (2), one can find, by using Corollary 4.10, an $\mathfrak{o}$-basis $z_{a_i}, \dots, z_{b_i}$ of $\mathrm{Gr}^{\mathrm{F}}_i L$ as in the statement of the assertion of (3) for $i = 1, \dots, r$. (Here, we note that $\mathrm{F}_i L$ is a free $\mathfrak{o}$-module of rank $n_1 + \cdots + n_i$.)

We prove assertion (3). Using Lemma 4.8, one can choose an element $y_j \in \mathrm{F}_i L$ for $j = a_i, \dots, b_i$ in such a way that $f(y_j) = x_j$ and the image of y_j in $\mathrm{Gr}^{\mathrm{F}}_i L$ is equal to z_j. Then the sequence $y_1, \dots, y_n$ of elements of L has the desired property. □

The following is well-known.

Lemma 4.13. *Let M be an $\mathfrak{o}$-module of finite length and $m_1, \dots, m_r$ be nonnegative integers. Then the number of filtrations $0 = \mathrm{F}_0 M \subset \cdots \subset \mathrm{F}_r M = M$ with $\mathrm{Gr}^{\mathrm{F}}_i M$ generated by exactly m_i elements for any $1 \leq i \leq r$ is invariant under the permutations of $m_1, \dots, m_r$.*

Outline of proof. First, reduce to the case where the permutation is an adjacent transposition. Then reduce to the case where $r = 2$. Finally, use the duality (Lemma 4.2) to treat this case. □

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5. The Mackey decomposition

In this section, we give the Mackey decomposition (Proposition 5.2) of the invariants by compact open subgroups of the form $\mathbb{K}_{n,\lambda}$. As an application, we give a reduction step in the proof of our main results.

5.1. Invariant subspaces of parabolically induced representations

Fix an integer $n \geq 1$. Let us consider the F-vector space F^n. We regard an element of F^n as a column vector. The group $G_n = \mathrm{GL}_n(F)$ acts on F^n from the left by the multiplication. Let $L_1, L_2 \subset F^n$ be $\mathfrak{o}$-lattices with $L_1 \supset L_2$. We denote by $\mathbb{K}_{L_1,L_2}$ the set of elements $g \in G_n$ satisfying the following conditions:

- We have $gL_1 = L_1$ and $gL_2 = L_2$.
- The endomorphism of the $\mathfrak{o}$-module L_1/L_2 induced by the multiplication by g is the identity map.

Then $\mathbb{K}_{L_1,L_2}$ is a compact open subgroup of G_n.

Lemma 5.1. *The G_n-conjugacy class of $\mathbb{K}_{L_1,L_2}$ depends only on n and an isomorphism class $[L_1/L_2]$ of the $\mathfrak{o}$-module L_1/L_2.*

Proof. Let L_1, L_2, L_1', L_2' be $\mathfrak{o}$-lattices of F^n such that $L_1 \supset L_2$, $L_1' \supset L_2'$ and L_1/L_2 is isomorphic to L_1'/L_2' as $\mathfrak{o}$-modules. Let us choose an isomorphism $L_1/L_2 \cong L_1'/L_2'$, and let f (respectively, f') denote the composite $L_1 \twoheadrightarrow L_1/L_2 \xrightarrow{\cong} L_1'/L_2'$ (respectively, the quotient map $L_1' \to L_1'/L_2'$). Then it follows from Lemma 4.9 that there exists an isomorphism $\alpha\colon L_1 \xrightarrow{\cong} L_1'$ satisfying $f = f' \circ \alpha$. By extending α to an automorphism $F^n \xrightarrow{\cong} F^n$ by F-linearity, we obtain an element $g \in G_n$ such that $\alpha(x) = gx$. It is then straightforward to check that $\mathbb{K}_{L_1',L_2'} = g\mathbb{K}_{L_1,L_2}g^{-1}$. This completes the proof. □

By abuse of notation, we denote the group $\mathbb{K}_{L_1,L_2}$ by $\mathbb{K}_{n,[L_1/L_2]}$. We note that, for $[M]$ in $|\mathcal{C}^n|$, the group $\mathbb{K}_{n,[M]}$ is well-defined only up to G_n-conjugation. If $\lambda = \mathrm{seq}_n([M])$, the G_n-conjugacy class of $\mathbb{K}_{n,[M]}$ is equal to the class of $\mathbb{K}_{n,\lambda}$. Indeed, if we set $L_1 = \mathfrak{o}^n$ and $L_1 = \oplus_{i=1}^n \mathfrak{p}^{\lambda_i}$ with $\lambda = (\lambda_1, \dots, \lambda_n)$, then we see that $\mathbb{K}_{L_1,L_2} = \mathbb{K}_{n,\lambda}$.

Fix a partition $\mathbf{n} = (n_1, \dots, n_r)$ of n with integers $n_1, \dots, n_r \geq 1$. Let $\pi_1, \dots, \pi_r$ be representations of $G_{n_1}, \dots, G_{n_r}$ of finite length, respectively. Consider the representation $\pi_1 \times \cdots \times \pi_r$ of G_n, which is parabolically induced from the representation $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ of the standard Levi subgroup $G_{n_1} \times \cdots \times G_{n_r}$ of G_n. Then for any $[M] \in |\mathcal{C}^n|$, the Mackey decomposition gives the following description of the $\mathbb{K}_{n,[M]}$-invariant part of $\pi_1 \times \cdots \times \pi_r$.

Proposition 5.2 (The Mackey decomposition). *There exists an isomorphism*

$$(\pi_1 \times \cdots \times \pi_r)^{\mathbb{K}_{n,[M]}} \cong \bigoplus_{\mathrm{F}_\bullet M} \pi_1^{\mathbb{K}_{n_1,[\mathrm{Gr}_1^{\mathrm{F}} M]}} \otimes \cdots \otimes \pi_r^{\mathbb{K}_{n_r,[\mathrm{Gr}_r^{\mathrm{F}} M]}}$$

of complex vector spaces. Here $\mathrm{F}_\bullet M$ in the direct sum above runs over the set of $\mathbf{n}$*-admissible filtrations of M: that is, the increasing filtrations*

$$0 = \mathrm{F}_0 M \subset \cdots \subset \mathrm{F}_r M = M$$

on M by $\mathfrak{o}$-submodules such that for $i = 1, \dots, r$, the $\mathfrak{o}$-module $\mathrm{Gr}_i^{\mathrm{F}} M = \mathrm{F}_i M/\mathrm{F}_{i-1} M$ is generated by at most n_i elements.

Proof. Let $P_{\mathbf{n}} \subset G_n$ denote the standard parabolic subgroup corresponding to the partition $\mathbf{n} = (n_1, \dots, n_r)$. Consider the quotient homomorphism $q\colon P_{\mathbf{n}} \to G_{n_1} \times \cdots \times G_{n_r}$. Let us choose a complete set $S \subset G_n$ of representatives of the double coset $P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{n,[M]}$.

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Then the Mackey decomposition yields an isomorphism

$$(\pi_1 \times \cdots \times \pi_r)^{\mathbb{K}_{n,[M]}} \cong \bigoplus_{g \in S} (\pi_1 \boxtimes \cdots \boxtimes \pi_r)^{q(P_{\mathbf{n}} \cap g\mathbb{K}_{n,[M]}g^{-1})}. \tag{5.1}$$

Let $\mathcal{F}_M$ denote the set of $\mathbf{n}$-admissible filtrations on M. In view of equation (5.1), it suffices to construct a bijection $\alpha\colon P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{n,[M]} \xrightarrow{\cong} \mathcal{F}_M$ satisfying the following property: If $P_{\mathbf{n}}g\mathbb{K}_{n,[M]}$ corresponds to the filtration $\mathrm{F}_\bullet M$ via α, then the subgroup $q(P_{\mathbf{n}} \cap g\mathbb{K}_{n,[M]}g^{-1})$ of $G_{n_1} \times \cdots \times G_{n_r}$ is a conjugate of the subgroup $\mathbb{K}_{n_1,[\mathrm{Gr}_1^{\mathrm{F}} M]} \times \cdots \times \mathbb{K}_{n_r,[\mathrm{Gr}_r^{\mathrm{F}} M]}$.

Lemma 5.3. *By choosing a pair (L_1, L_2) of $\mathfrak{o}$-lattices with $L_1 \supset L_2$ and an isomorphism $\gamma\colon L_1/L_2 \cong M$ of $\mathfrak{o}$-modules, we identify $\mathbb{K}_{n,[M]}$ with $\mathbb{K}_{L_1,L_2}$. We denote the composite $L_1 \twoheadrightarrow L_1/L_2 \xrightarrow{\gamma} M$ by f_1.*

(1) *Let $\mathcal{L}_M(F^n)$ be the set of pairs (L, f) of an $\mathfrak{o}$-lattice $L \subset F^n$ and a surjective homomorphism $f\colon L \twoheadrightarrow M$ of $\mathfrak{o}$-modules. Then there is a (canonical) bijection $G_n/\mathbb{K}_{n,[M]} \to \mathcal{L}_M(F^n)$ given by $g\mathbb{K}_{L_1,L_2} \mapsto (gL_1, y \mapsto f_1(g^{-1}y))$.*
(2) *There is a bijection from $G_n/P_{\mathbf{n}}$ to the set of $\mathbf{n}$-admissible filtrations on L_1 given by $hP_{\mathbf{n}} \mapsto \mathrm{F}^h_\bullet L_1 := L_1 \cap h(Fe_1 + \cdots + Fe_{b_\bullet})$, where $\{e_1, \ldots, e_n\}$ is the standard basis of F^n.*
(3) *Let $\mathcal{L}'_M(F^n)$ be the set of triples $(L, \mathrm{F}_\bullet L, f)$ of an $\mathfrak{o}$-lattice $L \subset F^n$, an $\mathbf{n}$-admissible filtration $\mathrm{F}_\bullet L$ on L and a surjective homomorphism $f\colon L \twoheadrightarrow M$ of $\mathfrak{o}$-modules. We let the group G_n act on $\mathcal{L}'_M(F^n)$ by $g.(L, \mathrm{F}_\bullet L, f) = (gL, g\mathrm{F}_\bullet L, y \mapsto f(g^{-1}y))$. Then there is a (canonical) bijection $P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{n,[M]} \to G_n\backslash\mathcal{L}'_M(F^n)$ given by sending $P_{\mathbf{n}}g\mathbb{K}_{L_1,L_2}$ to the G_n-orbit of $(gL_1, \mathrm{F}_\bullet gL_1, y \mapsto f_1(g^{-1}y))$, where $\mathrm{F}_i gL_1 := gL_1 \cap (Fe_1 + \cdots + Fe_{b_i})$.*

Proof. We show (1). We let the group G_n act from the left on the set $\mathcal{L}_M(F^n)$ by the rule $g.(L, f) = (gL, y \mapsto f(g^{-1}y))$. One can prove that the action of G_n on $\mathcal{L}_M(F^n)$ is transitive in the following way. Let (L, f) and (L', f') be two elements of $\mathcal{L}_M(F^n)$. Then by Lemma 4.9, there exists an isomorphism $\beta\colon L \xrightarrow{\cong} L'$ satisfying $f = f' \circ \beta$. By extending β to an automorphism of F^n by F-linearity, we obtain an element $g \in G_n$ such that $\beta(x) = gx$. Then we have $(L', f') = g.(L, f)$. Hence the map $g \mapsto g.(L_1, f_1)$ gives a surjective map $G_n \to \mathcal{L}_M(F^n)$. Since the stabiliser of (L_1, f_1) with respect to the action of G_n is equal to $\mathbb{K}_{L_1,L_2}$, it gives the desired bijection.

It is straightforward to check that this bijection does not depend on the choice of the triple (L_1, L_2, γ) in the following sense. Let (L'_1, L'_2, γ') be another choice. It follows from the proof of Lemma 5.1 that there exists $g \in G_n$ satisfying $gL_1 = L'_1$, $gL_2 = L'_2$ and $\gamma(y \mod L_2) = \gamma'(gy \mod L'_2)$ for all $y \in L_1$. Then for any such $g \in G_n$, we have $\mathbb{K}_{L'_1,L'_2} = g\mathbb{K}_{L_1,L_2}g^{-1}$, and the diagram

$$\begin{tikzcd}
G_n/\mathbb{K}_{L_1,L_2} \arrow[r] \arrow[d] & \mathcal{L}_M(F^n) \arrow[d, equal] \\
G_n/\mathbb{K}_{L'_1,L'_2} \arrow[r] & \mathcal{L}_M(F^n)
\end{tikzcd}$$

is commutative. Here the left vertical map sends $h\mathbb{K}_{L_1,L_2}$ to $hg^{-1}\mathbb{K}_{L'_1,L'_2}$. Hence we obtain (1).

Note that $G_n/P_{\mathbf{n}}$ is naturally identified with the set of partial flags $0 = V_0 \subset \cdots \subset V_r = F^n$ with $\dim(V_i/V_{i-1}) = n_i$ for $i = 1, \ldots, r$. Note that $(L_1 \cap V') \otimes_{\mathfrak{o}} F = V'$ for any subspace V' of F^n. On the other hand, if $\mathrm{F}_\bullet L_1$ is an $\mathbf{n}$-admissible filtration of L_1, since L_1 is a free $\mathfrak{o}$-module of rank $n = n_1 + \cdots + n_r$, each subquotient $\mathrm{Gr}_i^{\mathrm{F}} L_1$ is a free $\mathfrak{o}$-module of rank n_i for any i. Hence we have $L_1 \cap (\mathrm{F}_i L_1 \otimes_{\mathfrak{o}} F) = \mathrm{F}_i L_1$ for any i. Therefore we have (2).

Since the double cosets in $P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{n,[M]}$ are in one-to-one correspondence with the G_n-orbits in $(G_n/P_{\mathbf{n}}) \times (G_n/\mathbb{K}_{n,[M]})$ with respect to the diagonal left G_n-action, assertion (3) follows from (1) and (2). □

We continue the proof of Proposition 5.2. We identify $P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{n,[M]}$ with $G_n\backslash\mathcal{L}'_M(F^n)$ by Lemma 5.3. By sending the triple $(L, \mathrm{F}_\bullet L, f) \in \mathcal{L}'_M(F^n)$ to the filtration on M induced from $\mathrm{F}_\bullet L$ via f,

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we obtain a map $\alpha\colon P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{n,[M]} \to \mathcal{F}_M$. Let $\mathrm{F}^{\mathrm{st}}_{\bullet}\mathfrak{o}^n$ be the standard $\mathbf{n}$-admissible filtration on $\mathfrak{o}^n$: that is, the unique $\mathbf{n}$-admissible filtration on $\mathfrak{o}^n$ such that the standard basis of $\mathfrak{o}^n$ is a sequence compatible with $\mathrm{F}^{\mathrm{st}}_{\bullet}\mathfrak{o}^n$. Let us fix a surjective homomorphism $f\colon \mathfrak{o}^n \to M$, and let L denote its kernel. Then we can regard $\mathbb{K}_{n,[M]}$ as $\mathbb{K}_{\mathfrak{o}^n,L}$. In this case, one can describe the map α as follows. Let $s \in P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{\mathfrak{o}^n,L}$. Then by the Iwasawa decomposition, we have $s = P_{\mathbf{n}}k\mathbb{K}_{\mathfrak{o}^n,L}$ for some $k \in \mathrm{GL}_n(\mathfrak{o})$. Then $\alpha(s)$ is the filtration

$$0 = f(k^{-1}\mathrm{F}^{\mathrm{st}}_0\mathfrak{o}^n) \subset \cdots \subset f(k^{-1}\mathrm{F}^{\mathrm{st}}_r\mathfrak{o}^n) = M$$

on M. We note that $k^{-1}\mathrm{F}^{\mathrm{st}}_i\mathfrak{o}^n$ is the $\mathfrak{o}$-submodule of $\mathfrak{o}^n$ generated by the first b_i columns of k^{-1}.

Now let us choose a filtration $\mathrm{F}_{\bullet}M$ on M in $\mathcal{F}_M$. Let us fix a sequence $x_1,\ldots,x_n \in M$ compatible with $\mathrm{F}_{\bullet}M$. By considering the homomorphism $\mathfrak{o}^n \to M$ that sends the standard basis to the sequence $x_1,\ldots,x_n$, one can check that the map α is surjective. Suppose that two triples $(L,\mathrm{F}_{\bullet}L,f)$ and $(L',\mathrm{F}'_{\bullet}L',f')$ are sent to $\mathrm{F}_{\bullet}M$ via α. Let us choose a basis $y_1,\ldots,y_n$ of L and a basis $y'_1,\ldots,y'_n$ of L' as in assertion (2) of Lemma 4.12. By considering the change-of-basis matrix, we can see that the two triples are in the same G_n-orbit. This proves that the map α is injective. In conclusion, $\alpha\colon P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{n,[M]} \to \mathcal{F}_M$ is bijective.

