1

The pinch technique at one loop

In this chapter, we present in detail the pinch technique (PT) construction at one loop for a QCD-like theory, where there is no tree-level symmetry breaking (no Higgs mechanism). The analysis applies to any gauge group ($SU(N)$, exceptional groups, etc.); however, for concreteness, we will adopt the QCD terminology of quarks and gluons.

This introductory chapter and Chapter 2 go into both conventional technology and the pinch technique only at the one-loop level. Here, the reader will find an almost self-contained guide to the one-loop pinch technique with many calculational details plus some hints at the nonperturbative ideas used in later chapters (where nonperturbative effects will be studied by dressing the loops, i.e., using a skeleton expansion).

1.1 A brief history

Non-Abelian gauge theories (NAGTs) had been around for a long time when the pinch technique came into play [1, 2, 3, 4]. Their first use was in defining the one-loop PT gauge-boson propagator as a construct taken from some gauge-invariant object by combining parts of conventional Feynman graphs while preserving gauge invariance and other physical properties. The term *pinch technique* was introduced later [4], in a paper that extended the one-loop pinch technique to the three-gluon vertex. The name comes from a characteristic feature of the pinch technique, in which the needed parts of some Feynman graphs look as though a particular propagator line had been pinched out of existence. In all these early papers, only one-loop phenomena were studied, including a one-dressed-loop Schwinger–Dyson equation for the PT propagator. This equation showed how the infrared singularities arising because of asymptotic freedom (= infrared slavery) require dynamical gluon mass generation. Of course, the pinch technique should lead to unique results. These

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considerations followed from five requirements for all PT Green's functions not involving ghosts:

1. All Green's functions are independent of any gauge-fixing parameters.
2. All Green's functions are independent of the particular *S*-matrix process used to define them.
3. All Green's functions obey Ward identities of QED type, not involving ghosts.
4. All Green's functions obey dispersion relations in which there are no identifiable ghost contributions or threshholds.
5. The discontinuities (imaginary parts) of Green's functions can be calculated with the usual Cutkosky rules, consistent with unitarity for the *S*-matrix.

All these are properties of Green's functions in the background-field Feynman gauge, later shown to be equivalent to the pinch technique.

One remark concerning the imaginary parts and unitarity is in order. The photon propagator of QED satisfies a Källen–Lehmann representation with a positive spectral function, a property intimately related to the positivity of the beta function of QED. Because this beta function is negative for an asymptotically free theory, it is impossible to find a NAGT gauge-boson propagator with a positive spectral function, so unitarity holds in a generalized form, with some negative contributions to spectral functions. However, as pointed out in Section 1.7, special properties of the PT propagator allow its factorization into two terms, each obeying the Källen–Lehmann representation.[1] This factorization allows the rearrangement of PT Schwinger–Dyson equations into a form in which all necessary positivity constraints are realized.

At the beginning, how to extend the pinch technique to higher orders of perturbation theory was far from clear; the pioneering technology defined in the first papers would have been forbiddingly difficult for graphs with two or more loops. Fortunately, the problem of the all-order pinch technique has a solution that can be stated with remarkable simplicity: all that has to be done, as was shown [5, 6], is to calculate conventional Feynman graphs using the background-field methodology [7] in the Feynman gauge. The original proof was for NAGTs such as QCD, but it was extended [8] to all orders of electroweak theory. This work was inspired by remarks [9, 10, 11] to the effect that the original pinch technique and the background-field Feynman gauge gave exactly the same results at one loop in perturbation theory. This, of course, could have been a coincidence without much

[1] The product of two functions obeying the Källen–Lehmann representation need not obey it.

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meaning, but the all-order proof showed constructively how the PT requirements were satisfied at all orders in the background-field Feynman gauge.[2]

In roughly the same time period, string-theory workers [12] studied the off-shell extrapolation of string-theory amplitudes in the field theory, or zero Regge slope, limit. By imposing a consistent implementation of modular invariance, these workers showed that the off-shell gauge-theory amplitudes derived from string theory were automatically given in the background-field Feynman gauge–equivalent to the pinch technique.

The results showing the equivalence of the pinch technique and the background-field Feynman gauge set the stage for nonperturbative applications of the pinch technique, including the Schwinger–Dyson equations of the pinch technique and their consequences. The output of any PT calculation is not only independence of any gauge-fixing parameter but also freedom from contamination by unphysical objects. For example, if one tries to find the contributions of gauge-invariant condensates such as $\langle \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} \rangle$ to the usual gauge-boson propagator, one discovers that they are inextricably bound with nonphysical and gauge-dependent condensates involving the ghost fields. But for the PT propagator, only the gauge-invariant condensate, field-strength condensate emerges; there are no ghost contributions [13].

1.2 Notation and conventions

Unless explicitly stated otherwise, we adopt the conventions of Peskin and Schröeder [14]. Sometimes, such as in Chapters 7–9 and parts of Chapter 11, it is convenient to work in Euclidean space. The canonical gauge potential $A_\mu^a(x)$ is often combined in the Hermitian matrix form

$$A_\mu(x) = A_\mu^a(x) t^a, \tag{1.1}$$

where t^a are the $SU(N)$ generators satisfying the commutation relations

$$[t^a, t^b] = \mathrm{i} f^{abc} t^c, \tag{1.2}$$

with f^{abc} being the group's totally antisymmetric structure constants. The generators are normalized according to

$$\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}. \tag{1.3}$$

In the case of QCD, the fundamental representation is given by $t^a = \lambda^a/2$, where λ^a are the Gell–Mann matrices.

[2] And in no other background-field gauge; for other than the Feynman gauge, the original PT pinching rules would have to be applied to the background-field Green's functions to get those of the PT.

In Chapters 7, 8, and 9, dealing with nonperturbative phenomena, we combine the gauge potentials in the anti-Hermitean matrix form

$$A_\mu(x) = -\mathrm{i}gA^a_\mu(x)t^a,$$

in which case the matrix potential has a unit mass dimension in all space-time dimensions. The changes in all other definitions are trivial. This definition has many advantages when we go beyond perturbation theory.

The Lagrangian density for a general $SU(N)$ non-Abelian gauge theory is given by

$$\mathcal{L} = \mathcal{L}_{\mathrm{I}} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FPG}}. \tag{1.4}$$

$\mathcal{L}_{\mathrm{I}}$ represents the gauge invariant Lagrangian, namely,

$$\mathcal{L}_{\mathrm{I}} = -\frac{1}{4}G^{\mu\nu}_a G^a_{\mu\nu} + \bar{\psi}^i_{\mathrm{f}}\left(\mathrm{i}\gamma^\mu\mathcal{D}_\mu - m\right)_{ij}\psi^j_{\mathrm{f}}, \tag{1.5}$$

where $a = 1, \ldots, N^2 - 1$ (respectively, $i, j = 1, \ldots, N$) is the color index for the adjoint (respectively, fundamental) representation, and f is the flavor index. The matrix-covariant derivative and field strength are defined according to

$$\mathcal{D}_\mu = \partial_\mu - \mathrm{i}gA_\mu \tag{1.6}$$

$$\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] = -\mathrm{i}gG^a_{\mu\nu}t^a, \tag{1.7}$$

or, more explicitly,

$$(\mathcal{D}_\mu)_{ij} = \partial_\mu(I)_{ij} - \mathrm{i}gA^a_\mu(t^a)_{ij} \tag{1.8}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \tag{1.9}$$

with g being the (strong) coupling constant. Under a local (finite) gauge transformation $V = \exp[-\mathrm{i}\theta]$,

$$A_\mu \to V\frac{\mathrm{i}}{g}\partial_\mu V^\dagger + VA_\mu V^\dagger; \qquad G_{\mu\nu} \to VG_{\mu\nu}V^\dagger; \qquad \psi \to V\psi, \tag{1.10}$$

from which the invariance of $\mathcal{L}_{\mathrm{I}}$ follows. In terms of infinitesimal local gauge transformations,

$$\delta A^a_\mu = -\frac{1}{g}\partial_\mu\theta^a + f^{abc}\theta^b A^c_\mu; \qquad \delta_\theta\psi^i_{\mathrm{f}} = -\mathrm{i}\theta^a(t^a)_{ij}\psi^j_{\mathrm{f}}$$

$$\delta_\theta\bar{\psi}^i_{\mathrm{f}} = \mathrm{i}\theta^a\bar{\psi}^j_{\mathrm{f}}(t^a)_{ji}, \tag{1.11}$$

where $\theta^a(x)$ are the local infinitesimal parameters corresponding to the $SU(N)$ generators t^a.

