Glasgow Math. J. **51A** (2009) 157–163. © 2009 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004874. Printed in the United Kingdom

INTEGRABLE LATTICE MAPS: Q_V, A RATIONAL VERSION OF Q_4

CLAUDE M. VIALLET

LPTHE, Centre National de la Recherche Scientifique, UPMC Université Paris 06
Boîte 126/4 Place Jussieu, F-75252 Paris Cedex 05, France
e-mail: viallet@lpthe.jussieu.fr

Abstract. We describe a family of integrable lattice maps related to the known quad maps Q_4. The integrability criterion we use is the vanishing of the algebraic entropy. The family has the advantage of being parametrized rationally: all its parameters are unconstrained.

2000 *Mathematics Subject Classification.* 37K10, 52C99, 39A10.

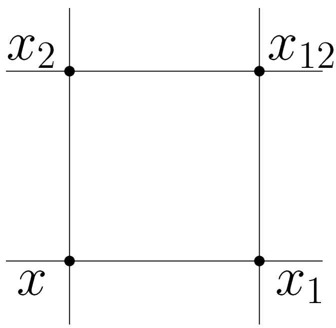
1. Contents. We give a rational form of a generic two-dimensional 'quad' map, containing the so-called Q_4 case [**1**–**4**, **9**, **13**], but whose coefficients are free. Its integrability is proved using the calculation of algebraic entropy.

We first explain the setting, i.e. what are two-dimensional lattice maps on a square lattice (quad maps), and describe two characteristics of integrability of such systems, respectively *Lax pair and consistency* [**6**, **13**], with the important (generic) example Q_4, and *vanishing of algebraic entropy* [**5**, **10**, **15**, **17**], which, as we will show, provides a natural generalisation of Q_4, baptised Q_V. We explain the factorization process of the iterates at the origin of the vanishing of the entropy, and present some directions for further investigations.

2. The setting. Consider a field x defined on a two-dimensional square lattice: at each vertex of the lattice, the value of x is related to the value at neighbouring vertices. The simplest possible relation links the values of x at the four corners of each elementary square plaquette by a multilinear relation

$$\begin{aligned} Q = {} & p_1 \cdot x\, x_1\, x_2\, x_{12} + p_2 \cdot x\, x_1\, x_2 + p_3 \cdot x\, x_1 x_{12} + p_4 \cdot x_1\, x_2\, x_{12} + p_5 \cdot x\, x_2\, x_{12} \\ & + p_6 \cdot x\, x_2 + p_7 \cdot x_1\, x_2 + p_8 \cdot x_2\, x_{12} + p_9 \cdot x\, x_1 + p_{10} \cdot x\, x_{12} + p_{11} \cdot x_1\, x_{12} \\ & + p_{12} \cdot x_2 + p_{13} \cdot x + p_{14} \cdot x_1 + p_{15} \cdot x_{12} + p_{16} = 0, \end{aligned} \tag{1}$$

so that any of the four corner values can be rationally expressed in terms of the other three.

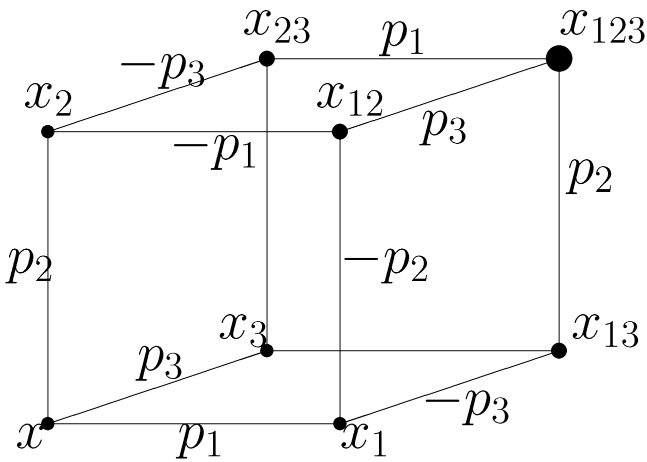


We will be interested in *global* properties of the evolutions defined by this relation.