Again, we realise $\mathbb{K}_{n,[M]}$ as $\mathbb{K}_{\mathfrak{o}^n,L}$ for a lattice $L \subset \mathfrak{o}^n$ with a surjection $f\colon \mathfrak{o}^n \twoheadrightarrow M$ such that $\operatorname{Ker} f = L$. Then by the Iwasawa decomposition, any $s \in P_{\mathbf{n}}\backslash G_n/\mathbb{K}_{\mathfrak{o}^n,L}$ is of the form $s = P_{\mathbf{n}}k_s\mathbb{K}_{\mathfrak{o}^n,L}$ for some $k_s \in \mathrm{GL}_n(\mathfrak{o})$. In this case, the corresponding triple is the G_n-orbit of $(\mathfrak{o}^n, \mathrm{F}^{\mathrm{st}}_{\bullet}\mathfrak{o}^n, f_s)$, where $f_s(y) = f(k_s^{-1}y)$. In particular, $\operatorname{Ker} f_s = k_sL$. Then

$$P_{\mathbf{n}} \cap k_s\mathbb{K}_{\mathfrak{o}^n,L}k_s^{-1} = \{p \in P_{\mathbf{n}} \cap \mathrm{GL}_n(\mathfrak{o}) \mid f_s \circ m(p) = f_s\},$$

where $m(p)\colon \mathfrak{o}^n \to \mathfrak{o}^n$ denotes the homomorphism given by the multiplication by p from the left. Recall that $\{e_1,\ldots,e_n\}$ is the standard basis of F^n. For $1 \le i \le r$, we set L_i to be the image of $k_sL \cap (\mathfrak{o}e_1 + \cdots + \mathfrak{o}e_{b_i})$ under the canonical projection

$$\mathfrak{o}^{b_i} = \mathfrak{o}e_1 + \cdots + \mathfrak{o}e_{b_i} \twoheadrightarrow \mathfrak{o}^{n_i} = \mathfrak{o}e_{a_i} + \cdots + \mathfrak{o}e_{b_i}.$$

Then $\mathfrak{o}^{n_i} \supset L_i$ are lattices in $F^{n_i} = Fe_{a_i} + \cdots + Fe_{b_i}$ such that $\mathfrak{o}^{n_i}/L_i \cong \mathrm{Gr}^{\mathrm{F}}_i M$, where $\mathrm{F}_{\bullet}M$ is the filtration corresponding to the G_n-orbit of $(\mathfrak{o}^n, \mathrm{F}^{\mathrm{st}}_{\bullet}\mathfrak{o}^n, f_s)$. Moreover, we have

$$q(P_{\mathbf{n}} \cap k_s\mathbb{K}_{\mathfrak{o}^n,L}k_s^{-1}) \subset \mathbb{K}_{\mathfrak{o}^{n_1},L_1} \times \cdots \times \mathbb{K}_{\mathfrak{o}^{n_r},L_r}.$$

We show that this inclusion is indeed an equality. Let $(k_1,\ldots,k_r) \in \mathbb{K}_{\mathfrak{o}^{n_1},L_1} \times \cdots \times \mathbb{K}_{\mathfrak{o}^{n_r},L_r}$ be given. Set $x_j = f_s(e_j) \in M$ for $1 \le j \le n$, and set $z_j = k_ie_j \in \mathfrak{o}^{n_i} = \mathrm{Gr}^{\mathrm{F}^{\mathrm{st}}}_i\mathfrak{o}^n$ for $a_i \le j \le b_i$. Since k_i fixes $\mathfrak{o}^{n_i} \ni x \mapsto f_s(x) \bmod \mathrm{F}_{i-1}M \in \mathrm{Gr}^{\mathrm{F}}_iM \cong \mathfrak{o}^{n_i}/L_i$, we see that the image of $f_s(z_j)$ in $\mathrm{Gr}^{\mathrm{F}}_iM$ is the same as the one of x_j. By assertion (3) of Lemma 4.12, one can take a sequence $e'_1,\ldots,e'_n \in \mathfrak{o}^n$ which is compatible with $\mathrm{F}^{\mathrm{st}}_{\bullet}\mathfrak{o}^n$ such that $x_j = f_s(e'_j)$ and the class of e'_j in $\mathrm{Gr}^{\mathrm{F}^{\mathrm{st}}}_i\mathfrak{o}^n = \mathfrak{o}^{n_i}$ is equal to z_j. Define $k \in G_n$ so that $e'_j = ke_j$ for $1 \le j \le n$. Since $Fe'_1 + \cdots + Fe'_{n_i} = Fe_1 + \cdots + Fe_{n_i}$ for $1 \le i \le r$, we have $k \in P_{\mathbf{n}}$. Moreover, since k preserves $\mathfrak{o}^n$ and $f_s(kx) = f_s(x)$, it also preserves $k_sL = \operatorname{Ker} f_s$. Hence $k \in \mathbb{K}_{\mathfrak{o}^n,k_sL} = k_s\mathbb{K}_{\mathfrak{o}^n,L}k_s^{-1}$. Since $q(k) = (k_1,\ldots,k_r)$, we conclude that $q(P_{\mathbf{n}} \cap k_s\mathbb{K}_{\mathfrak{o}^n,L}k_s^{-1}) = \mathbb{K}_{\mathfrak{o}^{n_1},L_1} \times \cdots \times \mathbb{K}_{\mathfrak{o}^{n_r},L_r}$. Namely, $q(P_{\mathbf{n}} \cap g\mathbb{K}_{n,[M]}g^{-1})$ is a $G_{n_1} \times \cdots \times G_{n_r}$-conjugate of $\mathbb{K}_{n_1,[\mathrm{Gr}^{\mathrm{F}}_1M]} \times \cdots \times \mathbb{K}_{n_r,[\mathrm{Gr}^{\mathrm{F}}_rM]}$. This completes the proof of Proposition 5.2. □

Remark 5.4. One can interpret the statement and the proof of Proposition 5.2 in terms of the topos theory. For more precise statements, see the previous paper of the second and third authors [18].

5.2. *Proof of the main theorems: a reduction step*

Let π be an irreducible representation of G_n. Then we can write $\pi = \pi' \times \pi_1 \times \cdots \times \pi_r$ as an irreducible parabolic induction such that

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- π' is an irreducible representation such that $L(s, \pi') = 1$;
- $\pi_i = Z(\mathfrak{m}_i)$ with $\mathfrak{m}_i$ of type χ_i for some unramified character χ_i of $F^\times$;
- if $i \neq j$, then $\chi_i\chi_j^{-1}$ is not of the form $|\cdot|^a$ for any $a \in \mathbb{Z}$.

If we knew Theorem 2.1 (respectively, Theorem 2.2) for π' and π_i for $1 \le i \le r$, by Proposition 5.2 and Corollary 4.7, we would obtain the same theorem for π. In other words, Theorems 2.1 and 2.2 are reduced to the following two cases:

- The case where $\pi = Z(\mathfrak{m})$ with $\mathfrak{m}$ of type χ for some unramified character χ of $F^\times$;
- The case where $L(s, \pi) = 1$.

We will deal with the first case in Section 6, and the second case will be treated in Sections 7 and 9.

6. Proof of the main theorems: the unipotent case

In this section, we prove Theorems 2.1 and 2.2 for $\pi = Z(\mathfrak{m})$ with $\mathfrak{m}$ of type χ for some unramified character χ of $F^\times$.

6.1. Proof of Theorem 2.1 for ladder representations of type χ

In this section, we prove Theorem 2.1 in the case where $\pi = Z([x_1, y_1]_\chi, \dots, [x_t, y_t]_\chi) \in \mathrm{Irr}(G_n)$ is of type χ with an unramified character χ of $F^\times$ such that π is a ladder representation: that is, $x_1 > \cdots > x_t$ and $y_1 > \cdots > y_t$. Recall from Example 2.5 (2) that

$$\lambda_\pi = \sum_{i=2}^{t} (0, \dots, 0, \underbrace{1, \dots, 1}_{\max\{y_i - x_{i-1} + 2, 0\}}) \in \Lambda_n.$$

For $[M] \in |\mathcal{C}^n|$, and for a partition $\mathbf{n} = (n_1, \dots, n_t)$ of n with $n_i \in \mathbb{Z}$, we set $\mathcal{N}_{\mathbf{n}}(M)$ to be the number of $\mathbf{n}$-admissible filtrations of M. Here, when $n_i < 0$ for some i, we understand that $\mathcal{N}_{\mathbf{n}}(M) = 0$.

Proposition 6.1. *We have*

$$\dim(\pi^{\mathbb{K}_{n,[M]}}) = \sum_{w \in S_t} \mathrm{sgn}(w) \mathcal{N}_{\mathbf{n}_w}(M),$$

where $\mathbf{n}_w = (y_1 - x_{w(1)} + 1, \dots, y_t - x_{w(t)} + 1)$.

Proof. By the determinantal formula [21], in the Grothendieck group of the category of representations of G_n of finite length, we have

$$\pi = \sum_{w \in S_t} \mathrm{sgn}(w) Z([x_{w(1)}, y_1]_\chi) \times \cdots \times Z([x_{w(t)}, y_t]_\chi).$$

Here, when $x = y + 1$ (respectively, $x > y + 1$), we formally set $Z([x, y]_\chi) = \mathbf{1}_{G_0}$ (respectively, $Z([x, y]_\chi) = 0$). Note that in [21], the determinantal formula was formulated using the Langlands classification, but by taking the Zelevinsky dual, it translates to the statement above.

Recall that for a compact open subgroup $\mathbb{K}$ of G_n, the functor $\pi \mapsto \pi^{\mathbb{K}}$ is exact. Hence, by Proposition 5.2, we have

$$\begin{aligned} \pi^{\mathbb{K}_{n,[M]}} &= \sum_{w \in S_t} \mathrm{sgn}(w) \left(\prod_{i=1}^{t} Z([x_{w(i)}, y_i]_\chi) \right)^{\mathbb{K}_{n,[M]}} \\ &= \sum_{w \in S_t} \mathrm{sgn}(w) \sum_{\mathrm{F}^w_\bullet M} \bigotimes_{i=1}^{t} Z([x_{w(i)}, y_i]_\chi)^{\mathbb{K}_{n_i,[\mathrm{Gr}_i^{\mathrm{F}^w} M]}}, \end{aligned}$$

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where $\mathrm{F}^w_\bullet M$ runs over the set of $\mathbf{n}_w$-admissible filtrations with $\mathbf{n}_w = (y_1 - x_{w(1)} + 1, \dots, y_t - x_{w(t)} + 1)$. Here, if $y_i - x_{w(i)} + 1 < 0$ for some i, we understand that there is no $\mathbf{n}_w$-admissible filtration. Since $Z([x_{w(i)}, y_i]_\chi)$ is a character which is trivial on $\mathrm{GL}_{y_i - x_{w(i)}+1}(\mathfrak{o})$, the dimension of $Z([x_{w(i)}, y_i]_\chi)^{\mathbb{K}_{n_i, [\mathrm{Gr}^{\mathrm{F}^w}_i M]}}$ is always one if $y_i - x_{w(i)} + 1 \geq 0$. Hence we obtain the assertion. □

Set $b = \max_{2 \leq i \leq t} \max\{y_i - x_{i-1} + 2, 0\}$. If $b = 0$, then π is unramified so that Theorem 2.1 is trivial for π. Hence we may assume that $b > 0$. Let $[M_\pi] \in |\mathcal{C}^n|$ be such that $\mathrm{seq}_n([M_\pi]) = \lambda_\pi$. Then $M_\pi \cong \oplus_{i=1}^b \mathfrak{o}/\mathfrak{p}^{a_i}$ for some $a_i \geq 1$.

Lemma 6.2. *If $[M] \leq [M_\pi]$, for any filtration $\mathrm{F}_\bullet M$, the $\mathfrak{o}$-module $[\mathrm{Gr}^{\mathrm{F}}_i M]$ can be generated by at most b elements.*

Proof. This follows from Lemma 4.3. □

Now we calculate the alternating sum on the right-hand side of Proposition 6.1. We will see that there are many nontrivial cancellations. See Section 6.2 below for an explicit example of this calculation.

Choose $2 \leq a \leq t$ such that $y_a - x_{a-1} + 2 = b$. The following lemma is a key in computing the alternating sum in the right-hand side of Proposition 6.1.

Lemma 6.3. *Suppose that $[M] \leq [M_\pi]$.*

(1) *Let X_1 be the subset of S_t consisting of w such that $w(k) \geq a - 1$ for any $k \geq a$. For $w \in S_t \setminus X_1$, take $1 \leq i, j \leq a - 1$ such that $w(i)$ achieves the largest value and $w(j)$ achieves the second-largest value among $\{w(1), \dots, w(a-1)\}$, and set $w' = w(i,j)$. Then $w(i), w(j) \geq a - 1$, and the map $w \mapsto w'$ is an involution on $S_t \setminus X_1$. Moreover, $\mathcal{N}_{\mathbf{n}_w}(M) = \mathcal{N}_{\mathbf{n}_{w'}}(M)$. In particular,*
$$\sum_{w \in S_t \setminus X_1} \mathrm{sgn}(w) \mathcal{N}_{\mathbf{n}_w}(M) = 0.$$

(2) *Let X_2 be the subset of X_1 consisting of w such that $w(k) \geq a$ for any $k > a$. For $w \in X_1 \setminus X_2$, take a unique $1 \leq i \leq a - 1$ such that $w(i) \geq a$, and set $w' = w(i, a)$. Then the map $w \mapsto w'$ is an involution on $X_1 \setminus X_2$. Moreover, $\mathcal{N}_{\mathbf{n}_w}(M) = \mathcal{N}_{\mathbf{n}_{w'}}(M)$. In particular,*
$$\sum_{w \in X_1 \setminus X_2} \mathrm{sgn}(w) \mathcal{N}_{\mathbf{n}_w}(M) = 0.$$

(3) *Let $S_{(a-1, t-a+1)}$ be the subgroup of S_t consisting of w such that $w(k) \geq a$ for any $k \geq a$, and set $X_3 = \{(a-1, w(a))w \mid w \in S_{(a-1,t-a+1)}\}$. Then*
$$X_2 = S_{(a-1,t-a+1)} \sqcup X_3.$$

(4) *Let X_4 be the subset of S_t consisting of w such that $w(a) = a - 1$ and $w(k) < a - 1$ for some $k > a$. Then $X_4 \subset S_t \setminus X_1$, and the involution in (1) preserves X_4. Moreover, the disjoint union $X_3 \sqcup X_4$ is equal to the subset of S_t consisting of w such that $w(a) = a - 1$.*

(5) *For $w \in X_4$, take $1 \leq i \leq a-1$ such that $w(i)$ achieves the largest value among $\{w(1), \dots, w(a-1)\}$, in particular $w(i) \geq a$. Set $\widetilde{w} = w(a, i)$ and $X_5 = \{\widetilde{w} \mid w \in X_4\}$. Then $X_5 \subset S_t \setminus X_1$, and the involution in (1) preserves X_5.*

Proof. We prove (1). Let $w \in S_t \setminus X_1$ and $1 \leq i, j \leq a - 1$ be as in the statement. Note that i and j depend on w, but the map $w \mapsto w'$ gives a well-defined involution on $S_t \setminus X_1$. Since there exists $k \geq a$ such that $w(k) < a - 1$, we notice that $w(i), w(j) \geq a - 1$. Hence
$$\min\{y_i - x_{w(i)} + 1, y_j - x_{w(j)} + 1, y_i - x_{w(j)} + 1, y_j - x_{w(i)} + 1\} \geq y_a - x_{a-1} + 2 = b.$$

By Lemma 6.2, we see that $\mathcal{N}_{\mathbf{n}_w}(M) = \mathcal{N}_{\mathbf{n}_{w'}}(M)$. Since $\mathrm{sgn}(w') = -\mathrm{sgn}(w)$, the last part follows. Hence we obtain (1).

We prove (2). When $w \in X_1 \setminus X_2$, there exists $k > a$ such that $w(k) = a - 1$. In particular, $w(a) \geq a$. Hence the map $w \mapsto w'$ gives a well-defined involution on $X_1 \setminus X_2$. By the same argument as in (1), we obtain (2).

Assertions (3) are (4) are obvious from the definitions.

We prove (5). Let $w \in X_4$. Then $\widetilde{w}(k) = w(k) < a - 1$ for some $k > a$ so that $\widetilde{w} \notin X_1$. Take $1 \leq i \leq a - 1$ as in the statement so that $\widetilde{w} = w(a, i)$. Note that $\widetilde{w}(a) = w(i) \geq a$. Let $1 \leq j_1, j_2 \leq a - 1$ be such that $\widetilde{w}(j_1)$ (respectively, $\widetilde{w}(j_2)$) achieves the largest (respectively, the second-largest) value among $\{\widetilde{w}(1), \dots, \widetilde{w}(a-1)\}$. Note that $a \leq \widetilde{w}(j_1) < w(i)$ and $\widetilde{w}(j_2) \geq a - 1$. If $\widetilde{w}(j_2) \geq a$, then j_1, j_2, i, a are all distinct from each other. In this case,

$$(\widetilde{w})' = \widetilde{w}(j_1, j_2) = w(a, i)(j_1, j_2) = w(j_1, j_2)(a, i) = \widetilde{w(j_1, j_2)}.$$

Hence we have $(\widetilde{w})' \in X_5$. If $\widetilde{w}(j_2) = a - 1$, then $j_2 = i$. In this case,

$$(\widetilde{w})' = \widetilde{w}(j_1, i) = w(a, i)(j_1, i) = w(j_1, i)(a, j_1) = w'(a, j_1) = \widetilde{w'}.$$

Hence we again have $(\widetilde{w})' \in X_5$. $\square$

Now we prove Theorem 2.1 for a ladder representation $\pi = Z([x_1, y_1]_\chi, \dots, [x_t, y_t]_\chi)$ of type χ with unramified character χ.

Proof of Theorem 2.1 for ladder representations of type χ. When $b = 0$, since π is unramified, the assertion is trivial. From now on, we assume that $b > 0$. In particular, one has $t \geq 2$.

Set

$$\pi' = Z([x_1, y_1]_\chi, \dots, [x_{a-2}, y_{a-2}]_\chi, [x_a, y_{a-1}]_\chi, [x_{a+1}, y_{a+1}]_\chi, \dots, [x_t, y_t]_\chi).$$

This is a ladder representation of some $G_{n'}$. We claim that

$$\dim(\pi^{\mathbb{K}_{n,[M]}}) = \sum_{M' \subset M} \dim(\pi'^{\mathbb{K}_{n',[M/M']}}),$$

where M' runs over the set of $\mathfrak{o}$-submodules of M generated by exactly b elements.

Suppose for a moment that this claim is true. Note that

$$\lambda_\pi = \lambda_{\pi'} + (0, \dots, 0, \underbrace{1, \dots, 1}_{b}).$$

By induction on t, we may assume that we have $\dim(\pi'^{\mathbb{K}_{n',[M/M']}}) = 0$ if $[M/M'] < [M_{\pi'}]$. In particular, $\dim(\pi^{\mathbb{K}_{n,[M]}}) = 0$ if $[M] < [M_\pi]$. Moreover, when $[M] = [M_\pi]$, by Corollary 4.7, there exists a unique $\mathfrak{o}$-submodule M' of M generated by exactly b elements such that $[M/M'] = [M_{\pi'}]$. Hence we have

$$\sum_{M' \subset M} \dim(\pi'^{\mathbb{K}_{n',[M/M']}}) = 1.$$

Therefore, the claim implies that

$$\dim(\pi^{\mathbb{K}_{n,[M]}}) = \begin{cases} 1 & \text{if } [M] = [M_\pi], \\ 0 & \text{if } [M] < [M_\pi]. \end{cases}$$

For the rest of the proof, we show the claim.

Let $X_1, X_2, X_3, X_4, X_5 \subset S_t$ be as in Lemma 6.3. We denote the inverse map of $S_{(a-1,t-a+1)} \ni w \mapsto (a-1, w(a))w \in X_3$ by $X_3 \ni w \mapsto \widetilde{w} \in S_{(a-1,t-a+1)}$. Then by Lemma 6.3 (1)–(3), we have

$$\begin{aligned}\dim(\pi^{\mathbb{K}_{n},[M]}) &= \sum_{w \in X_2} \operatorname{sgn}(w)\mathcal{N}_{\mathbf{n}_w}(M) \\ &= \sum_{w \in X_3} \operatorname{sgn}(\widetilde{w}) \left(\mathcal{N}_{\mathbf{n}_{\widetilde{w}}}(M) - \mathcal{N}_{\mathbf{n}_w}(M)\right).\end{aligned}$$

For $w \in X_3$, there exists $1 \leq i_0 \leq a-1$ uniquely such that $w(i_0) = \widetilde{w}(a) \geq a$. Since $w(a) = \widetilde{w}(i_0) = a-1$, we have

- $\min\{y_{i_0} - x_{w(i_0)} + 1, y_{i_0} - x_{\widetilde{w}(i_0)} + 1\} \geq y_a - x_{a-1} + 2 = b$;
- $y_a - x_{\widetilde{w}(a)} + 1 \geq b$, whereas $y_a - x_{w(a)} + 1 = b - 1$.

By Lemma 6.2, $\mathcal{N}_{\mathbf{n}_{\widetilde{w}}}(M) - \mathcal{N}_{\mathbf{n}_w}(M)$ is equal to the number of filtrations

$$0 = \mathrm{F}_0 M \subset \cdots \subset \mathrm{F}_t M = M$$

of M by $\mathfrak{o}$-submodules such that

- $\mathrm{Gr}_i^{\mathrm{F}} M$ is generated by at most $y_i - x_{w(i)} + 1$ elements for $i \neq a$;
- $\mathrm{Gr}_a^{\mathrm{F}} M$ is generated by exactly b elements.