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To quantize the theory, the gauge invariance needs to be broken; this breakup is achieved through a (covariant) gauge-fixing function $\mathcal{F}^a$, giving rise to the (covariant) gauge-fixing Lagrangian $\mathcal{L}_{\text{GF}}$ and its associated Faddeev–Popov ghost term $\mathcal{L}_{\text{FPG}}$. The most general way of writing these terms is through the Becchi–Rouet–Stora–Tyutin (BRST) operator s [15, 16] and the Nakanishi–Lautrup multipliers B^a [17, 18], which represent auxiliary, nondynamical fields that can be eliminated through their (trivial) equations of motion. Then, one gets

$$\mathcal{L}_{\text{GF}} = -\frac{\xi}{2}(B^a)^2 + B^a\mathcal{F}^a \tag{1.12}$$

$$\mathcal{L}_{\text{FPG}} = -\bar{c}^a s\mathcal{F}^a, \tag{1.13}$$

where

$$\delta_{\text{BRST}}\Phi = \epsilon s\Phi, \tag{1.14}$$

with ϵ being a Grassmann constant parameter and s being the BRST operator acting on the QCD fields according to

$$\begin{aligned} sA^a_\mu &= \partial_\mu c^a + gf^{abc}A^b_\mu c^c; & sc^a &= -\tfrac{1}{2}gf^{abc}c^b c^c \\ s\psi^i_{\text{f}} &= \mathrm{i}gc^a(t^a)_{ij}\psi^j_{\text{f}}; & s\bar{c}^a &= B^a \\ s\bar{\psi}^i_{\text{f}} &= -\mathrm{i}gc^a\bar{\psi}^j_{\text{f}}(t^a)_{ji}; & sB^a &= 0. \end{aligned} \tag{1.15}$$

From the preceding transformations, it is easy to show that the BRST operator is nilpotent: $s^2 = 0$. In addition, as a result, the sum of the gauge-fixing and Faddev–Popov terms can be written as a total BRST variation:

$$\mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FPG}} = s\left(\bar{c}^a\mathcal{F}^a - \frac{\xi}{2}\bar{c}^a B^a\right). \tag{1.16}$$

This, of course, is expected because of the well-known property that total BRST variations cannot appear in the physical spectrum of the theory, which in turn implies the ξ independence of the S-matrix elements and physical observables.

As far as the gauge-fixing function is concerned, there are several possible choices. The ubiquitous R_ξ gauges correspond to the covariant choice

$$\mathcal{F}^a_{R_\xi} = \partial^\mu A^a_\mu. \tag{1.17}$$

In this case, one has

$$\mathcal{L}_{\text{GF}} = \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \tag{1.18}$$

$$\mathcal{L}_{\text{FPG}} = \partial^\mu\bar{c}^a\partial_\mu c^a + gf^{abc}(\partial^\mu\bar{c}^a)A^b_\mu c^c; \tag{1.19}$$

the Feynman rules corresponding to such a gauge are reported in the appendix. One can also consider noncovariant gauge-fixing functions such as

$$\mathcal{F}_n^a = \frac{n^\mu n^\nu}{n^2}\partial_\mu A_\nu^a, \tag{1.20}$$

where n^μ is an arbitrary but constant four vector. In general, we can classify these gauges by the different values of n^2, i.e., $n^2 < 0$ (axial gauges), $n^2 = 0$ (light-cone gauge), and finally, $n^2 > 0$ (Hamilton or time-like gauge). Clearly, the gauge-fixing form of Eq. (1.20) does not work for the light-cone gauge, which needs a separate treatment, given in Section 1.6. In the other cases,

$$\mathcal{L}_{\text{GF}} = \frac{1}{2\xi(n^2)^2}(n^\mu n^\nu \partial_\mu A_\nu^a)^2 \tag{1.21}$$

$$\mathcal{L}_{\text{FPG}} = \frac{n^\mu n^\nu}{n^2}\left[\partial_\mu \bar{c}^a \partial_\nu c^a + g f^{abc}(\partial^\mu \bar{c}^a)A_\nu^b c^c\right]. \tag{1.22}$$

Notice that these noncovariant gauges, as well as the light-cone gauge, are ghost free because the ghosts decouple completely from the S-matrix in dimensional regularization.

Finally, because of the correspondence [9, 10, 11] between the PT and the particular class of gauges known as background field gauges [7], the latter will be described in depth in Chapter 2.

We end this section observing that when dealing with loop integrals, we will use dimensional regularization and employ the shorthand notation

$$\int_k \equiv \mu^\epsilon (2\pi)^{-d}\int d^d k, \tag{1.23}$$

where $d = 4 - \epsilon$ is the dimension of space-time and μ is the ’t Hooft mass scale, introduced to guarantee that the coupling constant is dimensionless in d dimensions. In addition, the standard result,

$$\int_k \frac{1}{k^2} = 0, \tag{1.24}$$

will be used often to set various terms appearing in the PT procedure to zero.

1.3 The basic one-loop pinch technique

We begin with some notation for propagators and a special decomposition for the free three-gluon vertex, a decomposition that also occurs in the background-field method.

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1.3.1 Origin of the longitudinal momenta

Consider the S-matrix element for the quark-quark elastic scattering process $q(p_1)q(r_1) \to q(p_2)q(r_2)$ in QCD. We have that $p_1 + r_1 = p_2 + r_2$ and set $q = r_2 - r_1 = p_1 - p_2$, with $s = q^2$ being the square of the momentum transfer. The longitudinal momenta responsible for triggering the kinematical re-arrangements characteristic of the pinch technique stem either from the bare gluon propagator $\Delta^{(0)}_{\alpha\beta}(k)$ or from the *external* bare (tree-level) three-gluon vertices, i.e., the vertices where the physical momentum transfer q is entering.

To study the origin of the longitudinal momenta in detail, first consider the gluon propagator $\Delta_{\alpha\beta}(k)$; after factoring out the trivial color factor δ^{ab}, in the R_ξ gauges, it takes the form

$$\mathrm{i}\Delta_{\alpha\beta}(q,\xi) = P_{\alpha\beta}(q)\Delta(q^2,\xi) + \xi\frac{q_\alpha q_\beta}{q^4}, \tag{1.25}$$

with $P_{\alpha\beta}(q)$ being the dimensionless transverse projector, defined as

$$P_{\alpha\beta}(q) = g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}. \tag{1.26}$$

The scalar function $\Delta(q^2,\xi)$ is related to the all-order gluon, self-energy

$$\Pi_{\alpha\beta}(q,\xi) = P_{\alpha\beta}(q)\Pi(q^2,\xi), \tag{1.27}$$

through

$$\Delta(q^2,\xi) = \frac{1}{q^2 + \mathrm{i}\Pi(q^2,\xi)}. \tag{1.28}$$

Because $\Pi_{\alpha\beta}$ has been defined in Eq. (1.28) with the imaginary factor i factored out in front, it is simply given by the corresponding Feynman diagrams in Minkowski space. The inverse of $\Delta_{\alpha\beta}$ can be found by requiring that

$$\Delta^{am}_{\alpha\mu}(q,\xi)(\Delta^{-1})^{\mu\beta}_{mb}(q,\xi) = \delta^{ab}g^{\beta}_{\alpha}, \tag{1.29}$$

and it is given by

$$-\mathrm{i}\Delta^{-1}_{\alpha\beta}(q,\xi) = P_{\alpha\beta}(q)\Delta^{-1}(q^2,\xi) + \frac{1}{\xi}q_\alpha q_\beta. \tag{1.30}$$

At tree level,

$$\mathrm{i}\Delta^{(0)}_{\alpha\beta}(q,\xi) = d(q^2)\left[\, g_{\alpha\beta} - (1-\xi)\frac{q_\alpha q_\beta}{q^2}\right] \tag{1.31}$$

$$d(q^2) = \frac{1}{q^2}. \tag{1.32}$$

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Evidently, the longitudinal (pinching) momenta are proportional to the combination $\lambda = 1 - \xi$ and vanish for the particular choice $\xi = 1$ (Feynman gauge) so that the free propagator is simply proportional to $g_{\alpha\beta}d(q^2)$. This is a particularly important feature of the Feynman gauge, which, as we will see, makes PT computations much easier. In this gauge, only longitudinal momenta from vertices can contribute to pinching at the one-loop level. The popular case $\xi = 0$ (Landau gauge) gives rise to a transverse $\Delta^{(0)}_{\alpha\beta}(k)$, which may have its advantages but really complicates the PT procedure at this level.