3. Integrability: Lax pair and consistency around the cube (CAC). Consider the archetypal case of discrete mKdV:

$$p_1\,(x\,x_1 - x_2\,x_{12}) + p_2\,(x\,x_2 - x_1\,x_{12}) = 0$$

It is possible to embed the two-dimensional cell into a three-dimensional one:



where one imposes a similar relation to all faces (the same for opposite faces).

$$p_i\,(x\,x_i - x_j\,x_{ij}) + p_j\,(x\,x_j - x_i\,x_{ij}) = 0, \qquad i, j = 1, 2, 3.$$

The higher-dimensional system is compatible, i.e. *the value of* x_{123} *is independent of the way it is calculated*. This is called consistency around the cube (CAC).

The major output of CAC is that it ensures the existence of a Lax pair, which is accepted as a proof of integrability [**6**, **13**].

4. Consistency around the cube: Q_4**.** While the defining plaquette relation is written on one cell, the CAC relation is written on a loop of cells, and is a *local* relation.

It is a very constraining equation, and is not easy to manipulate: if one takes the most general form of the defining relation Q, the expressions of x_{123} get quite difficult to handle, they are big.

We will be interested in the generic solution of CAC, i.e. the Adler solution [**1**]. Its form has been improved by Nijhoff [**13**] and Hietarinta [**9**]. It was shown to be the generic solution of CAC by Adler–Bobenko–Suris [**2–4**]. The solution was called Q_4. Its different avatars are respectively:

Adler's form:

$$k_0\,x\,x_1\,x_2\,x_{12} - k_1(x\,x_1\,x_2 + x_1\,x_2\,x_{12} + x\,x_2\,x_{12} + x\,x_1\,x_{12}) + k_2(x\,x_{12} + x_1\,x_2)$$
$$- k_3(x\,x_1 + x_2\,x_{12}) - k_4(x\,x_2 + x_1\,x_{12}) + k_5(x + x_1 + x_2 + x_{12}) + k_6 = 0$$

with $\quad k_0 = \alpha + \beta, \quad k_1 = \alpha\nu + \beta\mu, \quad k_2 = \alpha\nu^2 + \beta\mu^2, \quad k_5 = \dfrac{g_3}{2}k_0 + \dfrac{g_2}{4}k_1,$

$$k_6 = \frac{g_2^2}{16}k_0 + g_3 k_1,$$

https://doi.org/10.1017/S0017089508004874 Published online by Cambridge University Press

$$k_3 = \frac{\alpha\beta(\alpha+\beta)}{2(\nu-\mu)} - \alpha\nu^2 + \beta\left(2\mu^2 - \frac{g_2}{4}\right),$$
$$k_4 = \frac{\alpha\beta(\alpha+\beta)}{2(\mu-\nu)} - \beta\mu^2 + \alpha\left(2\nu^2 - \frac{g_2}{4}\right)$$
and $\alpha^2 = r(\mu), \qquad \beta^2 = r(\nu), \qquad r(z) = 4\,z^3 - g_2\,z - g_3.$

Nijhoff's form:

$$A((x-b)(x_2-b)-d)((x_1-b)(x_{12}-b)-d) + B((x-a)(x_1-a)-e)((x_2-a)(x_{12}-a)-e) = f$$

where (a, A), (b, B), $(c, C) = (b, B) - (a, A)$ on the curve $Z^2 = r(z)$,

and $\qquad d = (a-b)\,(c-b), \qquad e = (b-a)\,(c-a), \qquad f = A\;B\;C(a-b).$

Hietarinta's form:

$$sn(\alpha)\,sn(\beta)\,sn(\alpha+\beta)(k^2\,x\,x_1\,x_2\,x_{12}+1) + sn(\alpha+\beta)(x\,x_{12}+x_1\,x_2) - sn(\alpha)(x\,x_1+x_2\,x_{12}) - sn(\beta)(x\,x_2+x_1\,x_{12}) = 0.$$

All three forms are parametrized through elliptic functions. What we will see is that there is another form, where the parameters are free of any constraint. To see that, we will use the notion of algebraic entropy.