By Lemma 4.13, this number is equal to the number of pairs $(M', \mathrm{F}'_\bullet(M/M'))$, where $M' \subset M$ is an $\mathfrak{o}$-submodule generated by exactly b elements and $\mathrm{F}'_\bullet(M/M')$ is a filtration

$$0 = \mathrm{F}'_0(M/M') \subset \cdots \subset \mathrm{F}'_{a-1}(M/M') \subset \mathrm{F}'_{a+1}(M/M') \subset \cdots \subset \mathrm{F}'_t(M/M') = M/M'$$

of M/M' by $\mathfrak{o}$-submodules such that $\mathrm{Gr}_i^{\mathrm{F}'}(M/M')$ is generated by at most $y_i - x_{w(i)} + 1$ elements for $i \neq a$. Here, we set $\mathrm{Gr}_i^{\mathrm{F}'}(M/M') = \mathrm{F}'_i(M/M')/\mathrm{F}'_{i-1}(M/M')$ unless $i = a, a+1$, and $\mathrm{Gr}_{a+1}^{\mathrm{F}'}(M/M') = \mathrm{F}'_{a+1}(M/M')/\mathrm{F}'_{a-1}(M/M')$. Therefore,

$$\sum_{w \in X_2} \operatorname{sgn}(w)\mathcal{N}_{\mathbf{n}_w}(M) = \sum_{w \in X_3} \sum_{M' \subset M} \operatorname{sgn}(\widetilde{w})\mathcal{N}_{\mathbf{n}'_w}(M/M'),$$

where M' runs over the set of $\mathfrak{o}$-submodules of M generated by exactly b elements, and we set $\mathbf{n}'_w = (n_{w,1}, \ldots, n_{w,a-1}, n_{w,a+1}, \ldots, n_{w,t})$ with $n_{w,i} = y_i - x_{w(i)} + 1$ for $i \neq a$.

Note that $X_4 \cap X_5 = \emptyset$. By the same argument as above, we have

$$\begin{aligned}\sum_{w \in X_4 \sqcup X_5} \operatorname{sgn}(w)\mathcal{N}_{\mathbf{n}_w}(M) &= \sum_{w \in X_4} \operatorname{sgn}(\widetilde{w}) \left(\mathcal{N}_{\mathbf{n}_{\widetilde{w}}}(M) - \mathcal{N}_{\mathbf{n}_w}(M)\right) \\ &= \sum_{w \in X_4} \sum_{M' \subset M} \operatorname{sgn}(\widetilde{w})\mathcal{N}_{\mathbf{n}'_w}(M/M'),\end{aligned}$$

where M' runs over the set of $\mathfrak{o}$-submodules of M generated by exactly b elements, and $\mathbf{n}'_w$ is as above. However, by Lemma 6.3 (1), (4), (5), we see that the left-hand side is zero. Therefore,

$$\dim(\pi^{\mathbb{K}_{n},[M]}) = \sum_{M' \subset M} \sum_{w \in X_3 \sqcup X_4} \operatorname{sgn}(\widetilde{w})\mathcal{N}_{\mathbf{n}'_w}(M/M').$$

Next, we consider the alternating sum

$$\dim(\pi'^{\mathbb{K}_{n'},[M/M']}) = \sum_{w' \in S_{t-1}} \operatorname{sgn}(w')\mathcal{N}_{\mathbf{n}'_{w'}}(M/M').$$

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Here, we regard S_{t-1} as the set of bijective maps

$$w' \colon \{1, \ldots, a-1, a+1, \ldots, t\} \to \{1, \ldots, a-2, a, \ldots, t\}$$

by identifying $a-1$ and a. For $w \in X_3 \sqcup X_4$, define w' to be the restriction of w to $\{1, \ldots, a-1, a+1, \ldots, t\}$. Then we have a bijective map $X_3 \sqcup X_4 \to S_{t-1}$ since $X_3 \sqcup X_4$ is the subset of S_t consisting of w such that $w(a) = a-1$. Note that for $w \in X_3 \sqcup X_4$, the sign $\mathrm{sgn}(w')$ of w' as an element of S_{t-1} is equal to $\mathrm{sgn}(\widetilde{w})$.

Therefore,

$$\begin{aligned}
\dim(\pi^{\mathbb{K}_{n,[M]}}) &= \sum_{M' \subset M} \sum_{w \in X_3 \sqcup X_4} \mathrm{sgn}(\widetilde{w}) \mathcal{N}_{\mathbf{n}'_w}(M/M') \\
&= \sum_{M' \subset M} \sum_{w' \in S_{t-1}} \mathrm{sgn}(w') \mathcal{N}_{\mathbf{n}'_{w'}}(M/M') \\
&= \sum_{M' \subset M} \dim(\pi'^{\mathbb{K}_{n',[M/M']}}).
\end{aligned}$$

Hence we obtain the claim. This completes the proof of Theorem 2.1 for ladder representations of type χ. □

6.2. *Example of calculation of the alternating sum*

To understand the proof of Theorem 2.1 for ladder representations of type χ, the following explicit example may be helpful.

Example 6.4. For simplicity, we drop χ from the notation. Let us consider a ladder representation

$$\pi = Z([5,7],[3,6],[2,5],[0,3]) \in \mathrm{Irr}(G_{15}).$$

Then $\lambda_\pi = (0, \ldots, 0, 1, 3, 3, 3) \in \Lambda_{15}$ so that $M_\pi = \mathfrak{o}/\mathfrak{p} \oplus (\mathfrak{o}/\mathfrak{p}^3)^{\oplus 3}$. By the determinantal formula, we have

$$\begin{aligned}
\pi = {} & Z([5,7]) \times Z([3,6]) \times Z([2,5]) \times Z([0,3]) - Z([3,7]) \times Z([5,6]) \times Z([2,5]) \times Z([0,3]) \\
& - Z([5,7]) \times Z([3,6]) \times Z([0,5]) \times Z([2,3]) + Z([3,7]) \times Z([5,6]) \times Z([0,5]) \times Z([2,3]) \\
& - Z([5,7]) \times Z([2,6]) \times Z([3,5]) \times Z([0,3]) + Z([2,7]) \times Z([5,6]) \times Z([3,5]) \times Z([0,3]) \\
& + Z([5,7]) \times Z([0,6]) \times Z([3,5]) \times Z([2,3]) - Z([0,7]) \times Z([5,6]) \times Z([3,5]) \times Z([2,3]) \\
& - Z([5,7]) \times Z([0,6]) \times Z([2,5]) \times Z([3,3]) + Z([5,7]) \times Z([2,6]) \times Z([0,5]) \times Z([3,3]) \\
& + Z([0,7]) \times Z([5,6]) \times Z([2,5]) \times Z([3,3]) - Z([2,7]) \times Z([5,6]) \times Z([0,5]) \times Z([3,3]) \\
& - Z([2,7]) \times Z([3,6]) \times Z([5,5]) \times Z([0,3]) + Z([3,7]) \times Z([2,6]) \times Z([5,5]) \times Z([0,3]) \\
& + Z([0,7]) \times Z([3,6]) \times Z([5,5]) \times Z([2,3]) - Z([3,7]) \times Z([0,6]) \times Z([5,5]) \times Z([2,3]) \\
& + Z([2,7]) \times Z([0,6]) \times Z([5,5]) \times Z([3,3]) - Z([0,7]) \times Z([2,6]) \times Z([5,5]) \times Z([3,3]) \\
& + Z([2,7]) \times Z([3,6]) \times Z([0,5]) \times Z([5,3]) - Z([3,7]) \times Z([2,6]) \times Z([0,5]) \times Z([5,3]) \\
& - Z([0,7]) \times Z([3,6]) \times Z([2,5]) \times Z([5,3]) + Z([3,7]) \times Z([0,6]) \times Z([2,5]) \times Z([5,3]) \\
& + Z([0,7]) \times Z([2,6]) \times Z([3,5]) \times Z([5,3]) - Z([2,7]) \times Z([0,6]) \times Z([3,5]) \times Z([5,3]).
\end{aligned}$$

By Proposition 5.2, we have

$$\begin{aligned}
& \dim(\pi^{\mathbb{K}_{15,\lambda_\pi}}) \\
& = \mathcal{N}_{(3,4,4,4)}(M_\pi) - \mathcal{N}_{(5,2,4,4)}(M_\pi) - \mathcal{N}_{(3,4,6,2)}(M_\pi) + \mathcal{N}_{(5,2,6,2)}(M_\pi) \\
& - \mathcal{N}_{(3,5,3,4)}(M_\pi) + \mathcal{N}_{(6,2,3,4)}(M_\pi) + \mathcal{N}_{(3,7,3,2)}(M_\pi) - \mathcal{N}_{(8,2,3,2)}(M_\pi)
\end{aligned}$$

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$$
\begin{aligned}
&- \mathcal{N}_{(3,5,4,1)}(M_\pi) + \mathcal{N}_{(3,5,6,1)}(M_\pi) + \mathcal{N}_{(8,2,4,1)}(M_\pi) - \mathcal{N}_{(6,2,6,1)}(M_\pi)\\
&- \mathcal{N}_{(6,4,1,4)}(M_\pi) + \mathcal{N}_{(5,5,1,4)}(M_\pi) + \mathcal{N}_{(8,4,1,2)}(M_\pi) - \mathcal{N}_{(5,7,1,2)}(M_\pi)\\
&+ \mathcal{N}_{(6,7,1,1)}(M_\pi) - \mathcal{N}_{(8,5,1,1)}(M_\pi) + \mathcal{N}_{(6,4,6,-1)}(M_\pi) - \mathcal{N}_{(5,5,6,-1)}(M_\pi)\\
&- \mathcal{N}_{(8,4,4,-1)}(M_\pi) + \mathcal{N}_{(5,7,4,-1)}(M_\pi) + \mathcal{N}_{(8,5,3,-1)}(M_\pi) - \mathcal{N}_{(6,7,3,-1)}(M_\pi).
\end{aligned}
$$

By Lemma 6.2, we have

$$
\begin{aligned}
\dim(\pi^{\mathbb{K}_{15,\lambda_\pi}}) &= \mathcal{N}_{(3,4,4,4)}(M_\pi) - \mathcal{N}_{(4,2,4,4)}(M_\pi) - \mathcal{N}_{(3,4,4,2)}(M_\pi) + \mathcal{N}_{(4,2,4,2)}(M_\pi)\\
&\quad - \mathcal{N}_{(3,4,3,4)}(M_\pi) + \mathcal{N}_{(4,2,3,4)}(M_\pi) + \mathcal{N}_{(3,4,3,2)}(M_\pi) - \mathcal{N}_{(4,2,3,2)}(M_\pi)\\
&\quad - \mathcal{N}_{(3,4,4,1)}(M_\pi) + \mathcal{N}_{(3,4,4,1)}(M_\pi) + \mathcal{N}_{(4,2,4,1)}(M_\pi) - \mathcal{N}_{(4,2,4,1)}(M_\pi)\\
&\quad - \mathcal{N}_{(4,4,1,4)}(M_\pi) + \mathcal{N}_{(4,4,1,4)}(M_\pi) + \mathcal{N}_{(4,4,1,2)}(M_\pi) - \mathcal{N}_{(4,4,1,2)}(M_\pi)\\
&\quad + \mathcal{N}_{(4,4,1,1)}(M_\pi) - \mathcal{N}_{(4,4,1,1)}(M_\pi)\\
&= \mathcal{N}_{(3,4,4,4)}(M_\pi) - \mathcal{N}_{(4,2,4,4)}(M_\pi) - \mathcal{N}_{(3,4,4,2)}(M_\pi) + \mathcal{N}_{(4,2,4,2)}(M_\pi)\\
&\quad - \mathcal{N}_{(3,4,3,4)}(M_\pi) + \mathcal{N}_{(4,2,3,4)}(M_\pi) + \mathcal{N}_{(3,4,3,2)}(M_\pi) - \mathcal{N}_{(4,2,3,2)}(M_\pi).
\end{aligned}
$$

Note that if a filtration $\mathrm{F}_\bullet M_\pi$ satisfies that $\mathrm{Gr}^{\mathrm{F}}_3 M_\pi$ is generated by exactly 4 elements, then $\mathrm{Gr}^{\mathrm{F}}_i M_\pi$ for $i = 1,2,4$ can be generated by at most 3 elements by Lemmas 4.13, 4.3 and 6.2. Hence

$$
\begin{aligned}
\mathcal{N}_{(3,4,4,4)}(M_\pi) - \mathcal{N}_{(3,4,3,4)}(M_\pi) &= \mathcal{N}_{(3,3,4,3)}(M_\pi) - \mathcal{N}_{(3,3,3,3)}(M_\pi),\\
\mathcal{N}_{(4,2,4,4)}(M_\pi) - \mathcal{N}_{(4,2,3,4)}(M_\pi) &= \mathcal{N}_{(3,2,4,3)}(M_\pi) - \mathcal{N}_{(3,2,3,3)}(M_\pi),\\
\mathcal{N}_{(3,4,4,2)}(M_\pi) - \mathcal{N}_{(3,4,3,2)}(M_\pi) &= \mathcal{N}_{(3,3,4,2)}(M_\pi) - \mathcal{N}_{(3,3,3,2)}(M_\pi),\\
\mathcal{N}_{(4,2,4,2)}(M_\pi) - \mathcal{N}_{(4,2,3,2)}(M_\pi) &= \mathcal{N}_{(3,2,4,2)}(M_\pi) - \mathcal{N}_{(3,2,3,2)}(M_\pi).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dim(\pi^{\mathbb{K}_{15,\lambda_\pi}}) &= [(\mathcal{N}_{(3,3,4,3)}(M_\pi) - \mathcal{N}_{(3,3,3,3)}(M_\pi)) - (\mathcal{N}_{(3,2,4,3)}(M_\pi) - \mathcal{N}_{(3,2,3,3)}(M_\pi))]\\
&\quad - [(\mathcal{N}_{(3,3,4,2)}(M_\pi) - \mathcal{N}_{(3,3,3,2)}(M_\pi)) - (\mathcal{N}_{(3,2,4,2)}(M_\pi) - \mathcal{N}_{(3,2,3,2)}(M_\pi))].
\end{aligned}
$$

The right-hand side is equal to the number of filtrations

$$
0 = \mathrm{F}_0 M_\pi \subset \mathrm{F}_1 M_\pi \subset \mathrm{F}_2 M_\pi \subset \mathrm{F}_3 M_\pi \subset \mathrm{F}_4 M_\pi = M_\pi
$$

such that

- $\mathrm{Gr}^{\mathrm{F}}_2 M_\pi$ is generated by exactly 3 elements;
- $\mathrm{Gr}^{\mathrm{F}}_3 M_\pi$ is generated by exactly 4 elements;
- $\mathrm{Gr}^{\mathrm{F}}_4 M_\pi$ is generated by exactly 3 elements.

Since $M_\pi = \mathfrak{o}/\mathfrak{p} \oplus (\mathfrak{o}/\mathfrak{p}^3)^{\oplus 3}$, such a filtration exists uniquely and is given by

$$
\mathrm{F}_1 M_\pi = 0, \quad \mathrm{F}_2 M_\pi = (\mathfrak{p}^2/\mathfrak{p}^3)^{\oplus 3}, \quad \mathrm{F}_3 M_\pi = \mathfrak{o}/\mathfrak{p} \oplus (\mathfrak{p}^1/\mathfrak{p}^3)^{\oplus 3}, \quad \mathrm{F}_4 M_\pi = M_\pi.
$$

Therefore, we conclude that $\dim(\pi^{\mathbb{K}_{15,\lambda_\pi}}) = 1$, as desired.

6.3. Proof of Theorem 2.1 for general $Z(\mathfrak{m})$ of type χ

Now we consider $\pi = Z(\mathfrak{m})$ with $\mathfrak{m}$ of type χ for some unramified character χ of $F^\times$.

Lemma 6.5. *Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be multisegments. Then $Z(\mathfrak{m}_1 + \mathfrak{m}_2)$ appears as a subquotient of $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ with multiplicity one.*

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Proof. See [38, Proposition 2.3] (or [22, Proposition 3.5 (5)]). □

Recall that when $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$, we set $\mathrm{Card}(\mathfrak{m}) = r$.

Lemma 6.6. *Let $\mathfrak{m}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be multisegments. Suppose that $Z(\mathfrak{m})$ appears as a subquotient of $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$. Then $Z(\mathfrak{m}^-)$ appears as a subquotient of $Z(\mathfrak{m}_1^-) \times Z(\mathfrak{m}_2^-)$ if and only if* $\mathrm{Card}(\mathfrak{m}) = \mathrm{Card}(\mathfrak{m}_1) + \mathrm{Card}(\mathfrak{m}_2)$.

Proof. Suppose that $Z(\mathfrak{m}^-)$ appears as a subquotient of $Z(\mathfrak{m}_1^-) \times Z(\mathfrak{m}_2^-)$. By considering cuspidal supports, we have $l(\mathfrak{m}^-) = l(\mathfrak{m}_1^-) + l(\mathfrak{m}_2^-)$. For a similar reason, we have $l(\mathfrak{m}) = l(\mathfrak{m}_1) + l(\mathfrak{m}_2)$. Since $l(\mathfrak{m}) = l(\mathfrak{m}^-) + \mathrm{Card}(\mathfrak{m})$, we have the desired equality $\mathrm{Card}(\mathfrak{m}) = \mathrm{Card}(\mathfrak{m}_1) + \mathrm{Card}(\mathfrak{m}_2)$.

Conversely, suppose that the equality $\mathrm{Card}(\mathfrak{m}) = \mathrm{Card}(\mathfrak{m}_1) + \mathrm{Card}(\mathfrak{m}_2)$ holds. We set $c = \mathrm{Card}(\mathfrak{m})$. Then the cth derivatives of $Z(\mathfrak{m})$ and $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ are equal to $Z(\mathfrak{m}^-)$ and $Z(\mathfrak{m}_1^-) \times Z(\mathfrak{m}_2^-)$, respectively. Since the cth derivative is an exact functor (compare to [4, 3.2, 3.5]), the assertion follows. □

Proof of Theorem 2.1 for $\pi = Z(\mathfrak{m})$ of type χ. Let $\pi = Z(\mathfrak{m})$ be an irreducible representation of G_n, where $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ is a multisegment of type χ for some unramified character χ of $F^\times$. Let $t_\mathfrak{m}$ be the number of pairs of linked segments in $\{\Delta_1, \ldots, \Delta_r\}$. Note that $t_\mathfrak{m} \leq \binom{l(\mathfrak{m})}{2}$ since $r \leq l(\mathfrak{m})$.

We prove the claim by induction on the element $(l(\mathfrak{m}), t_\mathfrak{m})$ in the set $S = \{(l,t) \in \mathbb{Z}_{\geq 0}^2 \mid t \leq \binom{l}{2}\}$. Here we endow this set with the following total order. We have $(l,t) \leq (l',t')$ if and only if we have either $l < l'$, or $l = l'$ and $t \leq t'$. Note that for a fixed element $(l,t) \in S$, there are only finitely many elements in S that are less than (l,t).

Recall that we have a decomposition $\mathfrak{m} = \mathfrak{m}_{\max} + \mathfrak{m}^{\max}$ as in Section 2.3. We note that $Z(\mathfrak{m}_{\max})$ is a ladder representation. In particular, if $\mathfrak{m} = \mathfrak{m}_{\max}$, then we have the claim for $\mathfrak{m}$ (Section 6.1).