Next, we consider the conventional three-gluon vertex, to be denoted by $\Gamma^{amn}_{\alpha\mu\nu}(q, k_1, k_2)$, given by the following manifestly Bose-symmetric expression (all momenta are incoming, i.e., $q + k_1 + k_2 = 0$):

$$\mathrm{i}\Gamma^{amn}_{\alpha\mu\nu}(q, k_1, k_2) = gf^{amn}\Gamma_{\alpha\mu\nu}(q, k_1, k_2) \tag{1.33}$$
$$\Gamma_{\alpha\mu\nu}(q, k_1, k_2) = g_{\mu\nu}(k_1 - k_2)_\alpha + g_{\alpha\nu}(k_2 - q)_\mu + g_{\alpha\mu}(q - k_1)_\nu .$$

This vertex satisfies the standard Ward identities:

$$q^\alpha\Gamma_{\alpha\mu\nu}(q, k_1, k_2) = k_2^2 P_{\mu\nu}(k_2) - k_1^2 P_{\mu\nu}(k_1) \tag{1.34}$$
$$k_1^\mu\Gamma_{\alpha\mu\nu}(q, k_1, k_2) = q^2 P_{\alpha\nu}(q) - k_2^2 P_{\alpha\nu}(k_2) \tag{1.35}$$
$$k_2^\nu\Gamma_{\alpha\mu\nu}(q, k_1, k_2) = k_1^2 P_{\alpha\mu}(k_1) - q^2 P_{\alpha\mu}(q). \tag{1.36}$$

Unfortunately, the right-hand side is not the difference of inverse propagators, a defect that shows up in higher orders as the appearance of ghost terms in the identities, now called the Slavnov–Taylor identities.

But it is possible to decompose the vertex in a special way into two pieces, one of which satisfies a Ward identity of an elementary (ghost-free) type and the other contains the only longitudinal momenta capable of generating pinches [1, 19]. In the general ξ gauge, this decomposition, as applied to the vertex of Figure 1.1(b), is

$$\Gamma_{\mu\nu\alpha}(q, k_1, k_2) = \Gamma^{\xi}_{\mu\nu\alpha} + \Gamma^{\mathrm{P}\xi}_{\mu\nu\alpha}, \tag{1.37}$$

where

$$\begin{aligned}\Gamma^{\xi}_{\mu\nu\alpha}(q, k_1, k_2) = {} & (k_1 - k_2)_\alpha g_{\mu\nu} - 2q_\mu g_{\nu\alpha} + 2q_\nu g_{\mu\alpha} \\ & + \left(1 - \frac{1}{\xi}\right)[k_{2\nu} g_{\alpha\mu} - k_{1\mu} g_{\alpha\nu}],\end{aligned} \tag{1.38}$$

and

$$\Gamma^{\mathrm{P}\xi}_{\mu\nu\alpha}(q, k_1, k_2) = \frac{1}{\xi}[k_{2\nu} g_{\alpha\mu} - k_{1\mu} g_{\alpha\nu}]. \tag{1.39}$$

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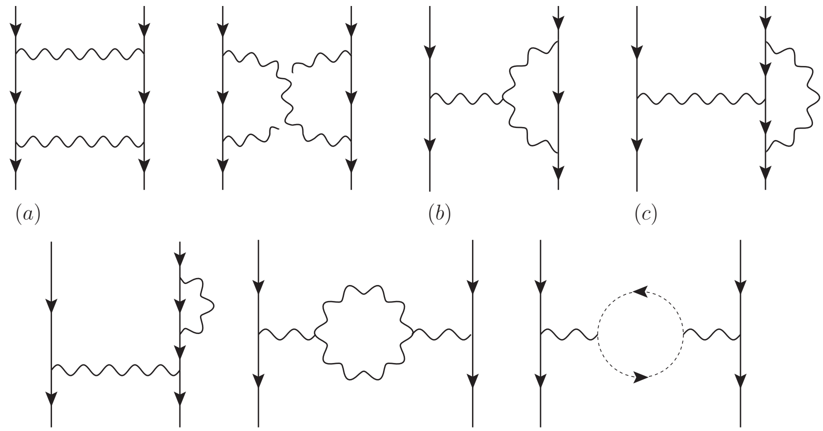

Figure 1.1. The diagrams contributing to the one-loop quark elastic scattering S-matrix element. (a) box contributions, (b) non-Abelian and (c) Abelian vertex contributions (two similar diagrams omitted), (d) quark self-energy corrections (three similar diagrams omitted), and (e) gluon self-energy contributions.

It is easy to check that Γ^{ξ} obeys the elementary Ward identity:

$$q^{\alpha}\Gamma^{\xi}_{\mu\nu\alpha}(q,k_1,k_2) = \Delta^{-1}_{\mu\nu}(k_2,\xi) - \Delta^{-1}_{\mu\nu}(k_1,\xi), \tag{1.40}$$

and that $\Gamma^{\mathrm{P}\xi}$ is the only part of the vertex that triggers pinches. In the pinch technique, (a trivial modification of) this ghost-free Ward identity holds to all orders and has, as a consequence, as in QED, the equality of the gluon wave function and vertex renormalization constants – a relation of great importance for further developments. Note that the vertex $\Gamma^{\xi}_{\alpha\mu\nu}(q,k_1,k_2)$ is Bose symmetric only with respect to the μ and ν legs. Evidently, the preceding decomposition assigns a special role to the q-leg, which is attached to two on-shell lines. In fact, this vertex Γ^{ξ} also occurs in the background-field method (see the appendix).[3]

It would be possible to carry out the (one-loop) PT manipulations with this vertex decomposition for any ξ, but, just as for the propagator, things simplify in the Feynman gauge, where a substantial part of Γ^{ξ} vanishes. Because we will use this gauge extensively, we record its vertex decomposition using the notation $\Gamma^{\mathrm{F}} = \Gamma^{\xi=1}$, $\Gamma^{\mathrm{P}\xi=1} = \Gamma^{\mathrm{P}}$. Then,

$$\Gamma_{\alpha\mu\nu}(q,k_1,k_2) = \Gamma^{\mathrm{F}}_{\alpha\mu\nu}(q,k_1,k_2) + \Gamma^{\mathrm{P}}_{\alpha\mu\nu}(q,k_1,k_2), \tag{1.41}$$

[3] Actually, in both the pinch technique and the background-field method, there are two kinds of vertices; at the one-loop level, only the one used here matters.

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with

$$\Gamma^{\mathrm{F}}_{\alpha\mu\nu}(q, k_1, k_2) = (k_1 - k_2)_\alpha g_{\mu\nu} + 2q_\nu g_{\alpha\mu} - 2q_\mu g_{\alpha\nu}, \tag{1.42}$$

$$\Gamma^{\mathrm{P}}_{\alpha\mu\nu}(q, k_1, k_2) = k_{2\nu} g_{\alpha\mu} - k_{1\mu} g_{\alpha\nu}, \tag{1.43}$$

and this allows $\Gamma^{\mathrm{F}}_{\alpha\mu\nu}(q, k_1, k_2)$ to satisfy the Ward identity

$$q^\alpha \Gamma^{\mathrm{F}}_{\alpha\mu\nu}(q, k_1, k_2) = (k_2^2 - k_1^2) g_{\mu\nu}, \tag{1.44}$$

where the right-hand side is the difference of two inverse propagators in the Feynman gauge.