5. Algebraic entropy. The space of initial data of the evolutions defined by relation (1) is infinite dimensional: indeed initial data ought to be given on a line which allows the calculation of the values at all points of the lattice. The simplest possible choice is to take a regular diagonal staircase going diagonally (for more details see [**17**]). We then have a notion of iteration of the evolution map, by calculating the values on diagonals moving away from the initial staircase. This defines a sequence of degrees d_n in terms of the initial data, and leads to the entropy

$$\epsilon = \lim_{n\to\infty} \frac{1}{n}\, log(d_n).$$

The outcome of our numerous experiments, as well as what we know for maps [**8**, **11**] leads to the claim that *integrability of the lattice map is equivalent to the vanishing of its entropy*.

6. Q_V. Apply this calculation to Q_4. The most general form of (1) having the same symmetries as Q_4 is

$$a_1\;xx_1x_2x_{12} + a_2\;(xx_2x_{12}+x_1x_2x_{12}+xx_1x_{12}+xx_1x_2) + a_3\;(xx_1+x_2x_{12}) + a_4\;(xx_{12}+x_1x_2) + a_5\;(x_1x_{12}+xx_2) + a_6\;(x+x_1+x_2+x_{12}) + a_7 = 0. \quad (2)$$

Since we use computer algebra to evaluate the sequence of degrees, it is much more efficient to work with integer coefficients. It is easy to find integer coefficients verifying the conditions fulfilled by $\{a_1,\ldots,a_7\}$. For example, choosing $r(z) = 4\,z^3 - 32\,z + 4$ and the points $(a, A) = (0, 2)$, $\quad (c, C) = (3, 4)$, $\quad (b, B) = (a, A) \oplus (c, C) =$

https://doi.org/10.1017/S0017089508004874 Published online by Cambridge University Press

$(-26/9, -2/27)$, we get the sequence $\{d_n\} = \{1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, \ldots\}$, that is to say the *quadratic growth*

$$d_n = 1 + n\,(n-1).$$

But we may also take this form *without any constraint on the coefficients* $\{a_1, \ldots, a_7\}$. *With arbitrary values of the parameters, we get the same quadratic growth as with constrained values*:

$$\{d_n\} = \{1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, \ldots\}$$

fitted with the generating function

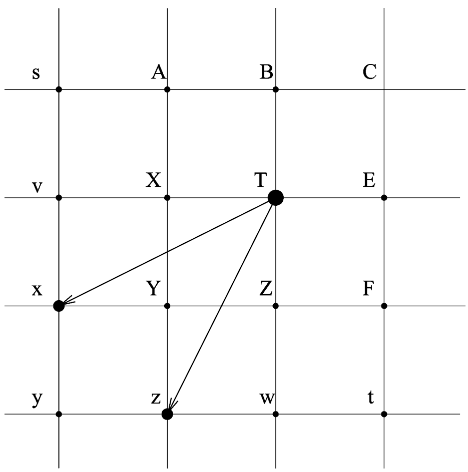
$$g(s) = \sum_{n=0}^{\infty} d_n\, s^n = \frac{1+s^2}{(1-s)^3} \quad \text{and} \quad d_n = 1 + n\,(n-1),$$

as we have checked with a number of randomly chosen parameters. This indicates *integrability of the unconstrained form*, with seven free homogeneous parameters (intersection of hyperplanes in the space of multilinear relations). This is what we call Q_V.

REMARK 1. The sequence of degrees verifies a finite recursion relation $d_n = 3\,d_{n-1} - 3\,d_{n-2} + d_{n-3}$ This means that the global behaviour of the sequence degrees is dictated by a local condition.

REMARK 2. One may wonder about the nature of the additional parameters of Q_V compared to Q_4. Three or these parameters come from the Moebius symmetry of the problem. Since there are seven homogeneous parameters in Q_V, we are left with exactly the three 'true' parameters of Q_4.