From now on, we assume that $\mathfrak{m}_{\max} \neq \mathfrak{m}$. Set

$$\Pi = Z(\mathfrak{m}_{\max}) \times Z(\mathfrak{m}^{\max}).$$

Since $l(\mathfrak{m}^{\max}) < l(\mathfrak{m})$, it follows from Proposition 5.2, Corollary 2.8 and the inductive hypothesis that

$$\dim(\Pi^{\mathbb{K}_{n,\lambda}}) = \begin{cases} 1 & \text{if } \lambda = \lambda_\mathfrak{m}, \\ 0 & \text{if } \lambda < \lambda_\mathfrak{m}. \end{cases}$$

It follows from Lemma 6.5 that $Z(\mathfrak{m})$ appears as a subquotient of Π. This implies that the $\mathbb{K}_{n,\lambda}$-invariant part of $Z(\mathfrak{m})$ is equal to zero if $\lambda < \lambda_\mathfrak{m}$. Hence it remains to show that the $\mathbb{K}_{n,\lambda_\mathfrak{m}}$-invariant part of $Z(\mathfrak{m})$ is one-dimensional. To do this, we may assume that Π is reducible, which implies that $t_\mathfrak{m} > 0$.

For an irreducible representation π of G_n and a representation σ of G_n of finite length, we write $\pi \dashv \sigma$ if π appears as a subquotient of σ. Let $\mathfrak{m}' \neq \mathfrak{m}$ be a multisegment, and suppose that $Z(\mathfrak{m}') \dashv \Pi$. It follows from [42, 7.1 Theorem] that $\mathfrak{m}'$ is obtained by successively applying elementary operations to $\mathfrak{m}$. In particular, we have $l(\mathfrak{m}') = l(\mathfrak{m})$ and $t_{\mathfrak{m}'} < t_\mathfrak{m}$. Hence, by the inductive hypothesis, we have

$$\dim(Z(\mathfrak{m}')^{\mathbb{K}_{n,\lambda'}}) = \begin{cases} 1 & \text{if } \lambda' = \lambda_{\mathfrak{m}'}, \\ 0 & \text{if } \lambda' < \lambda_{\mathfrak{m}'}. \end{cases}$$

Note that this implies $\lambda_{\mathfrak{m}'} \geq \lambda_\mathfrak{m}$. In fact, if $\lambda_{\mathfrak{m}'} < \lambda_\mathfrak{m}$, then the $\mathbb{K}_{n,\lambda_{\mathfrak{m}'}}$-invariant part of Π would be nonzero, which is a contradiction.

Now we claim that $\lambda_{\mathfrak{m}'} > \lambda_\mathfrak{m}$. For a proof by contradiction, suppose that $\lambda_{\mathfrak{m}'} = \lambda_\mathfrak{m}$. Since $l(\mathfrak{m}^{\mathrm{ram}}) = |\lambda_\mathfrak{m}|$ and $l(\mathfrak{m}'^{\mathrm{ram}}) = |\lambda_{\mathfrak{m}'}|$, by Proposition 2.7, we have

$$l(\mathfrak{m}'^{\mathrm{ram}}) = l(\mathfrak{m}^{\mathrm{ram}}) = l((\mathfrak{m}_{\max})^{\mathrm{ram}}) + l((\mathfrak{m}^{\max})^{\mathrm{ram}}).$$

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In particular, we have

$$\mathrm{Card}(\mathfrak{m}'^{\sharp}) = l(\mathfrak{m}') - l(\mathfrak{m}'^{\mathrm{ram}}) = l(\mathfrak{m}) - l(\mathfrak{m}^{\mathrm{ram}}) = \mathrm{Card}(\mathfrak{m}^{\sharp}).$$

By our assumption, we have $Z(\mathfrak{m}), Z(\mathfrak{m}') \dashv Z(\mathfrak{m}_{\max}) \times Z(\mathfrak{m}^{\max})$. Proposition 2.7 together with Lemma 6.5 implies that $Z(\mathfrak{m}^{\mathrm{ram}}) \dashv Z((\mathfrak{m}_{\max})^{\mathrm{ram}}) \times Z((\mathfrak{m}^{\max})^{\mathrm{ram}})$. By taking the Zelevinsky duals, we have $Z(\mathfrak{m}^{\sharp}), Z(\mathfrak{m}'^{\sharp}) \dashv Z((\mathfrak{m}_{\max})^{\sharp}) \times Z((\mathfrak{m}^{\max})^{\sharp})$, and $Z((\mathfrak{m}^{\sharp})^{-}) \dashv Z(((\mathfrak{m}_{\max})^{\sharp})^{-}) \times Z(((\mathfrak{m}^{\max})^{\sharp})^{-})$. Hence it follows from Lemma 6.6 that

$$\mathrm{Card}(\mathfrak{m}^{\sharp}) = \mathrm{Card}((\mathfrak{m}_{\max})^{\sharp}) + \mathrm{Card}((\mathfrak{m}^{\max})^{\sharp}).$$

Since we have seen that $\mathrm{Card}(\mathfrak{m}'^{\sharp}) = \mathrm{Card}(\mathfrak{m}^{\sharp})$, it again follows from Lemma 6.6 that $Z((\mathfrak{m}'^{\sharp})^{-}) \dashv Z(((\mathfrak{m}_{\max})^{\sharp})^{-}) \times Z(((\mathfrak{m}^{\max})^{\sharp})^{-})$. Again by taking the Zelevinsky duals, we see that

$$Z(\mathfrak{m}'^{\mathrm{ram}}) \dashv Z((\mathfrak{m}_{\max})^{\mathrm{ram}}) \times Z((\mathfrak{m}^{\max})^{\mathrm{ram}}).$$

This implies that $\mathfrak{m}'^{\mathrm{ram}}$ is obtained from $\mathfrak{m}^{\mathrm{ram}} = (\mathfrak{m}_{\max})^{\mathrm{ram}} + (\mathfrak{m}^{\max})^{\mathrm{ram}}$ by a successive chain of elementary operations.

Since we have assumed that $\lambda_{\mathfrak{m}'} = \lambda_{\mathfrak{m}}$, it follows that $\mathfrak{m}'^{\mathrm{ram}} = \mathfrak{m}^{\mathrm{ram}}$ and hence $(\mathfrak{m}'^{\sharp})^{-} = (\mathfrak{m}^{\sharp})^{-}$. Observe that for any integer $a \in \mathbb{Z}$, the number of segments in $\mathfrak{m}'$ that contain $\chi|\cdot|^{a}$ is equal to the number of segments in $\mathfrak{m}$ that contain $\chi|\cdot|^{a}$. Hence the equality $(\mathfrak{m}'^{\sharp})^{-} = (\mathfrak{m}^{\sharp})^{-}$ implies the equality $\mathfrak{m}'^{\sharp} = \mathfrak{m}^{\sharp}$. By taking the Zelevinsky duals, we obtain the equality $\mathfrak{m}' = \mathfrak{m}$, which is a contradiction. This completes the proof of the inequality $\lambda_{\mathfrak{m}'} > \lambda_{\mathfrak{m}}$.

Since $\mathfrak{m}' \neq \mathfrak{m}$ is an arbitrary multisegment satisfying $\mathfrak{m}' \dashv \Pi$, we see that the equation $\dim(\Pi^{\mathbb{K}_{n,\lambda_{\mathfrak{m}}}}) = 1$ implies $\dim(Z(\mathfrak{m})^{\mathbb{K}_{n,\lambda_{\mathfrak{m}}}}) = 1$. This completes the proof. □

6.4. *Proof of Theorem 2.2 for $Z(\mathfrak{m})$ of type χ*

In this section, we give a proof of Theorem 2.2 for $\pi = Z(\mathfrak{m})$ with $\mathfrak{m}$ of type χ, where χ is an unramified character of $F^{\times}$.

We consider the polynomial ring $R = \mathbb{Z}[x_1, x_2, \ldots]$ in countably many variables $\{x_i\}_{i \geq 1}$. For an $\mathfrak{o}$-module M of finite length, we define a homomorphism $\xi_M : R \longrightarrow \mathbb{Z}$ of $\mathbb{Z}$-modules as follows. We set $\xi_M(1) = 1$ if $M = 0$ and $\xi_M(1) = 0$ otherwise. For a monomial $x_{m_1} \cdots x_{m_s}$ in R, we define its image by ξ_M to be the number of increasing filtrations

$$0 = \mathrm{F}_0 M \subset \cdots \subset \mathrm{F}_s M = M$$

on M by $\mathfrak{o}$-submodules such that for $i = 1, \ldots, s$, the ith graded piece $\mathrm{Gr}_i^{\mathrm{F}} M$ is generated exactly by m_i elements. By Lemma 4.13, the homomorphism ξ_M is well-defined.

For an integer $m \geq 0$, we set

$$y_m = 1 + x_1 + \cdots + x_m \in R.$$

Lemma 6.7. *Let M be an $\mathfrak{o}$-module of finite length. Then the integer $\xi_M(y_{m_1} \cdots y_{m_s})$ is equal to the number $\mathcal{N}_{(m_1,\ldots,m_s)}(M)$ of $(m_1, \ldots, m_s)$-admissible filtrations on M.*

Proof. This is immediate from the definition of the homomorphism ξ_M and the definition of $(m_1, \ldots, m_s)$-admissible filtrations. □

By setting $\deg x_m = m$ for $m \geq 1$, we regard R as a graded ring. For any integer $m \geq 0$, let R_m denote the degree-m-part of R, and set

$$I_m = \bigoplus_{i \geq m} R_i.$$

Then I_m is an ideal of R, and we have $I_m \cdot I_{m'} \subset I_{m+m'}$.

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Lemma 6.8. *Let $m \geq 0$ be an integer, and let M be an $\mathfrak{o}$-module of length less than m. Then we have $\xi_M(I_m) = 0$.*

Proof. Let $f = x_{m_1} \cdots x_{m_s}$ be an arbitrary monomial that belongs to I_m. It suffices to show $\xi_M(f) = 0$. By definition of I_m, we have $m_1 + \cdots + m_s \geq m$. Suppose that there exists an increasing filtration

$$0 = \mathrm{F}_0 M \subset \cdots \subset \mathrm{F}_s M = M$$

on M by $\mathfrak{o}$-submodules such that for $i = 1, \dots, s$, the ith graded piece $\mathrm{Gr}_i^{\mathrm{F}} M$ is generated exactly by m_i elements. Then since $\mathrm{Gr}_i^{\mathrm{F}} M$ is of length at least m_i, the length of M is at least $m_1 + \cdots + m_s \geq m$, which is a contradiction. Hence by the definition of ξ_M, we have $\xi_M(f) = 0$ as desired. □

Now we prove Theorem 2.2 for $\pi = Z(\mathfrak{m})$ with $\mathfrak{m}$ of type χ.

Proof of Theorem 2.2 for $\pi = Z(\mathfrak{m})$ of type χ. Let us write

$$\mathfrak{m}^\sharp = \Delta_1 + \cdots + \Delta_s.$$

For $i = 1, \dots, s$, we set $\pi_i = Z(\Delta_i^\sharp)$. Let n and n_i be such that $\pi \in \mathrm{Irr}(G_n)$ and $\pi_i \in \mathrm{Irr}(G_{n_i})$. Then π appears as a subquotient of $\pi_1 \times \cdots \times \pi_s$, and we have $|\lambda_\pi| = |\lambda_{\pi_1}| + \cdots + |\lambda_{\pi_s}|$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$ be such that $|\lambda| < |\lambda_\pi|$. Then $\pi^{\mathbb{K}_{n,\lambda}}$ is a subquotient of $(\pi_1 \times \cdots \times \pi_s)^{\mathbb{K}_{n,\lambda}}$. Let $M = \mathfrak{o}/\mathfrak{p}^{\lambda_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\lambda_n}$. By Proposition 5.2, we have

$$(\pi_1 \times \cdots \times \pi_s)^{\mathbb{K}_{n,\lambda}} \cong \bigoplus_{\mathrm{F}_\bullet M} \pi_1^{\mathbb{K}_{n_1,[\mathrm{Gr}_1^{\mathrm{F}} M]}} \otimes \cdots \otimes \pi_r^{\mathbb{K}_{n_s,[\mathrm{Gr}_s^{\mathrm{F}} M]}},$$

where $\mathrm{F}_\bullet M$ runs over the set of increasing filtrations

$$0 = \mathrm{F}_0 M \subset \cdots \subset \mathrm{F}_s M = M$$

on M by $\mathfrak{o}$-submodules such that for $i = 1, \dots, s$, the $\mathfrak{o}$-module $\mathrm{Gr}_i^{\mathrm{F}} M = \mathrm{F}_i M / \mathrm{F}_{i-1} M$ is generated by at most n_i elements. Fix such a filtration $\mathrm{F}_\bullet M$. Since

$$\begin{aligned} &|\lambda_{\pi_1}| + \cdots + |\lambda_{\pi_s}| = |\lambda_\pi| \\ &> |\lambda| = \mathrm{length}_{\mathfrak{o}} M = \mathrm{length}_{\mathfrak{o}} \mathrm{Gr}_1^{\mathrm{F}} M + \cdots + \mathrm{length}_{\mathfrak{o}} \mathrm{Gr}_s^{\mathrm{F}} M, \end{aligned}$$

we have $|\lambda_{\pi_i}| > \mathrm{length}_{\mathfrak{o}} \mathrm{Gr}_i^{\mathrm{F}} M$ for some i. If we knew the claim for π_i for any $i = 1, \dots, s$, then we would have $(\pi_1 \times \cdots \times \pi_s)^{\mathbb{K}_{n,\lambda}} = 0$, which implies that $\pi^{\mathbb{K}_{n,\lambda}} = 0$. Hence we reduce the claim to the case where $s = 1$.

From now on, we assume that $s = 1$. Let us write $\Delta_1 = [1, n]_\chi$ for some unramified character χ. Then $\pi = Z([1,1]_\chi + \cdots + [n,n]_\chi)$ is an unramified twist of the Steinberg representation. By Tadić's determinantal formula [39], we have

$$\pi = \sum_{r=1}^{n} (-1)^{n-r} \sum_{0=n_0<n_1<\cdots<n_r=n} Z([n_0+1, n_1]_\chi) \times \cdots \times Z([n_{r-1}+1, n_r]_\chi)$$

in the Grothendieck group of the category of representations of G_n of finite length. Then it follows from Proposition 5.2 that for any $\mathfrak{o}$-module M of finite length, the dimension of the $\mathbb{K}_{n,[M]}$-invariant part $\pi^{\mathbb{K}_{n,[M]}}$ is equal to the number

$$\sum_{r=1}^{n} (-1)^{n-r} \sum_{0=n_0<n_1<\cdots<n_r=n} \mathcal{N}_{(n_1-n_0, \dots, n_r-n_{r-1})}(M).$$

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We set

$$f_n = \sum_{r=1}^{n} (-1)^{n-r} \sum_{0=n_0<n_1<\cdots<n_r=n} y_{n_1-n_0} \cdots y_{n_r-n_{r-1}} \in R. \tag{6.1}$$

Then it follows from Lemma 6.7 that for any $\mathfrak{o}$-module M of finite length, the dimension of $\pi^{\mathbb{K}_{n,[M]}}$ is equal to $\xi_M(f_n)$. Therefore, it suffices to prove that $\xi_M(f_n) = 0$ for any $\mathfrak{o}$-module M of length at most $n-2$. By Lemma 6.8, it suffices to show that f_n belongs to the ideal I_{n-1}.

Let us consider the ring $R[[t]]$ of formal power series in the variable t. We set

$$h = \sum_{i=1}^{\infty} y_i t^i \in tR[[t]].$$

Then f_n is equal to the coefficient of t^n in

$$F = (-1)^n \sum_{r=0}^{\infty} (-1)^r h^r.$$

Since

$$h = \frac{t + \sum_{i=1}^{\infty} x_i t^i}{1-t},$$

we have

$$F = \frac{(-1)^n}{1+h} = \frac{(-1)^n(1-t)}{1+\sum_{i=1}^{n} x_i t^i}.$$

Since the coefficients of t^i in $(1+\sum_{i=1}^{n} x_i t^i)^{-1}$ belongs to R_i for any $i \geq 0$, the claim follows. □

7. Proof of the main theorems: the case where $L(s,\pi) = 1$

In this section, we prove Theorem 2.2 for $\pi \in \mathrm{Irr}(G_n)$ with $L(s,\pi) = 1$, and we reduce Theorem 2.1 to the case of Speh representations.

7.1. Proof of Theorem 2.2 when $L(s,\pi) = 1$

First, we reduce Theorem 2.2 for π to the case where π is cuspidal. Let (π, V) be an irreducible representation of G_n such that $L(s,\pi) = 1$. Note that there exists a partition $n = n_1 + \cdots + n_r$ of n and cuspidal representations $\pi_1, \ldots, \pi_r$ of $G_{n_1}, \ldots, G_{n_r}$, respectively, such that the following conditions are satisfied:

- For $i = 1, \ldots, r$, we have $L(s,\pi_i) = 1$;
- π appears as a subquotient of the parabolic induction $\pi_1 \times \cdots \times \pi_r$;
- We have $|\lambda_\pi| = |\lambda_{\pi_1}| + \cdots + |\lambda_{\pi_r}|$.

Then by the same argument as in the proof of Theorem 2.2 for $\pi = Z(\mathfrak{m})$ of type χ in Section 6.4, we can reduce the claim for π to the ones for π_i for $i = 1, \ldots, r$: that is, the case where π is cuspidal.

To prove the claim for cuspidal π, we consider certain Hecke operators. Let $X_\lambda \subset M_n(\mathfrak{o})$ denote the subset of matrices $A = (a_{i,j}) \in M_n(\mathfrak{o})$ such that $a_{i,j} \equiv \delta_{i,j} \bmod \mathfrak{p}^{\lambda_i}$ for $1 \leq i, j \leq n$. Then

- X_λ contains $\mathbb{K}_{n,\lambda}$;
- X_λ is closed under the multiplication of matrices; and
- X_λ is bi-invariant under the action of $\mathbb{K}_{n,\lambda}$.

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We let $\mathcal{H}_\lambda$ denote the complex vector space of $\mathbb{C}$-valued compactly supported bi-$\mathbb{K}_{n,\lambda}$-invariant functions on G_n whose supports are contained in X_λ. Then $\mathcal{H}_\lambda$ has a structure of $\mathbb{C}$-algebra whose multiplication law is given by the convolution with respect to the Haar measure on G_n satisfying $\mathrm{vol}(\mathbb{K}_{n,\lambda}) = 1$. The unit element 1 of $\mathcal{H}_\lambda$ is equal to the characteristic function of $\mathbb{K}_{n,\lambda}$. Let $\mathfrak{a}_\lambda \subset \mathcal{H}_\lambda$ be the subspace of functions whose supports are contained in the complement $X_\lambda \setminus \mathbb{K}_{n,\lambda}$ of $\mathbb{K}_{n,\lambda}$ in X_λ. Then we have $\mathcal{H}_\lambda = \mathbb{C} \cdot 1 \oplus \mathfrak{a}_\lambda$, and $\mathfrak{a}_\lambda$ is a two-sided ideal of $\mathcal{H}_\lambda$.