1.3.2 The basic pinch operation

The term *pinch* arises from the operation of longitudinal momenta, such as in Γ^{P}, on vertices, which triggers Ward identities that lead to the cancellation of a preexisting propagator by an inverse propagator coming from the Ward identity. The resulting graph looks like a Feynman graph from which one line has been removed, as if it had been pinched out.

Whether acting on a vertex or a box diagram, as in Figure 1.1, the effect of the pinching momenta, regardless of their origin (gluon propagator or three-gluon vertex), is to trigger the elementary Ward identity

$$k_\nu \gamma^\nu = (\not{k} + \not{p} - m) - (\not{p} - m), \tag{1.45}$$

where the right-hand side (rhs) is the difference of two inverse tree-level quark propagators. The first of these terms cancels (pinches out) the internal tree-level fermion propagator $S^{(0)}(k + p)$, and the second term on the rhs vanishes when hitting the on-shell external leg. Diagrammatically speaking, an unphysical effective vertex appears in the place where $S^{(0)}(k + p)$ was, i.e., a vertex that does not appear in the original Lagrangian; as we will see, all such vertices cancel in the full, gauge-invariant amplitude.

First of all, it is immediate to verify the cancellation of the ξ-dependent terms at tree level. After extracting a kinematic factor of the form

$$\mathrm{i}\mathcal{V}^{a\alpha}(p_1, p_2) = \bar{u}(p_1)\mathrm{i}g t^a \gamma^\alpha u(p_2), \tag{1.46}$$

the tree-level amplitude reads

$$\mathcal{T}^{(0)} = \mathrm{i}\mathcal{V}^{a\alpha}(r_1, r_2)\mathrm{i}\Delta^{(0)}_{\alpha\beta}(q)\mathrm{i}\mathcal{V}^{a\beta}(p_1, p_2). \tag{1.47}$$

Then, because the on-shell spinors satisfy the equations of motion

$$\bar{u}(p)(\not{p} - m) = 0 = (\not{p} - m)u(p), \tag{1.48}$$

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the longitudinal part coming from $\Delta^{(0)}_{\alpha\beta}$ vanishes, and we obtain

$$\mathcal{T}^{(0)} = \mathrm{i}\mathcal{V}^{a\alpha}(r_1, r_2)d(q^2)\mathcal{V}^{a}_{\alpha}(p_1, p_2). \tag{1.49}$$

Next, let us concentrate on the two box diagrams, direct and crossed, shown in Figure 1.1(a). The sum of the two graphs gives

$$\begin{aligned}(a) = g^2 \int_k \bar{u}(r_1)\gamma^{\alpha} t^a S^{(0)}(r_2 - k)\gamma^{\rho} t^r u(r_2)\Delta^{(0)}_{\alpha\beta}(k-q)\Delta^{(0)}_{\rho\sigma}(k) \\ \times g^2 \bar{u}(p_1)\left\{\gamma^{\beta} t^a S^{(0)}(p_2+k)\gamma^{\sigma} t^r + \gamma^{\sigma} t^r S^{(0)}(p_1-k)\gamma^{\beta} t^a\right\} u(p_2).\end{aligned} \tag{1.50}$$

To see how the pinch technique works, we now study the action of the longitudinal momenta appearing in the product $\Delta^{(0)}_{\alpha\beta}(k-q)\Delta^{(0)}_{\rho\sigma}(k)$. Look, for example, at the term $k_\rho k_\sigma$ coming from $\Delta^{(0)}_{\rho\sigma}(k)$. Using Eqs. (1.45) and (1.48), we find that the contraction of k_σ with the term contained in the brackets in the second line on the rhs of Eq. (1.50) gives rise to the expression

$$\begin{aligned} g^2\bar{u}(p_1)k_\sigma \{\cdots\}^{\beta\sigma} u(p_2) &= g^2\bar{u}(p_1)\gamma^{\beta}\left\{t^a t^r - t^r t^a\right\} u(p_2) \\ &= \mathrm{i}g^2 f^{arn}\bar{u}(p_1)\gamma^{\beta} t^n u(p_2) \\ &= g f^{arn} P^{\beta}_{\nu}(q)\bar{u}(p_1)\mathrm{i}g\gamma^{\nu} t^n u(p_1) \\ &= \left[g f^{arn} P^{\beta}_{\nu}(q)\right]\mathrm{i}\mathcal{V}^{n\nu}(p_1, p_2). \end{aligned} \tag{1.51}$$

Notice that in the second step, we have used the commutation relation of Eq. (1.2), and in the third step, we have used the fact that for the on-shell process, longitudinal pieces proportional to $q_\beta q_\nu$ may be added without consequence since they vanish because of current conservation, thus converting g^{β}_{ν} to $P^{\beta}_{\nu}(q)$. The term in the last line of Eq. (1.51) couples to the external on-shell quarks as a propagator; evidently all reference to the internal (off-shell) quarks inside the brackets has disappeared. To continue the calculation, (1) multiply the result by k_ρ, (2) contract k_ρ with γ^ρ in the first line of Eq. (1.50), (3) employ again the Ward identity of Eq. (1.45), and (4) use the relation $\mathrm{i} f^{abc} t^a t^b = -1/2 C_A t^c$, where C_A is the Casimir eigenvalues of the adjoint fundamental representation.[4] The final result is a purely propagator-like term, i.e., a term that only depends on q (even though it originates from a box diagram) and couples to the external on-shell quarks as a propagator

[4] We denote with C_A (C_f) the Casimir eigenvalue of the adjoint (fundamental) representations. For $SU(N)$, $C_A = N$ and $C_f = (N^2-1)/2N$.

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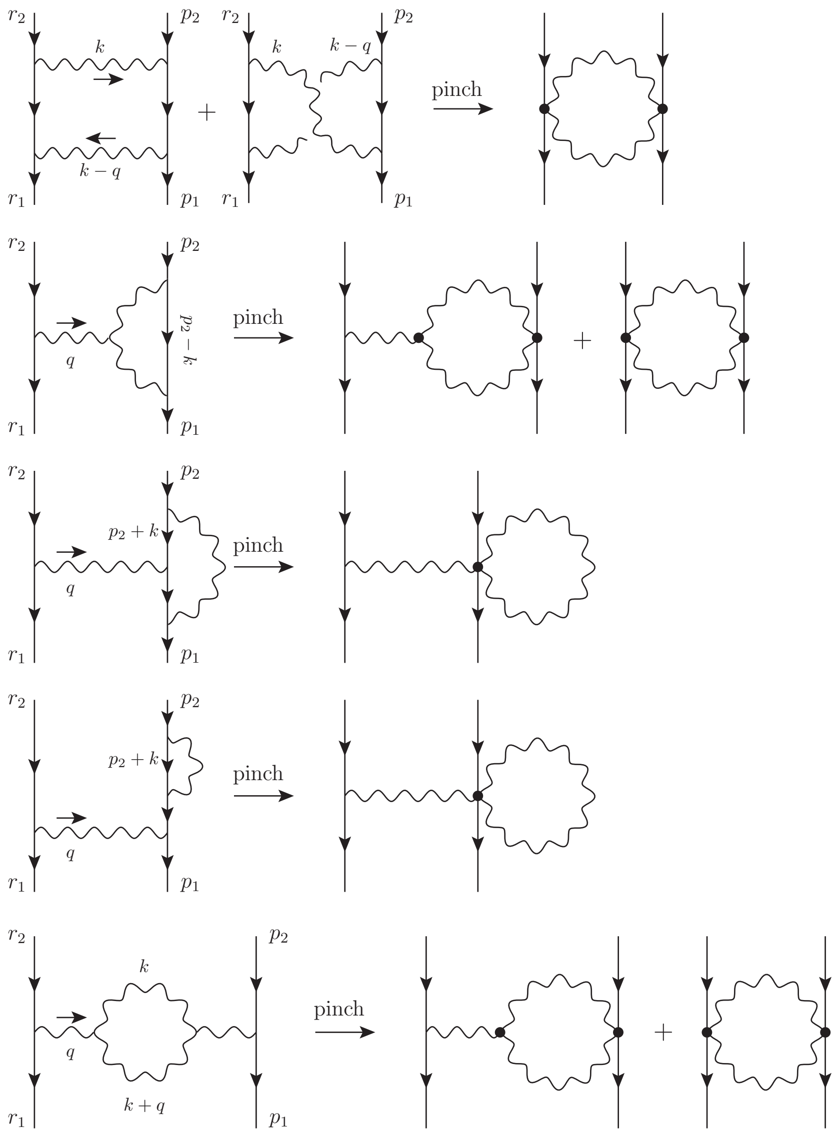

Figure 1.2. The pinching contributions coming from the different one-loop S-matrix diagrams.