7. Factorization. To analyse the origin of the entropy vanishing, one has to examine the factorization process, which explains the degree drop. Consider the following corner:



https://doi.org/10.1017/S0017089508004874 Published online by Cambridge University Press

Suppose initial data $(s, v, x, y, z, w, t, \ldots)$ are given on the two axes. One can calculate the degree d_{ij} at site ij and get, for Q_V

$$d_{ij} = 1 + 2\,i\,j.$$

The diagonal degree growth is quadratic $(d_n = 1 + 2\,n^2)$ = integrability.

If we now evaluate X, Y, Z, T for generic Q, with 16 independent coefficients, as in (1), we find

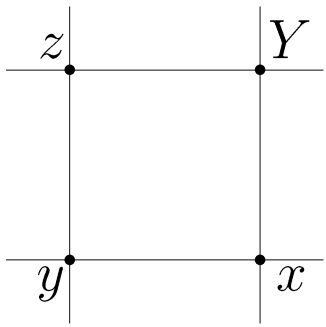
$$\deg(Y) = 1 + 1 + 1 = 3, \quad \deg(X) = \deg(Z) = \deg(Y) + 1 + 1 = 5$$
$$\deg(T) = \deg(X) + \deg(Y) + \deg(Z) = 13.$$

What happens with Q_V is that there is a factorization

$$T = \frac{H(x,z) \cdot P(x,y,z,u,v)}{H(x,z) \cdot Q(x,y,z,u,v)} \simeq \frac{P}{Q}$$

$$\deg(T) = \deg(X) + \deg(Y) + \deg(Z) - \deg(H) = 13 - 4 = 9.$$

The factor $H(x,z)$ defines a bi-quadratic (elliptic) curve. It appears naturally in the singularity analysis: suppose we look at the elementary plaquette



The relation Q gives a projective linear map $\varphi_{xz} : y \longrightarrow Y$, whose inverse φ^{-1} is projective linear. The composed map $\varphi \cdot \varphi^{-1}$ is proportional to the bi-quadratic $H(x,z)$, found in [**3**].

$$\begin{aligned} H(x,z) = {} & (p_{16}p_{10} - p_{15}p_{13}) + (-p_8p_6 + p_{12}p_5)\,x^2 + (p_7p_3 - p_2p_{11} - p_9p_4 + p_{14}p_1)\,z^2x \\ & + (-p_6p_4 - p_2p_8 + p_7p_5 + p_{12}p_1)\,x^2z + (-p_4p_2 + p_7p_1)\,x^2z^2 \\ & + (-p_{11}p_9 + p_{14}p_3)z^2 + (-p_2p_{15} - p_6p_{11} + p_7p_{10} - p_9p_8 + p_{12}p_3 \\ & + p_{16}p_1 - p_{13}p_4 + p_{14}p_5)xz + (p_{16}p_3 - p_{13}p_{11} + p_{14}p_{10} - p_9p_{15})\,z \\ & + (p_{16}p_5 + p_{12}p_{10} - p_{13}p_8 - p_6p_{15})\,x \end{aligned}$$

In the case of Q_V the drop at d_{22} is $13 - 9 = 4$. What factorizes from the iterate is precisely equation of the bi-quadratic $H(x,z)$. The elliptic curve of the known forms of Q_4 is lurking there.

REMARK 3. This does not account for the whole process, and higher-degree curves appear at later steps (total degree 16, degree 4 in x, y, z, and bi-quadratic in v, w). What may however happen is that, due to the specific form of the relation (2), it is sufficient to ensure that the first factorization happens to have them all.

This is the spirit of a systematic analysis we have performed, for quadratic relations and with the additional hypothesis that factors are made out of linear pieces (we know we will not find Q_4 this way). This produced 80 a priori different models. We have run an algebraic entropy test over those, and finally came out with a short list of integrable cases, and a list of models with non-vanishing entropy [**12**].