Let (π, V) be an irreducible representation of G_n. The action of G_n on V induces an action of $\mathcal{H}_\lambda$ on $V^{\mathbb{K}_{n,\lambda}}$. We let

$$\theta_V : \mathcal{H}_\lambda \to \mathrm{End}_{\mathbb{C}}(V^{\mathbb{K}_{n,\lambda}})$$

denote the induced homomorphism of $\mathbb{C}$-algebras. We set $\mathcal{H}_{\lambda,V} = \theta_V(\mathcal{H}_\lambda)$ and $\mathfrak{a}_{\lambda,V} = \theta_V(\mathfrak{a}_\lambda)$. Then $\mathcal{H}_{\lambda,V}$ is a finite-dimensional $\mathbb{C}$-algebra, and $\mathfrak{a}_{\lambda,V}$ is a two-sided ideal of $\mathcal{H}_{\lambda,V}$.

Lemma 7.1. *Suppose that π is cuspidal. Then any element $T \in \mathfrak{a}_{\lambda,V}$ is nilpotent.*

Proof. Since $V^{\mathbb{K}_{n,\lambda}}$ is finite-dimensional, it suffices to show that for any $v \in V^{\mathbb{K}_{n,\lambda}}$ and for any linear form $\widetilde{v}\colon V^{\mathbb{K}_{n,\lambda}} \to \mathbb{C}$, we have $\widetilde{v}(T^m v) = 0$ for any sufficiently large integer m.

Let us choose $\widetilde{T} \in \mathfrak{a}_\lambda$ satisfying $\theta_V(\widetilde{T}) = T$. For an integer $m \geq 0$, we let $X_\lambda^{\geq m}$ denote the subset of matrices $A \in X_\lambda$ satisfying $\det A \in \mathfrak{p}^m$. We note that $X_\lambda^{\geq 1}$ is equal to $X_\lambda \setminus \mathbb{K}_{n,\lambda}$. Since the product of any m matrices in $X_\lambda^{\geq 1}$ belongs to $X_\lambda^{\geq m}$, it follows that the mth power $\widetilde{T}^m$ of $\widetilde{T}$ is, as a function on G_n, supported on $X_\lambda^{\geq m} \cap G_n$.

Let $(\widetilde{\pi}, \widetilde{V})$ denote the contragredient representation of (π, V). We regard $\widetilde{v}$ as a vector in the $\mathbb{K}_{n,\lambda}$-invariant part $(\widetilde{V})^{\mathbb{K}_{n,\lambda}}$ of $\widetilde{V}$. Since π is cuspidal, the matrix coefficient

$$f(g) = \langle \pi(g)v, \widetilde{v} \rangle$$

of π is compactly supported modulo the centre Z_n of G_n. Observe that the intersection $G_n \cap \left(\bigcap_{m\geq 1} Z_n X_\lambda^{\geq m}\right)$ is empty. This implies that any subset K of G_n that is compact modulo Z_n does not intersect $X_\lambda^{\geq m}$ for any sufficiently large m. Thus, the function $f(g)$ is identically zero on $X_\lambda^{\geq m}$ for any sufficiently large m, which implies that $\widetilde{v}(T^m v) = 0$ as desired. □

Proof for Theorem 2.2 when $L(s, \pi) = 1$. As we have remarked above, we may and will assume that (π, V) is cuspidal.

Let us assume that $V^{\mathbb{K}_{n,\lambda}} \neq 0$. Since $V^{\mathbb{K}_{n,\lambda}}$ is finite-dimensional, one can take a minimal nonzero left $\mathcal{H}_{\lambda,V}$-submodule W of $V^{\mathbb{K}_{n,\lambda}}$. Lemma 7.1 implies that $\mathfrak{a}_{\lambda,V}$ is contained in the Jacobson radical of $\mathcal{H}_{\lambda,V}$. Hence any element of $\mathfrak{a}_{\lambda,V}$ acts as zero on W.

Let us choose nonzero vectors $w \in W$ and $\widetilde{w} \in (\widetilde{V})^{\mathbb{K}_{n,\lambda}}$ such that $\langle w, \widetilde{w} \rangle \neq 0$. Let $f(g)$ denote the matrix coefficient of π defined as

$$f(g) = \langle \pi(g)w, \widetilde{w} \rangle.$$

Let Φ denote the characteristic function of X_λ. Let us consider the zeta integral

$$Z(\Phi, s, f) = \int_{G_n} \Phi(g) |\det g|^s f(g) dg$$

of [8]. By definition, we have

$$Z(\Phi, s, f) = \sum_{m\geq 0} I_m q^{-ms},$$

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where

$$I_m = \int_{X_\lambda^{\geq m} \setminus X_\lambda^{\geq m+1}} f(g)dg$$
$$= \left\langle \int_{X_\lambda^{\geq m} \setminus X_\lambda^{\geq m+1}} \pi(g)wdg, \widetilde{w} \right\rangle$$

as a formal power series in q^{-s}. Since $\mathfrak{a}_\lambda$ annihilates w, it follows that $I_m = 0$ for $m \geq 1$. Hence

$$Z(\Phi, s, f) = I_0 = \left\langle \int_{\mathbb{K}_{n,\lambda}} \pi(k)wdk, \widetilde{w} \right\rangle = \left(\int_{\mathbb{K}_{n,\lambda}} dk \right) \langle w, \widetilde{w} \rangle$$

is a nonzero constant.

Let us consider the Fourier transform

$$\widehat{\Phi}(x) = \int_{M_n(F)} \Phi(y)\psi(xy)dy$$

of Φ with respect to ψ, where dy is the Haar measure on $M_n(F)$ that is self-dual with respect to ψ. Then $\widehat{\Phi}$ is supported on the subset $Y_\lambda \subset M_n(F)$ of matrices $B = (b_{i,j}) \in M_n(F)$ such that $b_{i,j} \in \mathfrak{p}^{-\lambda_j}$ for $1 \leq i, j \leq n$. We set $\check{f}(g) = f(g^{-1})$. Note that $\check{f}$ is a matrix coefficient of $(\widetilde{\pi}, \widetilde{V})$. Since $\det B \in \mathfrak{p}^{-|\lambda|}$ for any $B \in Y_\lambda$, it follows that the zeta integral $Z(\widehat{\Phi}, s, \check{f})$, as a formal power series in q^{-s}, belongs to $q^{|\lambda|s}\mathbb{C}[[q^{-s}]]$.

By our assumption, we have $L(s, \pi) = L(s, \widetilde{\pi}) = 1$. Hence it follows from the local functional equation that we have

$$Z\left(\widehat{\Phi}, 1 - s + \frac{n-1}{2}, \check{f}\right) = \varepsilon(s, \pi, \psi) Z\left(\Phi, s + \frac{n-1}{2}, f\right), \tag{7.1}$$

where $\varepsilon(s, \pi, \psi)$ denotes the ε-factor of π. It is known that $\varepsilon(s, \pi, \psi) = cq^{-|\lambda_\pi|s}$ for some nonzero constant c. Since the left-hand side is in $q^{-|\lambda|s}\mathbb{C}[[q^s]]$, we see that $|\lambda| \geq |\lambda_\pi|$. This proves Theorem 2.2 for π. □

As explained in Section 5.2, Proposition 5.2 and results in Section 6.4 and this subsection complete Theorem 2.2 in all cases.

7.2. Proof of Theorem 2.1: reduction to Speh representations

In this subsection, we prove Lemma 7.2. By this lemma, Theorem 2.1 for π with $L(s, \pi) = 1$ is reduced to the case where $\pi = Z(\Delta)$.

Lemma 7.2. *Let $\pi = Z(\mathfrak{m}) \in \mathrm{Irr}(G_n)$ be such that $L(s, \pi) = 1$. Write $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$. Assume that*

$$\dim(Z(\Delta_i)^{\mathbb{K}_{n_i,\lambda_i}}) = \begin{cases} 1 & \text{if } \lambda_i = \lambda_{\Delta_i}, \\ 0 & \text{if } \lambda_i < \lambda_{\Delta_i} \end{cases}$$

for $1 \leq i \leq r$, where n_i is such that $Z(\Delta_i) \in \mathrm{Irr}(G_{n_i})$. Then we have

$$\dim(\pi^{\mathbb{K}_{n,\lambda}}) = \begin{cases} 1 & \text{if } \lambda = \lambda_\pi, \\ 0 & \text{if } \lambda < \lambda_\pi. \end{cases}$$

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Proof. Set $\Pi = Z(\Delta_1) \times \cdots \times Z(\Delta_r)$. First, we claim that

$$\dim(\Pi^{\mathbb{K}_{n,\lambda}}) = \begin{cases} 1 & \text{if } \lambda = \lambda_\pi, \\ 0 & \text{if } \lambda < \lambda_\pi. \end{cases}$$

Write $\lambda_\pi = (\lambda_1, \ldots, \lambda_n)$, and consider $M = \oplus_{i=1}^n \mathfrak{o}/\mathfrak{p}^{\lambda_i}$. Then $\mathbb{K}_{n,\lambda_\pi}$ is conjugate to $\mathbb{K}_{n,[M]}$. By Proposition 5.2, we have

$$\Pi^{\mathbb{K}_{n,\lambda_\pi}} \cong \bigoplus_{\mathrm{F}_\bullet M} Z(\Delta_1)^{\mathbb{K}_{n_1,[\mathrm{Gr}_1^{\mathrm{F}} M]}} \otimes \cdots \otimes Z(\Delta_r)^{\mathbb{K}_{n_r,[\mathrm{Gr}_r^{\mathrm{F}} M]}},$$

where $\mathrm{F}_\bullet M$ runs over the set of $\mathbf{n}$-admissible filtrations with $\mathbf{n} = (n_1, \ldots, n_r)$. Since $\lambda_\pi = \lambda_{\Delta_1} + \cdots + \lambda_{\Delta_r}$, by Corollary 4.7, there exists a unique $\mathbf{n}$-admissible filtration $\mathrm{F}^0_\bullet M$ such that $\mathrm{seq}_n([\mathrm{Gr}_i^{\mathrm{F}^0} M]) = \lambda_{\Delta_i}$ for $1 \leq i \leq r$. Moreover, for any other filtration $\mathrm{F}_\bullet M$, it holds that $\mathrm{seq}_n([\mathrm{Gr}_i^{\mathrm{F}} M]) < \lambda_{\Delta_i}$ for some $1 \leq i \leq r$. Hence by our assumption, we have $Z(\Delta_1)^{\mathbb{K}_{n_1,[\mathrm{Gr}_1^{\mathrm{F}} M]}} \otimes \cdots \otimes Z(\Delta_r)^{\mathbb{K}_{n_r,[\mathrm{Gr}_r^{\mathrm{F}} M]}} = 0$, and

$$\dim\left(\Pi^{\mathbb{K}_{n,\lambda_\pi}}\right) = \dim\left(Z(\Delta_1)^{\mathbb{K}_{n_1,\lambda_{\Delta_1}}} \otimes \cdots \otimes Z(\Delta_r)^{\mathbb{K}_{n_r,\lambda_{\Delta_r}}}\right) = 1.$$

Conversely, suppose that $[M] \in |\mathcal{C}^n|$ satisfies $\Pi^{\mathbb{K}_{n,[M]}} \neq 0$. Then by Propositions 5.2, 4.4 and our assumption, we have

$$\mathrm{seq}_n([M]) \geq \lambda_{\Delta_1} + \cdots + \lambda_{\Delta_r} = \lambda_\pi.$$

In other words, if $\lambda < \lambda_\pi$, then $\Pi^{\mathbb{K}_{n,\lambda}} = 0$. Hence we obtain the claim.

In particular, since π is a subquotient of Π, we have $\pi^{\mathbb{K}_{n,\lambda}} = 0$ for $\lambda < \lambda_\pi$.

We show $\dim(\pi^{\mathbb{K}_{n,\lambda}}) = 1$ by induction on the number t_π of pairs of linked segments in $\{\Delta_1, \ldots, \Delta_r\}$. If $t_\pi = 0$, then by [42, 4.2 Theorem], Π is irreducible so that $\pi = \Pi$. In this case, the assertion is obtained above.

Now assume that $t_\pi > 0$. By [42, 7.1 Theorem], if $\pi' = Z(\mathfrak{m}') \in \mathrm{Irr}(G_n)$ is an irreducible constituent of Π, then the multisegment $\mathfrak{m}'$ is obtained from $\mathfrak{m}$ by a chain of elementary operations. In particular, if $\pi' \not\cong \pi$, we have $t_{\pi'} < t_\pi$. Moreover, since $L(s, \pi) = 1$, we see that $\lambda_{\pi'} > \lambda_\pi$. By the inductive hypothesis, we have $\pi'^{\mathbb{K}_{n,\lambda_\pi}} = 0$. Therefore, we have $\Pi^{\mathbb{K}_{n,\lambda_\pi}} = \pi^{\mathbb{K}_{n,\lambda_\pi}}$ since π appears in the irreducible constituents of Π with multiplicity one. It follows from the above claim that $\pi^{\mathbb{K}_{n,\lambda_\pi}}$ is one-dimensional. This completes the proof. □

Note that Theorem 2.1 for π is equivalent to the one for its unramified twist $\pi|\cdot|^c$. Therefore, we may assume that π has a unitary central character. In Section 9 below, we will prove Theorem 2.1 for $\pi = Z(\Delta)$ with a unitary central character such that $L(s, \pi) = 1$. The proof of this case is rather similar to the generic case in [14]. To carry out the proof, we will establish the theory of Rankin–Selberg integrals for $Z(\Delta)$ in Section 8.

Remark 7.3. We note that Lemma 7.2 does not work for π with $L(s, \pi) \neq 1$ since the equality $\lambda_\pi = \lambda_{\Delta_1} + \cdots + \lambda_{\Delta_r}$ does not hold in general. It is one of the two reasons we should treat the case where $L(s, \pi) = 1$ and the other case separately. The other reason will be explained in Remark 8.8 below.

8. Rankin–Selberg integrals for Speh representations

In [14], Jacquet–Piatetskii-Shapiro–Shalika proved Theorem 2.1 for π generic. The ingredients they used are the Rankin–Selberg integrals [15], which express the L-factors of the products of two generic representations of G_n and G_{n-1}. (They also have expressions for products of representations of groups of other ranks, but the one used for the study of local newforms is the one mentioned above.)

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In [20], Lapid and Mao introduced the Rankin–Selberg integrals for the products of Speh representations in the equal rank case. To prove Theorem 2.1 for Speh representations in the next section, we introduce the Rankin–Selberg integrals for the product of Speh representations in the case $G_{nm} \times G_{(n-1)m}$.

8.1. *Subgroups of* $\mathrm{GL}_{nm}(F)$

Fix positive integers m and n. In this subsection, we fix notations for some subgroups of $\mathrm{GL}_{nm}(F)$.

Set $G = G_{nm} = \mathrm{GL}_{nm}(F)$ and $K = \mathrm{GL}_{nm}(\mathfrak{o})$. Let $B = TN$ be the Borel subgroup of G consisting of upper triangular matrices, where T is the diagonal torus.

We write an element of G as $g = (g_{i,j})_{1\le i,j\le m}$ with $g_{i,j} \in M_n(F)$. Define

- L to be the subgroup of G consisting of block diagonal matrices: that is, $g = (g_{i,j})_{1\le i,j\le m} \in G$ with $g_{i,j} = 0$ for $i \ne j$;
- U to be the subgroup of G consisting of block upper unipotent matrices: that is, $g = (g_{i,j})_{1\le i,j\le m} \in G$ with $g_{i,i} = \mathbf{1}_n$ for $1 \le i \le m$ and $g_{i,j} = 0$ for $i > j$;
- S to be the subgroup of G consisting of $g = (g_{i,j})_{1\le i,j\le m} \in G$ such that each $g_{i,j}$ is a diagonal matrix;
- V to be the subgroup of G consisting of $g = (g_{i,j})_{1\le i,j\le m} \in G$ such that each $g_{i,j} - \delta_{i,j}\mathbf{1}_n$ is a strictly upper triangular matrix;
- D to be the subgroup of G consisting of $g = (g_{i,j})_{1\le i,j\le m} \in G$ such that each $g_{i,j}$ is of the form
$$g_{i,j} = \begin{pmatrix} g'_{i,j} & u_{i,j} \\ 0 & \delta_{i,j} \end{pmatrix}$$
for some $g'_{i,j} \in M_{n-1}(F)$ and $u_{i,j} \in F^{n-1}$.

Then $P = LU$ is the standard parabolic subgroup with $L \cong G_n \times \cdots \times G_n$ (m-times) as its Levi subgroup, and $Q = SV$ is a nonstandard parabolic subgroup with $S \cong G_m \times \cdots \times G_m$ (n-times) as its Levi subgroup.

We set $G' = G_{(n-1)m}$. We denote analogous subgroups by taking $'$, for example, $K' = \mathrm{GL}_{(n-1)m}(\mathfrak{o})$, $P' = L'U'$, $Q' = S'V'$ and so on. Define an embedding $\iota \colon G' \hookrightarrow G$ by
$$\iota(g') = \left(\begin{pmatrix} g'_{i,j} & 0 \\ 0 & \delta_{i,j} \end{pmatrix} \right)_{1\le i,j\le m},$$
where we write $g' = (g'_{i,j})_{1\le i,j\le m}$ with $g'_{i,j} \in M_{n-1}(F)$. Sometimes we identify G' with the image of ι. Note that G' is contained in D.

For example, when $n = 3$ and $m = 2$, the subgroups above are as follows:

$$L = \left(\begin{array}{ccc|ccc} * & * & * & & & \\ * & * & * & & & \\ * & * & * & & & \\ \hline & & & * & * & * \\ & & & * & * & * \\ & & & * & * & * \end{array}\right), \quad U = \left(\begin{array}{ccc|ccc} 1 & & & * & * & * \\ & 1 & & * & * & * \\ & & 1 & * & * & * \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right),$$

$$S = \left(\begin{array}{ccc|ccc} * & & & * & & \\ & * & & & * & \\ & & * & & & * \\ \hline * & & & * & & \\ & * & & & * & \\ & & * & & & * \end{array}\right), \quad V = \left(\begin{array}{ccc|ccc} 1 & * & * & & * & * \\ & 1 & * & & & * \\ & & 1 & & & \\ \hline & * & * & 1 & * & * \\ & & * & & 1 & * \\ & & & & & 1 \end{array}\right),$$

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$$D = \left(\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ & & 1 & & & \\ \hline * & * & * & * & * & * \\ * & * & * & * & * & * \\ & & & & & 1 \end{array}\right), \quad G' = \left(\begin{array}{ccc|ccc} * & * & & * & * & \\ * & * & & * & * & \\ & & 1 & & & \\ \hline * & * & & * & * & \\ * & * & & * & * & \\ & & & & & 1 \end{array}\right).$$

It is easy to see the following:

Lemma 8.1.

(1) *$D = VG'$ and $G' \cap V = V'$ so that $V \backslash D \cong V' \backslash G'$.*
(2) *$N \cap D = (N \cap V)N'$ and $(N \cap V) \cap N' = N' \cap V'$ so that $(N \cap V) \backslash (N \cap D) \cong (N' \cap V') \backslash N'$.*

Proof. Omitted. □

8.2. Two models of Speh representations

We introduce the Zelevinsky model and the Shalika model of a Speh representation. For the detail of these models and the relation between them, see [20, Section 3].