(see Figure 1.2). Armed with these observations, it is relatively easy to track down the action of all terms proportional to $(1-\xi)$; in fact, we can write the two boxes as follows:

$$(a) = (a)_{\xi=1} + \mathrm{i}\mathcal{V}^a_\alpha(r_1, r_2)\mathrm{i}d(q^2)\Pi^{\alpha\beta}_{\mathrm{box}}(q, \lambda)\mathrm{i}d(q^2)\mathrm{i}\mathcal{V}^a_\beta(p_1, p_2), \tag{1.52}$$

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Table 1.1. *Contributions of the box, vertex, and self-energy diagrams to the different ξ-dependent structures appearing in the various $\Pi^{\alpha\beta}_{(i)}(q,\lambda)$ terms generated during the PT process*

	$\lambda^2 \int_k \frac{k_\alpha k_\beta}{k^4(k+q)^4}$	$\lambda \int_k \frac{k_\alpha k_\beta}{k^4(k+q)^2}$	$\lambda \int_k \frac{1}{k^2(k+q)^4}$	$\lambda \int_k \frac{1}{k^4}$
$\Pi_{(a)}$	$\frac{1}{2}C_A$	0	$-C_A$	0
$2\Pi_{(b)}$	0	0	0	$C_A - 2C_f$
$2\Pi_{(c)}$	$-C_A$	$2C_A$	$2C_A$	$-2C_A$
$4\Pi_{(d)}$	0	0	0	$2C_f$
$\Pi_{(e)}$	$\frac{1}{2}C_A$	$-2C_A$	$-C_A$	C_A
Total	0	0	0	0

where the ξ-dependent propagator-like term $\Pi^{\alpha\beta}_{\text{box}}$ is given by

$$\Pi^{\alpha\beta}_{\text{box}}(q,\lambda) = \lambda g^2 C_A q^4 \left[\frac{\lambda}{2} P^{\alpha\mu}(q) P^{\beta\nu}(q) \int_k \frac{k_\mu k_\nu}{k^4(k+q)^4} - P^{\alpha\beta}(q) \int_k \frac{1}{k^4(k+q)^2} \right]. \tag{1.53}$$

It turns out that all the ξ-dependent parts isolated using the PT procedure are effectively propagator-like, whether they come from box-, vertex-, or propagator-like diagrams. So the general result is

$$(i) = (i)_{\xi=1} + \mathrm{i}\mathcal{V}^a_\alpha(r_1,r_2)\mathrm{i}d(q^2)\Pi^{\alpha\beta}_{(i)}(q,\lambda)\mathrm{i}d(q^2)\mathrm{i}\mathcal{V}^a_\beta(p_1,p_2), \tag{1.54}$$

where i runs over all possible different topologies appearing in Figure 1.1 ($i = a, b, c, d, e$). The value of the corresponding self-energy-like piece is shown in Table 1.1. The sum of each of its columns is zero, which explicitly shows at one loop the ξ-independence property of S-matrix elements.

In the PT framework, the ξ-dependent terms are eliminated in a very particular way. All ξ-dependent pieces turn out to be propagator-like so that all ξ-dependence has canceled, giving rise to subamplitudes that maintain their original kinematic identity (boxes, vertices, and self-energies) and are, in addition, individually ξ-independent. It is important to appreciate that the explicit cancellation carried out amounts effectively to choosing the Feynman gauge, $\xi = 1$, from the beginning. Of course, there is no doubt that this can be done for the entire physical amplitude considered; the point is that, thanks to the pinch technique, one may move from general ξ to $\xi = 1$ without compromising the notion of individual topologies. Such a notion would have been lost if, for instance, the demonstration of the ξ-independence involved integration over virtual momenta; had the integrations

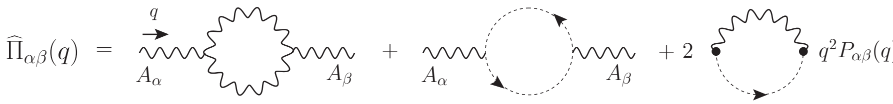

Figure 1.3. Diagrammatic representation of the one-loop pinch technique gluon self-energy $\widehat{\Pi}_{\alpha\beta}$ as the sum of the conventional gluon self-energy terms and the pinch contributions coming from the vertex.

been done first, one would have eventually succeeded in demonstrating the ξ-independence of the entire S-matrix element but would have missed out on the ability to identify ξ-independent subamplitudes, as we did. In addition, this result suggests that there is no loss of generality in choosing $\xi = 1$ from the beginning, thereby eliminating a major source of longitudinal pieces that are bound to cancel anyway through the special pinching procedure outlined earlier.

1.3.3 The pinch technique gluon self-energy at one loop

Next, we construct the PT gluon self-energy, to be denoted by $\widehat{\Pi}_{\alpha\beta}(q)$. It is given by the sum of the conventional and self-energy-like parts extracted from the two vertices, as shown in Figure 1.3, i.e.,

$$\widehat{\Pi}_{\alpha\beta}(q) = \Pi_{\alpha\beta}(q) + 2\Pi^{\mathrm{P}}_{\alpha\beta}(q). \tag{1.55}$$

Specifically, in a closed form,

$$\begin{aligned}\widehat{\Pi}_{\alpha\beta}(q) = \frac{1}{2}g^2 C_A &\left[\int_k \frac{\Gamma_{\alpha\mu\nu}\Gamma^{\mu\nu}_{\beta}}{k^2(k+q)^2} - \int_k \frac{k_\alpha(k+q)_\beta + k_\beta(k+q)_\alpha}{k^2(k+q)^2}\right] \\ &+ 2g^2 C_A \int_k \frac{q^2 P_{\alpha\beta}(q)}{k^2(k+q)^2},\end{aligned} \tag{1.56}$$

where we have symmetrized the ghost contribution for later convenience and neglected the fermion contribution.

It would be elementary to compute $\widehat{\Pi}_{\alpha\beta}$ directly from the rhs of Eq. (1.56). It is very instructive, however, to identify exactly the parts of the conventional $\Pi_{\alpha\beta}$ that combine with (and eventually cancel) the term $\Pi^{\mathrm{P}}_{\alpha\beta}$. To make this cancellation manifest, one may carry out the following rearrangement of the two elementary three-gluon vertices appearing in Eq. (1.56):

$$\Gamma\Gamma = \Gamma^{\mathrm{F}}\Gamma^{\mathrm{F}} + \Gamma^{\mathrm{P}}\Gamma + \Gamma\Gamma^{\mathrm{P}} - \Gamma^{\mathrm{P}}\Gamma^{\mathrm{P}}, \tag{1.57}$$

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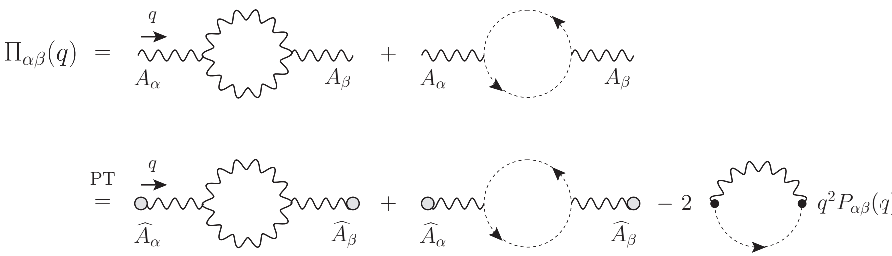