Again some local structure, extending over a finite range of elementary cells, ensures a global property (integrability), as may be seen form the existence of a finite recurrence relation on the degrees.

8. Conclusion and perspectives. In the setting we used, which is strongly constrained (multilinearity of the elementary relation, birationality of the evolution), a local property is good enough to ensure integrability.

About the rationality versus elliptic nature of the parametrization, the phenomenon is apparently the same as the one we saw [**10**] for the celebrated Baxter's solution of the Yang–Baxter equations. There exists a rational form of Baxter's R-matrix. It is gauge equivalent to the usual elliptic form, which reappears when one request a symmetric form of the solution.

This phenomenon invites us to examine again the 'Yang–Baxter maps' constructed from lattice maps [**7**, **14**, **16**].

Finally, Q_V will be useful if one wants to look at the possible "de-autonomisations" of Q_4.

REFERENCES

1. V. E. Adler, Bäcklund transformation for the Krichever–Novikov equation, *Int. Math. Res. Notices* **1** (1998), 1–4. arXiv:solv-int/9707015.

2. V. E. Adler, A. I. Bobenko and Y. B. Suris, *Classification of integrable equations on quad-graphs. The consistency approach*. Comm. Math. Phys. **233**(3) (2003), 513–543. arXiv:nlin.SI/0202024.

3. V. E. Adler, A. I. Bobenko and Y. B. Suris, Discrete nonlinear hyperbolic equations. Classification of integrable cases, *Funct. Anal. Appl.* (to appear). arXiv:0705.1663.

4. V. E. Adler and Y. B. Suris, Q_4: Integrable master equation related to an elliptic curve, *Int. Math. Res. Notices* **47** (2004), 2523–2553. arXiv:nlin.SI/0309030.

5. M. Bellon and C-M. Viallet, Algebraic Entropy, *Comm. Math. Phys.* **204** (1999), 425–437. chao-dyn/9805006.

6. A. I. Bobenko and Y. B. Suris, Integrable systems on quad-graphs, *Int. Math. Res. Notices* **11** (2002), 573–611. arXiv:nlin/0110004v1 [nlin.SI].

7. V. G. Drinfeld. On some unsolved problems in quantum group theory. In Quantum groups, volume 1510 of *Lecture Notes in Math*, pp. 1–8 (Springer, Berlin, 1992).

8. G. Falqui and C.-M. Viallet, Singularity, complexity, and quasi–integrability of rational mappings, *Comm. Math. Phys.* **154** (1993), 111–125. hep-th/9212105.

9. J. Hietarinta, Searching for CAC-maps, *J. Nonlinear Math. Phys.* **12** (2005), 223–230.

10. J. Hietarinta and C.-M. Viallet. *On the parametrization of solutions of the Yang–Baxter equations*. (q-alg/9504028) (1995). Available from http://arxiv.org/abs/q-alg/9504028.

11. J. Hietarinta and C.-M. Viallet, Singularity confinement and chaos in discrete systems, *Phys. Rev. Lett.* **81**(2) (1998), 325–328. solv-int/9711014.

12. J. Hietarinta and C-M. Viallet, Searching for integrable lattice maps using factorization. *P. Phys. A* **40** (2007), 12629–12643. arXiv:0705.1903.

13. F. Nijhoff, Lax pair for the Adler (lattice Krichever–Novikov) system, *Phys. Lett. A* **297** (2002), 49–58. arXiv:nlin.SI/0110027.

14. A. G. Tongas V. G. Papageorgiou, Yang–Baxter maps and multi-field integrable lattice equations, *J. Phys. A* **40** (2007), 12677–12690. arXiv:math/0702577.

15. S. Tremblay, B. Grammaticos and A. Ramani, Integrable lattice equations and their growth properties. *Phys. Lett. A* **278** (2001), 319–324.

16. A. P. Veselov, Yang–Baxter maps and integrable dynamics, *Phys. Lett. A* **314** (2003), 214–221.

17. C-M. Viallet. *Algebraic entropy for lattice equations*. Available from http://arxiv.org/abs/math-ph/0609043.