We define a function Ψ of $G = G_{nm}$ by

$$\Psi(g) = \psi\left(\sum_{\substack{1 \le i < nm \\ n \nmid i}} g_{i,i+1}\right).$$

We denote the restriction of Ψ to N (respectively, V) by the same symbol Ψ, which is a character of N (respectively, V).

Let π be an irreducible tempered representation of G_n. Then the parabolically induced representation

$$\pi|\cdot|^{-\frac{m-1}{2}} \times \pi|\cdot|^{-\frac{m-3}{2}} \times \cdots \times \pi|\cdot|^{\frac{m-1}{2}}$$

of G has a unique irreducible subrepresentation $\mathrm{Sp}(\pi, m)$. We call $\mathrm{Sp}(\pi, m)$ a *Speh representation*. Note that if $\pi = \rho$ is cuspidal, then $\mathrm{Sp}(\rho, m) = Z([-\frac{m-1}{2}, \frac{m-1}{2}]_\rho)$.

From now on, we set $\sigma = \mathrm{Sp}(\pi, m)$ for some irreducible tempered representation π of G_n. By [42, 8.3], we know that

$$\mathrm{Hom}_G(\sigma, \mathrm{Ind}_N^G(\Psi))$$

is one-dimensional. Following [20, Section 3.1], we write $\mathcal{W}^\psi_{\mathrm{Ze}}(\sigma)$ for the image of a nonzero element and call it the *Zelevinsky model* of σ.

In the case $m = 1$, the Zelevinsky model $\mathcal{W}^\psi(\pi) = \mathcal{W}^\psi_{\mathrm{Ze}}(\pi)$ is what is known as the *Whittaker model* of π. Note that the character Ψ is a generic character of N in this case, and the one-dimensionality above implies that every tempered representation π of G_n is generic.

As explained in [20, Section 3.1], for any $W \in \mathcal{W}^\psi_{\mathrm{Ze}}(\sigma)$, we have

$$W|_L \in \mathcal{W}^\psi(\pi|\cdot|^{\frac{(m-1)(n-1)}{2}}) \otimes \mathcal{W}^\psi(\pi|\cdot|^{\frac{(m-3)(n-1)}{2}}) \otimes \cdots \otimes \mathcal{W}^\psi(\pi|\cdot|^{-\frac{(m-1)(n-1)}{2}}).$$

By [32], we know that

$$\mathrm{Hom}_G(\sigma, \mathrm{Ind}_V^G(\Psi))$$

is also one-dimensional. Following [20, Section 3.1], we write $\mathcal{W}^\psi_{\mathrm{Sh}}(\sigma)$ for the image of a nonzero element and call it the *Shalika model* of σ. As explained in [20, Section 3.1], the usage of this terminology may not be a common one.

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We recall a theorem of Lapid and Mao.

Theorem 8.2 [20, Theorem 4.3]. *For $W_1, W_2 \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$, the integral*

$$\mathcal{B}(W_1, W_2, s) = \int_{V\backslash D} W_1(g)\overline{W_2(g)}|\det g|^s dg$$

converges for $\mathrm{Re}(s) > -1$ *and admits meromorphic continuation to the complex plane. Moreover,* $(W_1, W_2) \mapsto \mathcal{B}(W_1, W_2, 0)$ *is a G-invariant inner product on* $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$.

Proof. See [20, Theorem 4.3]. See also [20, Propositions 4.1, 6.2]. □

Note that $\mathcal{W}^{\psi}_{\mathrm{Ze}}(\sigma)$ and $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$ are isomorphic to each other since both are isomorphic to σ. We can give isomorphisms explicitly as follows.

Proposition 8.3 [20, Lemmas 3.8, 3.11]. *Let $W_{\mathrm{Ze}} \in \mathcal{W}^{\psi}_{\mathrm{Ze}}(\sigma)$ and $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$. Then W_{Ze} (respectively, W_{Sh}) is compactly supported on $(V \cap N)\backslash V$ (respectively, $(N \cap V)\backslash(N \cap D)$). Moreover, an isomorphism $\mathcal{T} = \mathcal{T}^{\psi} : \mathcal{W}^{\psi}_{\mathrm{Ze}}(\sigma) \xrightarrow{\sim} \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$ is given by the integral*

$$\mathcal{T}W_{\mathrm{Ze}}(g) = \int_{(V\cap N)\backslash V} W_{\mathrm{Ze}}(ug)\Psi(u)^{-1} du.$$

The inverse of $\mathcal{T}$ is given by the integral

$$\mathcal{T}^{-1}W_{\mathrm{Sh}}(g) = \int_{(N\cap V)\backslash(N\cap D)} W_{\mathrm{Sh}}(ug)\Psi(u)^{-1} du.$$

Proof. See [20, Lemmas 3.8, 3.11]. □

8.3. Rankin–Selberg integrals in the Zelevinsky models

For irreducible tempered representations π and π' of G_n and G_{n-1}, respectively, we have Speh representations $\sigma = \mathrm{Sp}(\pi, m) \in \mathrm{Irr}(G)$ and $\sigma' = \mathrm{Sp}(\pi', m) \in \mathrm{Irr}(G')$. For $W \in W^{\psi}_{\mathrm{Ze}}(\sigma)$, $W' \in W^{\psi^{-1}}_{\mathrm{Ze}}(\sigma')$ and $s \in \mathbb{C}$, consider the integral

$$I_m(s, W, W') = \int_{N'\backslash G'} W(\iota(g))W'(g)|\det g|^{s-\frac{m}{2}} dg.$$

We call this *the Rankin–Selberg integral in the Zelevinsky models.*

Lemma 8.4. *Formally, $I_m(s, W, W')$ is equal to*

$$\int_{P'\backslash G'} \left(\int_{(N'\cap L')\backslash L'} W(\iota(lg))W'(lg)|\det l|^{s-\frac{m}{2}} \delta_{P'}^{-1}(l) dl\right) |\det g|^{s-\frac{m}{2}} dg.$$

Proof. This follows from a well-known integral formula. □

When $m = 1$, several properties of $I_1(s, W, W')$ were obtained in [15]. The following is a generalisation of [15, (2.7) Theorem], whose proof is analogous to that of [20, Theorem 5.1].

Theorem 8.5. *Let π and π' be irreducible tempered representations of G_n and G_{n-1}, respectively. We denote the central character of π' by $\omega_{\pi'}$.*

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(1) *The integral* $I_m(s, W, W')$ *is absolutely convergent for* $\mathrm{Re}(s) \gg 0$.
(2) *The function*

$$\left(\prod_{i=1}^{m} L(s-m+i, \pi \times \pi')\right)^{-1} I_m(s, W, W')$$

is in $\mathbb{C}[q^{-s}, q^{s}]$. *In particular, it is entire.*
(3) *The functional equation*

$$I_m(m-s, \widetilde{W}, \widetilde{W}') = \omega_{\pi'}(-1)^{(n-1)m} \left(\prod_{i=1}^{m} \gamma(s-m+i, \pi \times \pi', \psi)\right) I_m(s, W, W')$$

holds, where $\widetilde{W}(g) = W(w_{nm}{}^{t}g^{-1}w'_n)$ *and* $\widetilde{W}'(g') = W'(w_{(n-1)m}{}^{t}g'^{-1}w'_{n-1})$ *with*

$$w_{nm} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}, \quad w'_n = \begin{pmatrix} & & \mathbf{1}_n \\ & \iddots & \\ \mathbf{1}_n & & \end{pmatrix} \in G.$$

Here $\gamma(s, \pi \times \pi', \psi)$ *is the gamma factor defined by*

$$\gamma(s, \pi \times \pi', \psi) = \varepsilon(s, \pi \times \pi', \psi) \frac{L(1-s, \widetilde{\pi} \times \widetilde{\pi}')}{L(s, \pi \times \pi')}.$$

Proof. When $m = 1$, the assertions are [15, (2.7) Theorem].

Note that $\delta_{P'}(l) = \prod_{i=1}^{m} |\det l_i|^{(m+1-2i)(n-1)}$ for $l = \mathrm{diag}(l_1, \dots, l_m) \in L'$. Moreover,

$$\prod_{i=1}^{m} |\det l_i|^{\frac{(m+1-2i)(1-n)}{2}} W(\iota(lg)) \in \mathcal{W}^{\psi}(\pi)^{\otimes m} \quad \text{and}$$
$$\prod_{i=1}^{m} |\det l_i|^{\frac{(m+1-2i)(2-n)}{2}} W'(lg) \in \mathcal{W}^{\psi^{-1}}(\pi')^{\otimes m}$$

for fixed $g \in G'$. It follows that the inner integral of Lemma 8.4 is of the form

$$\sum_{\alpha,\beta} \prod_{i=1}^{m} I_1(s-m+i, W_{i,\alpha}, W'_{i,\beta})$$

for some $W_{i,\alpha} \in \mathcal{W}^{\psi}(\pi)$ and $W'_{i,\beta} \in \mathcal{W}^{\psi^{-1}}(\pi')$ (depending on g). Hence we obtain assertions (1) and (2).

We prove assertion (3). For $g \in G'$ and $l = \mathrm{diag}(l_1, \dots, l_m) \in L'$ with $l_i \in \mathrm{GL}_{n-1}(F)$, we note that

$$w'_n{}^{t}\iota(g)^{-1}w'_n = \iota\left(w'_{n-1}{}^{t}g^{-1}w'_{n-1}\right),$$
$$w_{nm}{}^{t}\iota(l)^{-1}w'_n = \mathrm{diag}\left(w_n \begin{pmatrix} {}^{t}l_m^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \dots, w_n \begin{pmatrix} {}^{t}l_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Hence we have

$$\widetilde{W}(\iota(lg)) = W(w_{nm}{}^{t}\iota(l)^{-1}w'_n \cdot w'_n{}^{t}\iota(g)^{-1}w'_n)$$
$$= W\left(\mathrm{diag}\left(w_n \begin{pmatrix} {}^{t}l_m^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \dots, w_n \begin{pmatrix} {}^{t}l_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \iota(w'_{n-1}{}^{t}g^{-1}w'_{n-1})\right).$$

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Similarly, we have

$$\begin{aligned}\widetilde{W}'(lg) &= W'(w_{(n-1)m}{}^t l^{-1} w'_{n-1} \cdot w'_{n-1}{}^t g^{-1} w'_{n-1}) \\ &= W'(\mathrm{diag}(w_{n-1}{}^t l_m^{-1}, \ldots, w_{n-1}{}^t l_1^{-1}) \cdot w'_{n-1}{}^t g^{-1} w'_{n-1}).\end{aligned}$$

Moreover, the map $g \mapsto \theta(g) := w'_{n-1}{}^t g^{-1} w'_{n-1}$ is a homeomorphism on $P'\backslash G'$ such that $d\theta(g) = dg$ and $|\det \theta(g)| = |\det g|^{-1}$. Hence

$$\begin{aligned}&I_m(m-s, \widetilde{W}, \widetilde{W}') \\ &= \int_{P'\backslash G'} \left(\int_{(N'\cap L')\backslash L'} \widetilde{W}(\iota(lg))\widetilde{W}'(lg) |\det l|^{-(s-\frac{m}{2})} \delta_{P'}^{-1}(l) dl \right) |\det g|^{-(s-\frac{m}{2})} dg \\ &= \int_{P'\backslash G'} \left(\int_{(N'\cap L')\backslash L'} \widetilde{W}(\iota(lg))\widetilde{W}'(lg) \prod_{i=1}^m |\det l_i|^{\frac{(m+1-2i)(3-2n)}{2} - s + i - \frac{1}{2}} dl \right) |\det g|^{-(s-\frac{m}{2})} dg \\ &= \omega_{\pi'}(-1)^{(n-1)m} \left(\prod_{i=1}^m \gamma(s-m+i, \pi\times\pi', \psi) \right) \\ &\times \int_{P'\backslash G'} \left(\int_{(N'\cap L')\backslash L'} W(\iota(lg))W'(lg) \prod_{i=1}^m |\det l_i|^{\frac{(m+1-2i)(3-2n)}{2} + s - m + i - \frac{1}{2}} dl \right) |\det g|^{s-\frac{m}{2}} dg \\ &= \omega_{\pi'}(-1)^{(n-1)m} \left(\prod_{i=1}^m \gamma(s-m+i, \pi\times\pi', \psi) \right) I_m(s, W, W').\end{aligned}$$

Here, in the third equation, we made the change of variables $l_i \mapsto l_{m+1-i}$ and $g \mapsto \theta(g)$. This completes the proof. □

Lemma 8.6. *For any $W' \in W_{\mathrm{Ze}}^{\psi^{-1}}(\sigma')$ with $W'(\mathbf{1}_{(n-1)m}) \neq 0$, there exists $W \in W_{\mathrm{Ze}}^{\psi}(\sigma)$ such that $I_m(s, W, W') = 1$ for all $s \in \mathbb{C}$.*

Proof. By [20, Corollary 3.15], the space $\{W|_D \mid W \in W_{\mathrm{Ze}}^{\psi}(\sigma)\}$ contains the compact induction $\mathrm{ind}_{N\cap D}^D(\Psi)$. Hence the assertion follows by taking $W \in W_{\mathrm{Ze}}^{\psi}(\sigma)$ such that $W|_D$ is supported on $(N\cap D)\Omega$ for a small neighbourhood Ω of $\mathbf{1}_{nm} \in D$. □

Proposition 8.7. *The $\mathbb{C}$-span of the integrals $I_m(s, W, W')$ for $W \in W^{\psi}(\sigma)$ and $W' \in W^{\psi^{-1}}(\sigma')$ is a fractional ideal of $\mathbb{C}[q^{-s}, q^s]$, which is generated by $P_m(q^{-s})^{-1}$ for some $P_m(X) \in \mathbb{C}[X]$ with $P_m(0) = 1$. Moreover, $P_1(q^{-s}) = L(s, \pi\times\pi')^{-1}$ and $P_m(X)$ divides $\prod_{i=1}^m P_1(q^{m-i}X)$.*

Proof. Note that

$$I_m(s, \iota(h)W, hW') = |\det h|^{-(s-\frac{m}{2})} I_m(s, W, W')$$

for $h \in G'$, where $(\iota(h)W)(g) = W(g\iota(h))$ and $(hW')(g') = W'(g'h)$. Hence the $\mathbb{C}$-span of the integrals $I_m(s, W, W')$ is a fractional ideal of $\mathbb{C}[q^{-s}, q^s]$. The other assertions follow from Lemma 8.6 and Theorem 8.5 (2). □

Remark 8.8. One might expect that $P_m(X) = \prod_{i=1}^m P_1(q^{m-i}X)$, but we do not know if this holds in general. This is a reason we cannot prove Theorem 9.1 below for $\sigma = \mathrm{Sp}(\pi, m)$ when $L(s,\pi) \neq 1$ by a method similar to that in [14]. However, as an application of Theorem 2.1, we will prove the equation $P_m(X) = \prod_{i=1}^m P_1(q^{m-i}X)$ when π' is unramified (see Theorem 9.1 below).

8.4. Rankin–Selberg integrals in the Shalika models

Now we translate the results for the Zelevinsky models obtained in the previous subsection to those for the Shalika models.

Recall that $\sigma = \mathrm{Sp}(\pi, m) \in \mathrm{Irr}(G)$ and $\sigma' = \mathrm{Sp}(\pi', m) \in \mathrm{Irr}(G')$. For $W_{\mathrm{Sh}} \in W^{\psi}_{\mathrm{Sh}}(\sigma)$, $W'_{\mathrm{Sh}} \in W^{\psi^{-1}}_{\mathrm{Sh}}(\sigma')$ and $s \in \mathbb{C}$, consider the integral

$$Z_m(s, W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}) = \int_{V'\backslash G'} W_{\mathrm{Sh}}(\iota(g))W'_{\mathrm{Sh}}(g)|\det g|^{s-\frac{m}{2}}\,dg.$$

We call this *the Rankin–Selberg integral in the Shalika models.*

Proposition 8.9. *If $W_{\mathrm{Sh}} = \mathcal{T}^{\psi} W_{\mathrm{Ze}}$ and $W'_{\mathrm{Sh}} = \mathcal{T}^{\psi^{-1}} W'_{\mathrm{Ze}}$, we have*

$$Z_m(s, W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}) = I_m(s, W_{\mathrm{Ze}}, W'_{\mathrm{Ze}}).$$

Proof. By Lemma 8.1 and Proposition 8.3, we have

$$\begin{aligned}
Z_m(s, W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}) &= \int_{V'\backslash G'} W_{\mathrm{Sh}}(\iota(g)) \left(\int_{(V'\cap N')\backslash V'} W'_{\mathrm{Ze}}(ug)\Psi(u)\,du\right) |\det g|^{s-\frac{m}{2}}\,dg \\
&= \int_{V'\backslash G'} \int_{(V'\cap N')\backslash V'} W_{\mathrm{Sh}}(\iota(ug))W'_{\mathrm{Ze}}(ug)|\det(ug)|^{s-\frac{m}{2}}\,du dg \\
&= \int_{(V'\cap N')\backslash G'} W_{\mathrm{Sh}}(\iota(g))W'_{\mathrm{Ze}}(g)|\det g|^{s-\frac{m}{2}}\,dg \\
&= \int_{N'\backslash G'} \int_{(V'\cap N')\backslash N'} W_{\mathrm{Sh}}(\iota(ug))W'_{\mathrm{Ze}}(ug)|\det(ug)|^{s-\frac{m}{2}}\,du dg \\
&= \int_{N'\backslash G'} \left(\int_{(N\cap V)\backslash (N\cap D)} W_{\mathrm{Sh}}(\iota(ug))\Psi(u)^{-1}\,du\right) W'_{\mathrm{Ze}}(g)|\det g|^{s-\frac{m}{2}}\,dg \\
&= I_m(s, W_{\mathrm{Ze}}, W'_{\mathrm{Ze}}).
\end{aligned}$$

This proves the proposition. □

Therefore, assertions similar to those in Theorem 8.5, Lemma 8.6 and Proposition 8.7 hold for $Z_m(s, W_{\mathrm{Sh}}, W'_{\mathrm{Sh}})$. Here, we note the following. If $W_{\mathrm{Ze}} \in \mathcal{W}^{\psi}_{\mathrm{Ze}}(\sigma)$, we define $\widetilde{W}_{\mathrm{Ze}} \in \mathcal{W}^{\psi^{-1}}_{\mathrm{Ze}}(\widetilde{\sigma})$, where $\widetilde{\sigma}$ is the contragredient representation of σ, by $\widetilde{W}_{\mathrm{Ze}}(g) = W_{\mathrm{Ze}}(w_{nm}{}^t g^{-1} w'_n)$. One can easily check that

$$\mathcal{T}^{\psi^{-1}} \widetilde{W}_{\mathrm{Ze}}(g) = \mathcal{T}^{\psi} W_{\mathrm{Ze}}(w_{nm}{}^t g^{-1} w'_n).$$

Hence we define $\widetilde{W}_{\mathrm{Sh}} \in \mathcal{W}^{\psi^{-1}}_{\mathrm{Sh}}(\widetilde{\sigma})$ for $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$ by $\widetilde{W}_{\mathrm{Sh}}(g) = W_{\mathrm{Sh}}(w_{nm}{}^t g^{-1} w'_n)$.

8.5. The case where π' is unramified

In the following section, we need sharper results when π' is unramified.