Figure 1.4. The conventional one-loop gluon self-energy before (first line) and after (second line) the pinch technique rearrangement. A shaded circle at the end of an external gluon line denotes that the corresponding gluon behaves as if it were a background gluon.

where, in this symbolic equation, all Lorentz indices have been suppressed, and the product of any two Γs means

$$\Gamma\Gamma \to \Gamma_{\alpha\mu\nu}\Gamma_{\beta}^{\mu\nu}. \tag{1.58}$$

Dropping terms leading to tadpolelike diagrams (which vanish by dimensional regularization), one then finds

$$\Gamma^{\mathrm{P}}\Gamma + \Gamma\Gamma^{\mathrm{P}} = -4P_{\alpha\beta}(q)q^2 - 2k_\alpha k_\beta - 2(k+q)_\alpha(k+q)_\beta \tag{1.59}$$

$$\Gamma^{\mathrm{P}}\Gamma^{\mathrm{P}} = 2k_\alpha k_\beta + (k_\alpha q_\beta + q_\alpha k_\beta). \tag{1.60}$$

We see that the conventional gluon self-energy can be cast in the following form (see also Figure 1.4):

$$\begin{aligned}\Pi^{(1)}_{\alpha\beta}(q) = \frac{1}{2}g^2 C_A \left[\int_k \frac{\Gamma^{\mathrm{F}}_{\alpha\mu\nu}\Gamma^{\mathrm{F}\,\mu\nu}_{\beta}}{k^2(k+q)^2} - 2\int_k \frac{(2k+q)_\alpha(2k+q)_\beta}{k^2(k+q)^2}\right] \\ - 2g^2 C_A \int_k \frac{q^2 P_{\alpha\beta}(q)}{k^2(k+q)^2}.\end{aligned} \tag{1.61}$$

It is easy to prove, using the vanishing of one-loop tadpoles, that each term appearing in the preceding equation is individually conserved so that we have the first ghost-free Ward identity:

$$q^\alpha \widehat{\Pi}_{\alpha\beta} = 0. \tag{1.62}$$

The PT re-arrangement has created three manifestly transverse structures, all admitting a unique diagrammatic representation and field theoretical interpretation: the first two terms have in fact precisely the structure of the background-field Feynman gauge at one loop (studied in Chapter 2), whereas the last term represents the one-loop version of a very special auxiliary function that will be thoroughly studied

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when extending the algorithm to the Schwinger-Dyson equations of non-Abelian gauge theories (Chapter 6). It exactly cancels the pinching contribution coming from the vertex (see Eq. (1.56)) so that one is left with the result

$$\widehat{\Pi}_{\alpha\beta}(q) = \frac{1}{2}g^2 C_A \left[\int_k \frac{\Gamma^{\mathrm{F}}_{\alpha\mu\nu}\Gamma^{\mathrm{F}\,\mu\nu}_{\beta}}{k^2(k+q)^2} - \int_k \frac{2(2k+q)_\alpha(2k+q)_\beta}{k^2(k+q)^2} \right]. \tag{1.63}$$

Using

$$\Gamma^{\mathrm{F}}_{\alpha\mu\nu}\Gamma^{\mathrm{F}\,\mu\nu}_{\beta} = d(2k+q)_\alpha(2k+q)_\beta + 8q^2 P_{\alpha\beta}(q), \tag{1.64}$$

and

$$\int_k \frac{(2k+q)_\alpha(2k+q)_\beta}{k^2(k+q)^2} = -\left(\frac{1}{d-1}\right)\int_k \frac{q^2 P_{\alpha\beta}(q)}{k^2(k+q)^2}, \tag{1.65}$$

the one-loop PT propagator can be written in the simple form

$$\widehat{\Pi}_{\alpha\beta}(q) = \frac{1}{2}g^2 C_A \left(\frac{7d-6}{d-1}\right)\int_k \frac{q^2 P_{\alpha\beta}(q)}{k^2(k+q)^2}, \tag{1.66}$$

valid at $d = 3, 4$. Writing

$$\widehat{\Pi}_{\alpha\beta}(q) = P_{\alpha\beta}(q)\widehat{\Pi}(q^2), \tag{1.67}$$

and following the standard integration rules for the Feynman integral, we obtain the following for the unrenormalized $\widehat{\Pi}$ in $d = 4$:

$$\widehat{\Pi}(q^2) = \mathrm{i}bg^2q^2\left[\frac{2}{\epsilon} + \ln 4\pi - \gamma_E - \ln\frac{q^2}{\mu^2} + \frac{67}{33}\right], \tag{1.68}$$

where γ_E is the Euler–Mascheroni constant ($\gamma_E \approx 0.57721$) and

$$b = \frac{11C_A}{48\pi^2} \tag{1.69}$$

is the gauge-invariant one-loop coefficient of the β function of QCD ($\beta = -bg^3$) in the absence of quark loops[5] (for the $d = 3$ case, see Chapter 9).

The gauge-invariant constant b in front of the logarithm corresponds to an analogous gauge-invariant number in the vacuum polarization of QED, where the corresponding coefficient is $-\alpha/3\pi$; of course, the difference in the sign occurs because QCD is asymptotically free whereas QED is not. That the PT gluon propagator captures the leading renormalization group (RG) logarithms is a direct consequence of the Ward identity Eq. (1.92) and the consequent relation $\widehat{Z}_1 = \widehat{Z}_2$. (This means [3] that

[5] For comparison, the standard one-loop Feynman self-energy replaces b by $(C_A/32\pi^2)(13/3-\xi)$, which is obviously gauge dependent and not yielding the correct coefficient b even in the Feynman gauge.

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the product of the unrenormalized charge g_0^2 and the unrenormalized PT propagator is the same as this product of renormalized quantities – again, just as in QED – and so this product defines a coupling constant-propagator combination that is not only gauge invariant but renormalization group invariant.) Indeed, if $\widehat{Z}_1 = \widehat{Z}_2$, then the charge renormalization constant, Z_g, and the wave function renormalization of the PT gluon self-energy, $\widehat{Z}_A$, are related by $Z_g = \widehat{Z}_A^{-1/2}$, exactly as in QED.

Finally, notice that since both the $\Gamma^{\mathrm{F}}\Gamma^{\mathrm{F}}$ term and the ghostlike term are separately conserved, the ghostlike term is no longer needed to satisfy the Ward identity for the proper self-energy (Eq. (1.62)). One might ask what quantitative difference it makes to drop the ghostlike term, which, in a Schwinger–Dyson context, would amount to a *truncation* of the series. The answer it that without the ghostlike term, the proper self-energy has exactly the same transverse form but is 10/11 times the self-energy with the ghostlike term. Interestingly, the pinch technique already offers at one loop the ability to truncate gauge invariantly, i.e., preserving the transversality of the truncated answer. This will have profound consequences when addressing the issue of devising a gauge-invariant truncation scheme for the Schwinger–Dyson equations of QCD (Chapter 6).

1.4 Another way to the pinch technique

So far, to develop the pinch technique, we have used a specific S-matrix element for quark-quark scattering. The question naturally arises: is the PT result found this way independent of the process used to define it? This is surely what we must expect from any sensible definition of Green's functions. The answer is yes, as we indicate in the next subsection. Another natural question, given the answer to the first question, is whether one can define the pinch technique in an intrinsically process-independent way. Again, the answer is yes. We do not make any reference to background-field techniques, where the answer to both questions is clearly yes.