Let π' be an irreducible unramified representation of G_{n-1} with Satake parameter $(x_1, \dots, x_{n-1}) \in (\mathbb{C}^{\times})^{n-1}/S_{n-1}$. Hence π' is the unique irreducible unramified constituent of

$$I(s_1, \dots, s_{n-1}) = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{n-1}},$$

where s_j is a complex number such that $q^{-s_j} = x_j$. Since the principal series $I(s_1, \dots, s_{n-1})$ is generic and unramified, there exists a unique Whittaker function $W^0(x_1, \dots, x_{n-1}) \in \mathcal{W}^{\psi^{-1}}(I(s_1, \dots, s_{n-1}))$ such that $W^0(k_1; x_1, \dots, x_{n-1}) = 1$ for any $k_1 \in \mathrm{GL}_{n-1}(\mathfrak{o})$. When π' is tempered, that is, $|x_j| = 1$ for any $1 \le j \le n-1$, the function $W^0(x_1, \dots, x_{n-1})$ belongs to $\mathcal{W}^{\psi^{-1}}(\pi')$. Note that $W^0(x_1, \dots, x_{n-1})$ is a Hecke eigenfunction whose Hecke eigenvalues are uniquely determined by $(x_1, \dots, x_{n-1}) \in (\mathbb{C}^\times)^{n-1}/S_{n-1}$.

Recall that $G' = G_{(n-1)m}$, $K' = G_{(n-1)m}(\mathfrak{o})$ and that $P' = L'U'$ is the standard parabolic subgroup of G' with $L' \cong G_{n-1} \times \cdots \times G_{n-1}$ (m-times). Let $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ with $x_{i,j} \in \mathbb{C}^\times$. We can define a function

$$W^0_{\mathrm{Ze}}(\underline{x}) \colon G' \to \mathbb{C}$$

by

$$W^0_{\mathrm{Ze}}(ulk; \underline{x}) = \Psi^{-1}(u)\delta_{P'}^{\frac{1}{2}}(l) \prod_{i=1}^{m} W^0(l_i; x_{i,1}, \dots, x_{i,n-1})$$

for $u \in U'$, $l = \mathrm{diag}(l_1, \dots, l_m) \in L'$ and $k \in K'$. (Here, we note that $\Psi(u) = 1$ for $u \in U'$.) As in [20, Lemma 3.8], $W^0_{\mathrm{Ze}}(\underline{x})$ is compactly supported on $(V' \cap N')\backslash V'$. We set

$$W^0_{\mathrm{Sh}}(g; \underline{x}) = \int_{(V' \cap N')\backslash V'} W^0_{\mathrm{Ze}}(ug; \underline{x})\Psi(u)du.$$

If $\underline{x} = {}^t(q^{-\frac{m-1}{2}}x_j, q^{-\frac{m-3}{2}}x_j, \dots, q^{\frac{m-1}{2}}x_j)_{1 \le j \le n-1}$ with $|x_j| = 1$ for any $1 \le j \le n-1$, then $W^0_{\mathrm{Ze}}(\underline{x}) \in \mathcal{W}^{\psi^{-1}}_{\mathrm{Ze}}(\mathrm{Sp}(\pi', m))$, where π' is the irreducible unramified representation of G_{n-1} with Satake parameter $(x_1, \dots, x_{n-1})$. In general, $W^0_{\mathrm{Ze}}(g; \underline{x}) = l(g \cdot f^0)$ for some $l \in \mathrm{Hom}_{N'}(I(s_1, \dots, s_{(n-1)m}), \Psi)$, where $s_1, \dots, s_{(n-1)m}$ are complex numbers such that

$$\{q^{-s_1}, \dots, q^{-s_{(n-1)m}}\} = \{x_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n-1\}$$

as multisets, and $f^0 \in I(s_1, \dots, s_{(n-1)m})^{K'}$. Note that $W^0_{\mathrm{Ze}}(\underline{x})$ is a Hecke eigenfunction whose Hecke eigenvalues are uniquely determined by $(s_1, \dots, s_{(n-1)m}) \in \mathbb{C}^{(n-1)m}/S_{(n-1)m}$.

Lemma 8.10. *The Hecke eigenspace in* $\mathrm{Ind}_{N'}^{G'}(\Psi)^{K'}$ *with Hecke eigenvalues determined by* $(s_1, \dots, s_{(n-1)m})$ *is spanned by* $W^0_{\mathrm{Ze}}(\underline{x})$ *for* $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ *such that* $\{q^{-s_1}, \dots, q^{-s_{(n-1)m}}\} = \{x_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n-1\}$ *as multisets.*

Proof. Since Ψ is trivial on $U' \subset N'$, we have a canonical isomorphism

$$\mathrm{Hom}_{N'}(I(s_1, \dots, s_{(n-1)m}), \Psi) \cong \mathrm{Hom}_{N' \cap L'}(\mathrm{Jac}_{P'}(I(s_1, \dots, s_{(n-1)m})), \Psi),$$

where $\mathrm{Jac}_{P'}$ is the unnormalised Jacquet functor along $P' = L'U'$. Note that $\Psi|_{N' \cap L'}$ is a generic character. Moreover, by the Geometric Lemma of Bernstein–Zelevinsky [4, 2.12], the semisimplification of $\mathrm{Jac}_{P'}(I(s_1, \dots, s_{(n-1)m}))$ is equal to

$$\delta_{P'}^{\frac{1}{2}} \otimes \left(\bigoplus_{\underline{x}} I(s_{1,1}, \dots, s_{1,n-1}) \boxtimes \cdots \boxtimes I(s_{m,1}, \dots, s_{m,n-1}) \right),$$

where $\underline{x} = (x_{i,j})$ runs over $M_{m,n-1}(\mathbb{C})/(S_{n-1})^m$ such that $\{q^{-s_1}, \dots, q^{-s_{(n-1)m}}\} = \{x_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n-1\}$, and $s_{i,j}$ is a complex number such that $q^{-s_{i,j}} = x_{i,j}$. Hence $\dim \mathrm{Hom}_{N'}(I(s_1, \dots, s_{(n-1)m}), \Psi)$ is less than or equal to the number of choices of such $\underline{x}$. This proves the claim. □

Let π be an irreducible tempered representation of G_n, and set $\sigma = \mathrm{Sp}(\pi, m)$. For $W_{\mathrm{Ze}} \in \mathcal{W}^{\psi}_{\mathrm{Ze}}(\sigma)$ and $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$, one can consider the integrals $I_m(s, W_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x}))$ and $Z_m(s, W_{\mathrm{Sh}}, W^0_{\mathrm{Sh}}(\underline{x}))$ defined

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by the same integrals in the previous two subsections. By the same arguments as in these subsections, we can prove the following theorem. We omit the proof of it.

Theorem 8.11. *The integrals $I_m(s, W_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x}))$ and $Z_m(s, W_{\mathrm{Sh}}, W^0_{\mathrm{Sh}}(\underline{x}))$ have the following properties:*

(1) *The integral $I_m(s, W_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x}))$ is absolutely convergent for* $\mathrm{Re}(s) \gg 0$.
(2) *The function*

$$\left(\prod_{i=1}^{m}\prod_{j=1}^{n-1} L\left(s + s_{i,j} - \frac{m-1}{2}, \pi\right)\right)^{-1} I_m(s, W_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x}))$$

is in $\mathbb{C}[q^{-s}, q^s]$, where $s_{i,j}$ is a complex number such that $q^{-s_{i,j}} = x_{i,j}$. In particular, it is entire.
(3) *The functional equation*

$$\begin{aligned} &I_m(m-s, \widetilde{W}_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x}^{-1})) \\ &= \left(\prod_{i=1}^{m}\prod_{j=1}^{n-1} \gamma\left(s + s_{i,j} - \frac{m-1}{2}, \pi, \psi\right)\right) I_m(s, W_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x}))\end{aligned}$$

holds, where $\widetilde{W}_{\mathrm{Ze}}(g) = W_{\mathrm{Ze}}(w_{nm}{}^t g^{-1} w'_n)$ and $\underline{x}^{-1} = (x_{i,j}^{-1})$.
(4) *If $W_{\mathrm{Sh}} = \mathcal{T}^{\psi} W_{\mathrm{Ze}}$, then*

$$I_m(s, W_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x})) = Z_m(s, W_{\mathrm{Sh}}, W^0_{\mathrm{Sh}}(\underline{x})).$$

Proof. Omitted. □

9. Essential vectors for Speh representations

We continue to use the notations in the previous section. Recall that ψ is unramified: that is, ψ is trivial on $\mathfrak{o}$ but nontrivial on $\mathfrak{p}^{-1}$. Let π be an irreducible tempered representation of G_n, and set $\sigma = \mathrm{Sp}(\pi, m)$. In this section, we define a notion of essential vectors and prove Theorem 2.1 for Speh representations.

9.1. Essential vectors

The following theorem is a generalisation of [14, (4.1) Théorème].

Theorem 9.1. *Let the notation be as above. There exists a unique function $W^{\mathrm{ess}}_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$ such that*

(1) *$W^{\mathrm{ess}}_{\mathrm{Sh}}(g \cdot \iota(k)) = W^{\mathrm{ess}}_{\mathrm{Sh}}(g)$ for any $g \in G$ and $k \in K'$;*
(2) *for all $s \in \mathbb{C}$ and $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ with $x_{i,j} \in \mathbb{C}^\times$,*

$$Z_m(s, W^{\mathrm{ess}}_{\mathrm{Sh}}, W^0_{\mathrm{Sh}}(\underline{x})) = \prod_{i=1}^{m}\prod_{j=1}^{n-1} L\left(s + s_{i,j} - \frac{m-1}{2}, \pi\right),$$

where $s_{i,j}$ is a complex number such that $q^{-s_{i,j}} = x_{i,j}$.

Definition 9.2. We call the unique function $W^{\mathrm{ess}}_{\mathrm{Sh}}$ the *essential vector* of $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$.

First we consider existence. Here, we show it only when $L(s, \pi) = 1$. The general case will be proven in Section 9.3 below.

Proof of the existence statement in Theorem 9.1 when $L(s, \pi) = 1$. Note that $Q' = S'V'$ is conjugate to a standard parabolic subgroup of G' by an element of K'. Hence we have the Iwasawa decomposition

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$G' = Q'K'$. Define a smooth function φ of $D = V\iota(G')$ by $\mathrm{Supp}(\varphi) = V\iota(K')$ and $\varphi(u \cdot \iota(k)) = \Psi(u)$ for $u \in V$ and $k \in K'$. Then $\varphi \in \mathrm{ind}_V^D(\Psi)$ and

$$\int_{V'\backslash G'} \varphi(g) W^0_{\mathrm{Sh}}(g; \underline{x}) |\det g|^{s-\frac{m}{2}} dg = 1$$

for all $s \in \mathbb{C}$ and $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ with $x_{i,j} \in \mathbb{C}^\times$. By [20, Corollary 3.15], one can take $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$ such that $W_{\mathrm{Sh}}|_D = \varphi$. Then $Z_m(s, W_{\mathrm{Sh}}, W^0_{\mathrm{Sh}}(\underline{x})) = 1$ holds for all $s \in \mathbb{C}$ and $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ with $x_{i,j} \in \mathbb{C}^\times$. By replacing W_{Sh} with

$$\int_{K'} W_{\mathrm{Sh}}(g \cdot \iota(k)) dk,$$

we may assume that W_{Sh} is right $\iota(K')$-invariant. Then W_{Sh} satisfies the conditions in Theorem 9.1. This completes the proof of the existence statement in Theorem 9.1 when $L(s, \pi) = 1$. □

We now prove the uniqueness statement (in general).

Proof of the uniqueness statement in Theorem 9.1. Let $L^2(V'\backslash G'; \Psi)$ denote the space of functions φ on G' such that $\varphi(vg) = \Psi(v)\varphi(g)$ for $v \in V'$ and $g \in G'$, and φ is square-integrable on $V'\backslash G'$. Define Π to be the closure of the subspace of $L^2(V'\backslash G'; \Psi)$ consisting of smooth functions φ_{Sh} of G' such that

$$\varphi_{\mathrm{Sh}}(g) = \int_{(V'\cap N')\backslash V'} \varphi_{\mathrm{Ze}}(vg)\Psi(v)^{-1} dv$$

for some smooth function φ_{Ze} that satisfies $\varphi_{\mathrm{Ze}}(ug) = \Psi(u)\varphi_{\mathrm{Ze}}(g)$ for $u \in N'$ and $g \in G'$.

Lemma 9.3. *Let φ be a smooth function on G' such that*

(1) $\varphi \in \Pi$;
(2) $\varphi(gk) = \varphi(g)$ *for* $g \in G'$ *and* $k \in K'$;
(3) *for any* $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ *with* $x_{i,j} \in \mathbb{C}^\times$,

$$\int_{V'\backslash G'} \varphi(g) W^0_{\mathrm{Sh}}(g; \underline{x}) dg = 0.$$

Then $\varphi = 0$.

Proof. This is an analogue of [14, (3.5) Lemme]. Consider the direct integral expression of the unitary representation Π of G':

$$\Pi \cong \int^{\oplus}_{\pi' \in \mathrm{Irr}_{\mathrm{unit}}(G')} \pi' d\mu(\pi'),$$

where $\mathrm{Irr}_{\mathrm{unit}}(G')$ is the set of equivalence classes of irreducible unitary representations of G' and μ is a certain Borel measure on it. For almost all π', there exists a G'-equivariant intertwining operator $A_{\pi'} : \Pi \longrightarrow \pi'$ such that

$$(\varphi_1, \varphi_2)_{L^2(V'\backslash G'; \Psi)} = \int_{\pi'} (A_{\pi'}\varphi_1, A_{\pi'}\varphi_2)_{\pi'} d\mu(\pi')$$

for $\varphi_1, \varphi_2 \in \Pi \subset L^2(V'\backslash G'; \Psi)$, where $(\cdot, \cdot)_{\pi'}$ is a G'-invariant inner product on π'.

Now we assume that $\varphi \neq 0$. Then there exists $\pi' \in \mathrm{Irr}(G')$ such that $A_{\pi'}\varphi \neq 0$. Since φ is right K'-invariant, $A_{\pi'}\varphi$ belongs to the subspace of π' consisting of K'-fixed vectors. Then using Lemma 8.10,

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we see that $(A_{\pi'}\varphi, A_{\pi'}\varphi)_{\pi'}$ is a linear combination of integrals of the form

$$\int_{V'\backslash G'} \varphi(g) W^0_{\mathrm{Sh}}(g; \underline{x}) dg$$

for some $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ with $x_{i,j} \in \mathbb{C}^\times$. This contradicts Condition (3). $\square$

We continue the proof of the uniqueness statement in Theorem 9.1. Now suppose that two functions $W_1, W_2 \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$ satisfy the conditions of Theorem 9.1. Set $W = W_1 - W_2$, which is square-integrable on $V'\backslash G'$ by Theorem 8.2. Note that $W = \mathcal{T}^{\psi} W_{\mathrm{Ze}}$ for some $W_{\mathrm{Ze}} \in \mathcal{W}^{\psi}_{\mathrm{Ze}}(\sigma)$. We define V'' to be the subgroup of V consisting of $v = (v_{i,j})$ with $v_{i,j} \in M_n(F)$ such that $v_{i,j}$ is of the form

$$v_{i,j} = \begin{pmatrix} \delta_{i,j}\mathbf{1}_{n-1} & u_{i,j} \\ 0 & \delta_{i,j} \end{pmatrix}$$

for $u_{i,j} \in F^{n-1}$. Then V' normalises V'' and $V = V'V''$. Hence

$$\begin{aligned} W(\iota(g)) &= \int_{(V\cap N)\backslash V} W_{\mathrm{Ze}}(u\cdot\iota(g))\Psi(u)^{-1} du \\ &= \int_{(V'\cap N)\backslash V'} \left(\int_{(V''\cap N)\backslash V''} W_{\mathrm{Ze}}(u\cdot\iota(vg))\Psi(u)^{-1} du \right) \Psi(v)^{-1} dv. \end{aligned}$$

Since $\iota(G')$ normalises V'', and since the action of $\iota(N')$ on V'' does not change the invariant measure on $(V''\cap N)\backslash V''$, if we set

$$\varphi_{\mathrm{Ze}}(g') = \int_{(V''\cap N)\backslash V''} W_{\mathrm{Ze}}(u\cdot\iota(g'))\Psi(u)^{-1} du$$

for $g' \in G'$, then $\varphi_{\mathrm{Ze}}(u'g') = \Psi(u')\varphi_{\mathrm{Ze}}(g')$ for $u' \in N'$ and $g' \in G'$. Therefore, we have $W \circ \iota \in \Pi$. Hence we can apply Lemma 9.3 to $W \circ \iota$, and we obtain that $W \circ \iota = 0$. Since $D = V'\iota(G')$, it follows that $W|_D = 0$. By Theorem 8.2, we conclude that $W = 0$, as desired. This completes the proof of the uniqueness statement in Theorem 9.1. $\square$

Corollary 9.4. *Let $W \in \mathcal{W}^{\psi}_{\mathrm{Ze}}(\sigma)$. If W is right $\iota(K')$-invariant, and if $W|_L = 0$, then $W = 0$.*

Proof. By the assumptions, one has $I_m(s, W, W^0_{\mathrm{Ze}}(\underline{x})) = 0$ for all $s \in \mathbb{C}$ and $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ with $x_{i,j} \in \mathbb{C}^\times$. By the same argument as in the proof of the uniqueness statement in Theorem 9.1, we have $\mathcal{T}^{\psi} W = 0$, and hence $W = 0$. $\square$

As an application, we have a part of Theorem 2.1 for Speh representations. Recall from Example 2.5 (4) that

$$\lambda_\sigma = (\underbrace{0, \ldots, 0}_{(n-1)m}, \underbrace{c_\pi, \ldots, c_\pi}_{m}) \in \Lambda_{nm},$$

where c_π is the conductor of π.

Proposition 9.5. *Let $\lambda \in \Lambda_{nm}$. If $\lambda < \lambda_\sigma$, then $\sigma^{\mathbb{K}_{nm,\lambda}} = 0$.*

Proof. If $\lambda < \lambda_\sigma$, then the first $(n-1)m$ components of λ are 0. Hence there exists a compact subgroup K_λ of G conjugate to $\mathbb{K}_{nm,\lambda}$ such that

- $K_\lambda \supset \iota(K')$;
- $K_\lambda \cap L \supset \mathbb{K}_{n,\lambda_1} \times \cdots \times \mathbb{K}_{n,\lambda_m}$ with $\lambda_i \in \Lambda_n$ of the form $\lambda_i = (0, \ldots, 0, a_i)$ such that $0 \le a_i < c_\pi$ for some $1 \le i \le m$.