1.4.1 Process independence

It is important to stress at this point that the only completely off-shell Green's function involved in the previous construction was the gluon self-energy; instead, the quark-gluon vertex has the incoming gluon off shell and the two quarks on shell, whereas the box has all four incoming quarks on shell. These latter quantities were also made ξ independent in the process of constructing the fully off-shell, ξ-independent gluonic two-point function. Similarly, the construction of a fully off-shell PT quark self-energy requires its embedding in a process such as quark-gluon elastic scattering. The generalization of the methodology is now clear; for

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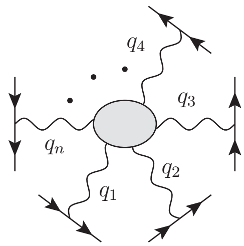

Figure 1.5. S-matrix embedding necessary for constructing a ξ-independent, fully off-shell gluonic n-point function.

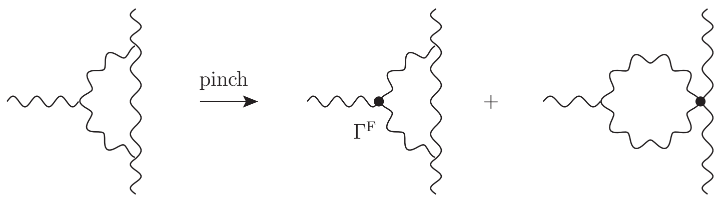

Figure 1.6. The pinching procedure when the embedding particles are *on-shell* gluons. Despite appearances, the vertex by which the pinching contribution is connected to the external gluons is a three-gluon vertex.

example, for constructing a ξ-independent, fully off-shell gluonic n-point function (i.e., with n off-shell gluons), one must consider the entire ξ-independent process consisting of n pairs of quarks, $q(p_1)q(k_1)$, $q(p_2)q(k_2)$, $\ldots$, $q(p_n)q(k_n)$ and hook each gluon A_i to one pair of test quarks; the off-shell momentum transfer q_i of the ith gluonic leg will be $q_i = p_i - k_i$ (see Figure 1.5). Note, however, that one may equally well use gluons as external test particles or as even fictitious scalars carrying color. Provided that the embedding process is ξ-independent, the answer that the pinch technique furnishes for a given fully off-shell n-point function is unique; that is, it is independent of the embedding process. The PT Green's functions are process independent or universal – a property of Green's functions that can hardly be violated. The universality of the one-loop gluon self-energy has been demonstrated through explicit computations using a variety of external test particles. For example, when gluons are used as external test particles, the pinching isolates propagator-like pieces that are attached to the external gluons through a tree-level three-gluon vertex (see Figure 1.6). In this case, the analog of the quark-gluon vertex $\widehat{\Gamma}^a_\alpha$ is the one-loop vertex with one off-shell and two on-shell gluons, which, as we will see in Section 1.5.2, is the one-loop generalization of Γ^{F}. This latter vertex should not be confused with the PT three-gluon vertex with all three gluons off shell, which can be constructed by embedding it into a six-quark process (one pair for each leg), to be discussed in Section 1.4.2. The distinction between

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these two three-gluon vertices is crucial and will be made more explicit later on; in addition, a more precise field-theoretic notation will be adopted that will allow us to distinguish the three-gluon vertices unambiguously.

We emphasize that the PT construction is not restricted to the use of on-shell S-matrix amplitudes and works equally well inside, for example, a gauge-invariant current correlation function or a Wilson loop. This fact is particularly relevant for the correct interpretation of the correspondence between the pinch technique and the background-field method to be discussed in Chapter 2. Actually, in the first PT calculation ever [3], the set of one-loop Feynman diagrams studied were the ones contributing to the *gauge-invariant* Green's function:

$$G(x, y) = \left\langle 0 \left| T \left\{ \mathrm{Tr} \left[\Phi(x)\Phi^{\dagger}(x) \right] \mathrm{Tr} \left[\Phi(y)\Phi^{\dagger}(y) \right] \right\} \right| 0 \right\rangle, \tag{1.70}$$

where $\Phi(x)$ is a matrix describing a set of scalar test particles in an appropriate representation of the gauge group. In this case, the special momentum with respect to which the vertex decomposition of Eq. (1.41) should be carried out (i.e., the equivalent of q in that same equation) is the momentum transfer between the two sides of the scalar loop (i.e., one should count loops as if the Φ loop had been opened at x and y). The advantage of using an S-matrix amplitude is purely operational: the PT construction becomes more expeditious because several terms can be set to zero directly owing to the equation of motion of the on-shell test particles. Instead, in the case of a Wilson loop, one would have to carry out the additional step of demonstrating explicitly their cancellation against other similar terms.

1.4.2 Intrinsic pinch technique

The central achievement of the previous sections has been the construction of the ξ-independent, off-shell gluon self-energy, $\widehat{\Pi}_{\alpha\beta}$, through its embedment into a physical S-matrix element, corresponding to quark-quark elastic scattering. This was accomplished by identifying propagator-like pieces from the vertices and boxes contributing to the embedding process and reassigning them to the conventional gluon self-energy, $\Pi_{\alpha\beta}$. This procedure has been carried out for a general value of the gauge-fixing parameter ξ, leading to a unique answer that is most economically reached by choosing the Feynman gauge from the beginning. Thus $\widehat{\Pi}_{\alpha\beta}$ is obtained by adding to $\Pi_{\alpha\beta}$ the propagator-like pieces $2\Pi^{\mathrm{P}}_{\alpha\beta}$ extracted from the vertices, as shown in Eq. (1.56). In the analysis following Eq. (1.56), it became clear that these latter terms cancel very precise terms of the conventional self-energy $\Pi_{\alpha\beta}$, furnishing, finally, $\widehat{\Pi}_{\alpha\beta}$. Specifically, after the vertex decomposition of Eq. (1.57), the terms Γ^{P} acted on the corresponding Γ, triggering the Ward identities (1.34), (1.35), and (1.36): the term $2\Pi^{\mathrm{P}}_{\alpha\beta}$ cancels against the terms of the Ward identities

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that are proportional to $q^2 P_{\alpha\beta}$. This observation motivates the following, more expeditious, course of action: instead of identifying the propagator-like pieces from the various graphs, focus on $\Pi_{\alpha\beta}$, carry out the decomposition of Eq. (1.57), and discard the terms coming from the Ward identities that are proportional to $q^2 P_{\alpha\beta}$; what is left is then the PT answer.

This alternative, and completely equivalent, approach to pinching was first introduced in [4] and is known as the *intrinsic pinch technique*. Its main virtue is that it avoids as much as possible the embedment of the Green's function under construction into a physical amplitude. As we shall see later, the intrinsic approach is particularly suited for extending the PT construction in the context of the Schwinger–Dyson equation.

1.5 Pinch technique vertices

After the propagator, the next step is, of course, three-leg vertices. We can extract one of the vertices, the quark-gluon vertex, from the graphs of Figure 1.1 in much the same way that we found the pinch technique gluon propagator. The other vertex under consideration here, the three-gluon vertex, is much more complicated and needs a nontrivial extension of the work we have already done. We begin with the quark-gluon vertex.

1.5.1 The one-loop pinch technique quark-gluon vertex and its Ward identity

Let us now turn to the longitudinal terms contained in the pinching part $\Gamma^{\mathrm{P}}_{\alpha\mu\nu}$ of the three-gluon vertex (see Eq. (1.43)) appearing in the non-Abelian vertex graph and two such vertices inside the gluon self-energy graph. One may ask at this point, what is the purpose of carrying the PT decomposition of the vertex given that one has already achieved ξ-independent structures? The answer is that the effect of the pinching momenta of $\Gamma^{\mathrm{P}}_{\alpha\mu\nu}$ is to make the effective ξ-independent Green's functions satisfy ghost-free Ward identities instead of the usual Slavnov–Taylor identities.