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Let $W \in \mathcal{W}_{\mathrm{Ze}}^{\psi}(\sigma)^{K_\lambda}$. Since $\pi^{\mathbb{K}_{n,\lambda_i}} = 0$ by [14, (5.1) Théorème], we see that $W|_L = 0$. It follows from Corollary 9.4 that $W = 0$. Hence $\sigma^{\mathbb{K}_{nm,\lambda}} \cong \sigma^{K_\lambda} = 0$. □

9.2. Properties of essential vectors

Recall that $G = \mathrm{GL}_{nm}(F)$ and $K = \mathrm{GL}_{nm}(\mathfrak{o})$. For a positive integer a, define $K(a) \subset K$ to be the subgroup consisting of $k = (k_{i,j})_{1\le i,j\le m} \in K$ with $k_{i,j} \in M_n(\mathfrak{o})$ such that the last row of $k_{i,j}$ is congruent to $(0,\dots,0,\delta_{i,j})$ mod $\mathfrak{p}^a$ for $1 \le i, j \le m$. Put another way, if we denote by $D(\mathfrak{o}/\mathfrak{p}^a)$ the image of $D \cap K$ under $K \to \mathrm{GL}_{nm}(\mathfrak{o}/\mathfrak{p}^a)$, then $K(a)$ is the inverse image of $D(\mathfrak{o}/\mathfrak{p}^a)$. Note that $K(a)$ is conjugate to $\mathbb{K}_{nm,\lambda}$ with

$$\lambda = (\underbrace{0,\dots,0}_{(n-1)m}, \underbrace{a,\dots,a}_{m}) \in \Lambda_{nm}$$

by an element of K.

Let π be an irreducible tempered representation of G_n, and set $\sigma = \mathrm{Sp}(\pi, m)$. We prove the following proposition in this subsection. It together with Proposition 9.5 contains Theorem 2.1 for σ when $L(s,\sigma) = 1$.

Proposition 9.6. *Suppose that $L(s,\pi) = 1$. Then $\mathcal{W}_{\mathrm{Sh}}^{\psi}(\sigma)^{K(c_\pi)}$ is the one-dimensional vector space spanned by the essential vector $W_{\mathrm{Sh}}^{\mathrm{ess}}$.*

The proof of Proposition 9.6 is analogous to that of [14, (5.1) Théorème]. Suppose that $L(s,\pi) = 1$. For $d \in \mathbb{Z}$ and $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\sigma)$, we consider

$$Z_{m,d}(W_{\mathrm{Sh}}; \underline{x}) = \int_{V'\backslash\{g\in G' \mid |\det g| = q^{-d}\}} W_{\mathrm{Sh}}(\iota(g)) W_{\mathrm{Sh}}^{0}(g;\underline{x}) |\det g|^{-\frac{m}{2}}\, dg.$$

Note that

$$Z_{m,d}(W_{\mathrm{Sh}}; x\underline{x}) = x^d Z_{m,d}(W_{\mathrm{Sh}}; \underline{x}),$$

where $x\underline{x} = (xx_{i,j})$ if $\underline{x} = (x_{i,j})$.

Lemma 9.7. *There is an integer $d(W_{\mathrm{Sh}})$ such that $Z_{m,d}(W_{\mathrm{Sh}};\underline{x}) = 0$ for any $d < d(W_{\mathrm{Sh}})$ and $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ with $x_{i,j} \in \mathbb{C}^\times$.*

Proof. By (the proof of) Proposition 8.9, it is enough to show an analogous assertion for $I_m(s, W_{\mathrm{Ze}}, W_{\mathrm{Ze}}^0(\underline{x}))$ with $W_{\mathrm{Ze}} \in \mathcal{W}_{\mathrm{Ze}}^{\psi}(\sigma)$. Let $g \in G'$ with $|\det g| = q^{-d}$ such that $W_{\mathrm{Ze}}(\iota(g))W_{\mathrm{Ze}}^0(g;\underline{x}) \neq 0$. We take $k' \in K'$, $u' \in N'$ and $a_1, \dots, a_{(n-1)m} \in \mathbb{Z}$ such that

$$g = u' \begin{pmatrix} \varpi^{a_1} & & \\ & \ddots & \\ & & \varpi^{a_{(n-1)m}} \end{pmatrix} k'.$$

Since

$$W_{\mathrm{Ze}}^0(g;\underline{x}) = C \prod_{j=1}^{m} W^0\left(\begin{pmatrix} \varpi^{a_{(n-1)(j-1)+1}} & & \\ & \ddots & \\ & & \varpi^{a_{(n-1)j}} \end{pmatrix}; x_{i,1}, \dots, x_{i,n-1}\right)$$

for some $C \neq 0$ (depending on $a_1, \dots, a_{(n-1)m}$), we must have $a_{(n-1)(j-1)+1} \ge \cdots \ge a_{(n-1)j}$ for any $1 \le j \le m$. In particular, $d = \sum_{i=1}^{(n-1)m} a_i \ge \sum_{j=1}^{m} (n-1) a_{(n-1)j}$.

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For $l \geq 0$, let $V''(\mathfrak{p}^l)$ be the subgroup of K consisting of $(k_{i,j})_{1\leq i,j\leq m}$ with $k_{i,j} \in M_n(\mathfrak{o})$ such that $k_{i,j}$ is of the form

$$k_{i,j} = \begin{pmatrix} \delta_{i,j}\mathbf{1}_{n-1} & u_{i,j} \\ 0 & \delta_{i,j} \end{pmatrix}$$

for $u_{i,j} \in (\mathfrak{p}^l)^{n-1}$. Since W_{Ze} is smooth, one can take sufficiently large l such that W_{Ze} is right $V''(\mathfrak{p}^l)$-invariant. Note that $\iota(K')$ acts on $V''(\mathfrak{p}^l)$. In particular, for any $z_1, \dots, z_m \in \mathfrak{p}^l$, we can take $u \in V''(\mathfrak{p}^l)$ such that $(\iota(k') \cdot u \cdot \iota(k')^{-1})_{nj-1,nj} = z_j$ for $1 \leq j \leq m$. Then we have

$$\begin{aligned} W_{\mathrm{Ze}}(\iota(g)) &= W_{\mathrm{Ze}}\left(\iota(u')\iota\begin{pmatrix} \varpi^{a_1} & & \\ & \ddots & \\ & & \varpi^{a_{n-1}} \end{pmatrix}\iota(k')u\right) \\ &= \left(\prod_{j=1}^m \psi(\varpi^{a_{(n-1)j}} z_j)\right) W_{\mathrm{Ze}}(\iota(g)). \end{aligned}$$

Since $z_1, \dots, z_m \in \mathfrak{p}^l$ are arbitrary, if $W_{\mathrm{Ze}}(\iota(g)) \neq 0$, then we must have $a_{(n-1)j} \geq -l$ for $1 \leq j \leq m$. In conclusion, we have $d \geq \sum_{j=1}^m (n-1)a_{(n-1)j} \geq -(n-1)ml$. This completes the proof of the lemma. □

By the proof of this lemma, one can take $d(W_{\mathrm{Sh}}) = -(n-1)ml$ if W_{Sh} is right $V''(\mathfrak{p}^l)$-invariant. In particular, if $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\sigma)^{K(a)}$, then we can take $d(W_{\mathrm{Sh}}) = 0$ and $d(\widetilde{W}_{\mathrm{Sh}}) = -(n-1)ma$.

Now, if we set $x = q^{-s}$, we have

$$Z_m(s, W_{\mathrm{Sh}}, W_{\mathrm{Sh}}^0(\underline{x})) = \sum_{d\in\mathbb{Z}} x^d Z_{m,d}(W_{\mathrm{Sh}}; \underline{x}) = \sum_{d\geq d(W_{\mathrm{Sh}})} x^d Z_{m,d}(W_{\mathrm{Sh}}; \underline{x}).$$

If we replace π, W_{Sh} and ψ with $\widetilde{\pi}$, $\widetilde{W}_{\mathrm{Sh}}$ and ψ^{-1}, respectively, since $\widetilde{W}_{\mathrm{Sh}}^0(\underline{x}) = W_{\mathrm{Sh}}^0(\underline{x}^{-1})$ (with respect to ψ), we have

$$Z_m(s, \widetilde{W}_{\mathrm{Sh}}, \widetilde{W}_{\mathrm{Sh}}^0(\underline{x})) = \sum_{d\geq d(\widetilde{W}_{\mathrm{Sh}})} x^d Z_{m,d}(\widetilde{W}_{\mathrm{Sh}}; \underline{x}^{-1}),$$

and hence

$$Z_m(m-s, \widetilde{W}_{\mathrm{Sh}}, \widetilde{W}_m^0(\underline{x})) = \sum_{d\geq d(\widetilde{W}_{\mathrm{Sh}})} q^{-m}x^{-d} Z_{m,d}(\widetilde{W}_{\mathrm{Sh}}; \underline{x}^{-1}).$$

By the functional equation (Theorem 8.11 (3)), using the assumption that $L(s,\pi) = 1$, we have

$$\begin{aligned} &\sum_{d\geq d(\widetilde{W}_{\mathrm{Sh}})} q^{-m}x^{-d} Z_{m,d}(\widetilde{W}_{\mathrm{Sh}}; \underline{x}^{-1}) \\ &= \left(\prod_{i=1}^m \prod_{j=1}^{n-1} \varepsilon\left(s + s_{i,j} - \frac{m-1}{2}, \pi, \psi\right)\right) \sum_{d\geq d(W_{\mathrm{Sh}})} x^d Z_{m,d}(W_{\mathrm{Sh}}; \underline{x}) \end{aligned}$$

as a formal power series of x, where $s_{i,j}$ is a complex number such that $x_{i,j} = q^{-s_{i,j}}$. If we write $\varepsilon(s,\pi,\psi) = \varepsilon_0 q^{-c_\pi s} = \varepsilon_0 x^{c_\pi}$, we have

$$\prod_{i=1}^m \prod_{j=1}^{n-1} \varepsilon\left(s + s_{i,j} - \frac{m-1}{2}, \pi, \psi\right) = \varepsilon_0^{m(n-1)} q^{\frac{c_\pi m(m-1)(n-1)}{2}} x^{c_\pi m(n-1)} \prod_{i=1}^m \prod_{j=1}^{n-1} x_{i,j}^{c_\pi}.$$

In particular, if $d > d'(W_{\mathrm{Sh}}) := -c_\pi m(n-1) - d(\widetilde{W}_{\mathrm{Sh}})$, we must have $Z_{m,d}(W_{\mathrm{Sh}};\underline{x}) = 0$. Hence we obtain the following.

Proposition 9.8. *Assume that $L(s,\pi) = 1$. Write $\varepsilon(s,\pi,\psi) = \varepsilon_0 q^{-c_\pi s}$. For $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\sigma)$, let $d(W_{\mathrm{Sh}})$ and $d(\widetilde{W}_{\mathrm{Sh}})$ be the constants in Lemma 9.7, and set $d'(W_{\mathrm{Sh}}) = -c_\pi m(n-1) - d(\widetilde{W}_{\mathrm{Sh}})$. Then*

$$Z_m(s, W_{\mathrm{Sh}}, W^0_{\mathrm{Sh}}(\underline{x})) = \sum_{d(W_{\mathrm{Sh}}) \le d \le d'(W_{\mathrm{Sh}})} x^d Z_{m,d}(W_{\mathrm{Sh}};\underline{x})$$

is a finite sum. Moreover, we have a functional equation

$$\begin{aligned}
&\sum_{d(\widetilde{W}_{\mathrm{Sh}}) \le d \le d'(\widetilde{W}_{\mathrm{Sh}})} q^{-m} x^{-d} Z_{m,d}(\widetilde{W}_{\mathrm{Sh}};\underline{x}^{-1}) \\
&= \varepsilon_0^{m(n-1)} q^{\frac{c_\pi m(m-1)(n-1)}{2}} x^{c_\pi m(n-1)} \left(\prod_{i=1}^{m} \prod_{j=1}^{n-1} x_{i,j}^{c_\pi} \right) \sum_{d(W_{\mathrm{Sh}}) \le d \le d'(W_{\mathrm{Sh}})} x^d Z_{m,d}(W_{\mathrm{Sh}};\underline{x}).
\end{aligned}$$

Now we prove Proposition 9.6.

Proof of Proposition 9.6. First, we show that the essential vector $W^{\mathrm{ess}}_{\mathrm{Sh}}$ is $K(c_\pi)$-invariant. By Proposition 9.8, we have

$$Z_m(m-s, \widetilde{W}^{\mathrm{ess}}_{\mathrm{Sh}}, \widetilde{W}^0_{\mathrm{Sh}}(\underline{x})) = \varepsilon_0^{m(n-1)} q^{\frac{c_\pi m(m-1)(n-1)}{2}} x^{c_\pi m(n-1)} \left(\prod_{i=1}^{m} \prod_{j=1}^{n-1} x_{i,j}^{c_\pi} \right),$$

where $x = q^{-s}$. Set $a = \varpi^{c_\pi} \mathbf{1}_{(n-1)m}$, which is in the centre of G'. We notice that $|\det a^{-1}|^{\frac{m}{2}-s} = q^{\frac{c_\pi m^2(n-1)}{2}} x^{c_\pi m(n-1)}$ and

$$\widetilde{W}^0_{\mathrm{Sh}}(ga^{-1};\underline{x}) = \left(\prod_{i=1}^{m} \prod_{j=1}^{n-1} x_{i,j}^{c_\pi} \right) \widetilde{W}^0_{\mathrm{Sh}}(g;\underline{x}).$$

If we define $W'_{\mathrm{Sh}} \in \mathcal{W}^{\psi^{-1}}_{\mathrm{Sh}}(\widetilde{\sigma})$ by

$$W'_{\mathrm{Sh}}(g) = W^{\mathrm{ess}}_{\mathrm{Sh}}(g \cdot \iota(a)),$$

then it is right $\iota(K')$-invariant and

$$Z_m(m-s, W'_{\mathrm{Sh}}, \widetilde{W}^0_{\mathrm{Sh}}(\underline{x})) = q^{\frac{c_\pi m^2(n-1)}{2}} x^{c_\pi m(n-1)} \left(\prod_{i=1}^{m} \prod_{j=1}^{n-1} x_{i,j}^{c_\pi} \right).$$

By Lemma 9.3 and Theorem 8.2, we see that $\widetilde{W}^{\mathrm{ess}}_{\mathrm{Sh}} = C W'_{\mathrm{Sh}}$ for some constant C. Hence

$$W^{\mathrm{ess}}_{\mathrm{Sh}}(g) = C \widetilde{W}'_{\mathrm{Sh}}(g) = C W^{\mathrm{ess}}_{\mathrm{Sh}}(w_{nm}{}^t g^{-1} w'_n \cdot \iota(\varpi^{c_\pi} \mathbf{1}_{(n-1)m})).$$

Since $W^{\mathrm{ess}}_{\mathrm{Sh}}$ is right $V''(\mathfrak{o})$-invariant, it follows that $W^{\mathrm{ess}}_{\mathrm{Sh}}$ is right ${}^tV''(\mathfrak{p}^{c_\pi})$-invariant. Therefore, we conclude that $W^{\mathrm{ess}}_{\mathrm{Sh}}$ is right $K(c_\pi)$-invariant.

Next, we show that $\dim(\sigma^{K(c_\pi)}) = 1$. If $\mathcal{W}_{\mathrm{Ze}} \in \mathcal{W}^{\psi}_{\mathrm{Ze}}(\sigma)^{K(c_\pi)}$, we have

$$W_{\mathrm{Ze}}|_L \in \left(\bigotimes_{i=1}^{m} \mathcal{W}^{\psi}(\pi|\cdot|^{\frac{(m+1-2i)(n-1)}{2}}) \right)^{K(c_\pi)\cap L},$$

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where the right-hand side is one-dimensional and is spanned by the tensor product of essential vectors. Hence $Z_m(s, \mathcal{T}^\psi W_{\mathrm{Ze}}, W^0_{\mathrm{Sh}}(\underline{x})) = I_m(s, W_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x}))$ does not depend on $s \in \mathbb{C}$ and $\underline{x} = (x_{i,j}) \in M_{m,n-1}(\mathbb{C})$ with $x_{i,j} \in \mathbb{C}^\times$. Using the uniqueness statement in Theorem 9.1, we conclude that $\mathcal{T}^\psi W_{\mathrm{Ze}}$ is a constant multiple of $W^{\mathrm{ess}}_{\mathrm{Sh}}$. □

Since $\mathbb{K}_{nm,\lambda_\sigma}$ is conjugate to $K(c_\pi)$, by Propositions 9.5 and 9.6, we complete the proof of Theorem 2.1 for $\sigma = \mathrm{Sp}(\pi, m)$ such that $L(s,\pi) = 1$. As explained in Section 5.2, this together with results in Sections 6.1, 6.4 and Lemma 7.2 completes Theorem 2.1 in all cases.

To prove Theorem 9.1 in Section 9.3, we use the following special case of Theorem 2.1.

Corollary 9.9. *Let π be an irreducible tempered representation of G_n, and set $\sigma = \mathrm{Sp}(\pi, m)$. Then we have*

$$\dim(\sigma^{\mathbb{K}_{nm,\lambda}}) = \begin{cases} 1 & \text{if } \lambda = \lambda_\sigma, \\ 0 & \text{if } \lambda < \lambda_\sigma. \end{cases}$$

9.3. Proof of Theorem 9.1: the case where $L(s,\pi) \neq 1$

Finally, we prove the existence statement in Theorem 9.1 in general. Before doing it, we state the following consequence of Corollary 9.9.

Corollary 9.10. *Let π be an irreducible tempered representation of G_n, and set $\sigma = \mathrm{Sp}(\pi, m)$. Then the restriction map $W_{\mathrm{Ze}} \mapsto W_{\mathrm{Ze}}|_L$ gives an isomorphism of one-dimensional vector spaces*

$$\mathcal{W}^\psi_{\mathrm{Ze}}(\sigma)^{K(c_\pi)} \xrightarrow{\cong} \mathcal{W}^\psi(\pi|\cdot|^{\frac{(m-1)(n-1)}{2}})^{K(c_\pi)} \otimes \cdots \otimes \mathcal{W}^\psi(\pi|\cdot|^{-\frac{(m-1)(n-1)}{2}})^{K(c_\pi)}.$$

Proof. Since the compact open subgroup $K(c_\pi)$ is conjugate to $\mathbb{K}_{nm,\lambda_\sigma}$, we conclude that $\sigma^{K(c_\pi)}$ is one-dimensional. By Lemma 9.4, the restriction map $W_{\mathrm{Ze}} \mapsto W_{\mathrm{Ze}}|_L$ is injective on $\mathcal{W}^\psi_{\mathrm{Ze}}(\sigma)^{K(c_\pi)}$. Since the image is in

$$\left(\bigotimes_{i=1}^m \mathcal{W}^\psi(\pi|\cdot|^{\frac{(m+1-2i)(n-1)}{2}})\right)^{K(c_\pi)\cap L},$$

which is one-dimensional, we obtain the desired isomorphism. □

Proof of the existence statement in Theorem 9.1. By Lemma 8.4 and Corollary 9.10 together with [14, (4.1) Théorème], we can find $W^{\mathrm{ess}}_{\mathrm{Ze}} \in \mathcal{W}^\psi_{\mathrm{Ze}}(\sigma)^{K(c_\pi)}$ such that

$$I_m(s, W^{\mathrm{ess}}_{\mathrm{Ze}}, W^0_{\mathrm{Ze}}(\underline{x})) = \prod_{i=1}^m \prod_{j=1}^{n-1} L\left(s + s_{i,j} - \frac{m-1}{2}, \pi\right).$$

Then $W^{\mathrm{ess}}_{\mathrm{Sh}} = \mathcal{T}^\psi \mathcal{W}^{\mathrm{ess}}_{\mathrm{Ze}}$ satisfies the conditions in Theorem 9.1. □

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