This is best seen in the case of the one-loop quark-gluon vertex $\Gamma^a_\alpha(p_1, p_2)$, composed by the graphs of Figures 1.1(b) and 1.1(c), now written (after the gauge-fixing parameter cancellations described earlier) in the Feynman gauge. It is well known that the QED counterpart of $\Gamma^a_\alpha(p_1, p_2)$, namely, the photon-electron vertex $\Gamma_\alpha(p_1, p_2)$, satisfies to all orders (and for every ξ) the Ward identity

$$q^\alpha \Gamma_\alpha(p_1, p_2) = \mathrm{i}e\left[S_e^{-1}(p_1) - S_e^{-1}(p_2)\right], \tag{1.71}$$

$$\mathrm{i}H^a(p,q) = \; -gt^a \quad +$$

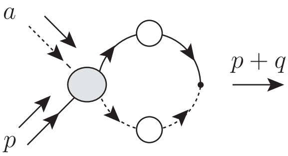

Figure 1.7. The auxiliary function H appearing in the quark-gluon vertex Slavnov–Taylor identity. The shaded blob represents the (connected) ghost-fermion kernel appearing in the usual QCD skeleton expansion.

where S_e is the (all-order) electron propagator; Eq. (1.71) is the naive, all-order generalization of the tree-level identity (1.45).

The quark-gluon vertex $\Gamma_\alpha^a(p_1, p_2)$ also obeys the Ward identity of Eq. (1.45) at tree level (multiplied by t^a):

$$q^\alpha \Gamma_\alpha^a(p_1, p_2) = \mathrm{i}gt^a \left[S^{-1}(p_1) - S^{-1}(p_2)\right]. \tag{1.72}$$

However, at higher orders, it obeys a Slavnov–Taylor identity that is not the naive generalization of this tree-level Ward identity. Instead, $\Gamma_a^\alpha(p_1, p_2)$ satisfies the Slavnov–Taylor identity [20]

$$q^\alpha \Gamma_\alpha^a(p_1, p_2) = \left[q^2 D^{aa'}(q)\right]\left[S^{-1}(p_2)H^{a'}(q, p_1) + \bar{H}^{a'}(p_1, q)S^{-1}(p_2)\right], \tag{1.73}$$

where $D^{aa'}(q)$ and $S(p)$ represent the full ghost and quark propagator, respectively, and H^a is a composite operator defined as (see also Figure 1.7)

$$\begin{aligned}\mathrm{i}S(p)\mathrm{i}D^{aa'}(q)\mathrm{i}H^a(p, q) = -gt^d \int d^4x \int d^4y\, \mathrm{e}^{\mathrm{i}p\cdot x}\, \mathrm{e}^{\mathrm{i}q\cdot y} \\ \times \left\langle 0 \middle| T\left\{\bar{q}(x)\bar{c}^{a'}(y)\left[c^d(0)q(0)\right]\right\} \middle| 0 \right\rangle,\end{aligned} \tag{1.74}$$

where T denotes the time-ordered product of fields and $\bar{H}$ is the Hermitean conjugate of H. At tree level, H_{ij}^a reduces to $H_{ij}^{(0)a} = t_{ij}^a$.

After these general considerations, let us carry out the decomposition of Eq. (1.41) to the non-Abelian vertex of Figure 1.1(b). Then let us write, suppressing again the color indices,

$$(b)_{\xi=1} = \mathrm{i}\mathcal{V}_a^\alpha \mathrm{i}d(q^2)\bar{u}(p_1)\mathrm{i}\widetilde{\Gamma}_\alpha^a(p_1, p_2)u(p_2), \tag{1.75}$$

and concentrate on the (one-loop) non-Abelian contribution to the quark-gluon vertex $\widetilde{\Gamma}_\alpha^a$. We have

$$\mathrm{i}\widetilde{\Gamma}_\alpha^a(p_1, p_2) = \frac{1}{2}g^3 C_A t^a \int_k \frac{\left[\Gamma_{\alpha\mu\nu}^{\mathrm{F}} + \Gamma_{\alpha\mu\nu}^{\mathrm{P}}\right]\gamma^\nu S^{(0)}(p_2 - k)\gamma^\mu}{k^2(k+q)^2}, \tag{1.76}$$

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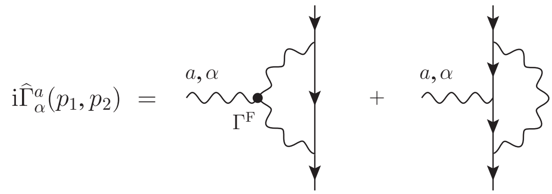

Figure 1.8. Diagrammatic representation of the pinch technique quark-gluon vertex at one loop.

where, in this case,

$$\Gamma^{\mathrm{F}}_{\alpha\mu\nu} = g_{\mu\nu}(2k+q)_\alpha + 2q_\nu g_{\alpha\mu} - 2q_\mu g_{\alpha\nu} \tag{1.77}$$

$$\Gamma^{\mathrm{P}}_{\alpha\mu\nu} = -(k+q)_\nu g_{\alpha\mu} - k_\mu g_{\alpha\nu}. \tag{1.78}$$

Despite appearances, if we use the Dirac equations of motion, the part of the vertex graph containing Γ^{P} is in fact purely propagator-like:

$$\int_k \frac{\Gamma^{\mathrm{P}}_{\alpha\mu\nu}\gamma^\nu S^{(0)}(p_2-k)\gamma^\mu}{k^2(k+q)^2} \xrightarrow[\text{Dirac Eq.}]{\text{PT}} 2\gamma_\alpha \int_k \frac{1}{k^2(k+q)^2}. \tag{1.79}$$

So also, in this case, one obtains from the one-loop, quark-gluon vertex a propagator-like contribution given by

$$\Pi^{\mathrm{P}}_{\alpha\beta}(q) = g^2 C_A \int_k \frac{q^2 P_{\alpha\beta}(q)}{k^2(k+q)^2}. \tag{1.80}$$

As before, this term plus the equal one coming from the mirror vertex ought to be re-assigned to the PT self-energy. Let us then concentrate on the remaining terms in the vertex. In fact, the part of the vertex graph containing Γ^{F} remains unchanged because it has no longitudinal momenta. Adding it to the usual Abelian-like graph, we obtain the one-loop PT quark-gluon vertex, to be denoted by $\widehat{\Gamma}^a_\alpha$, given by (see Figure 1.8)

$$\begin{aligned}\mathrm{i}\widehat{\Gamma}^a_\alpha(p_1, p_2) = g^3 t^a \Bigg[&\frac{1}{2} C_A \int_k \frac{\Gamma^{\mathrm{F}}_{\alpha\mu\nu}\gamma^\nu S^{(0)}(p_2-k)\gamma^\mu}{k^2(k+q)^2} \\ &+ \left(C_f - \frac{C_A}{2}\right)\int_k \frac{\gamma^\mu S^{(0)}(p_1+k)\gamma_\alpha S^{(0)}(p_2+k)\gamma_\mu}{k^2}\Bigg].\end{aligned} \tag{1.81}$$

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Now it is easy to derive the Ward identity that $\widehat{\Gamma}^a_\alpha(p_1, p_2)$ satisfies. Using Eq. (1.44), we get

$$\begin{aligned} q^\alpha \widehat{\Gamma}^a_\alpha(p_1, p_2) &= -\mathrm{i}g^3 C_f t^a \left[\int_k \frac{\gamma^\mu S^{(0)}(p_2+k)\gamma_\mu}{k^2} - \int_k \frac{\gamma^\mu S^{(0)}(p_1+k)\gamma_\mu}{k^2} \right] \\ &= \mathrm{i}g t^a \left\{ \widehat{\Sigma}(p_1) - \widehat{\Sigma}(p_2) \right\}. \end{aligned} \tag{1.82}$$

Clearly, Eq. (1.82) is the naive generalization of Eq. (1.72) at one loop, i.e., the Ward identity satisfied by Γ^a_α at tree level; this makes the analogy with Eq. (1.71) fully explicit. An immediate consequence of Eq. (1.82) is that the renormalization constants of $\widehat{\Gamma}^a_\alpha$ and $\widehat{\Sigma}$, to be denoted by $\widehat{Z}_1$ and $\widehat{Z}_2$, respectively, are related by the relation $\widehat{Z}_1 = \widehat{Z}_2$, which is none other than the textbook relation $Z_1 = Z_2$ of QED realized in a non-Abelian context.

A direct comparison of the Slavnov–Taylor identity (1.73), satisfied by the conventional vertex Γ^a_α, with the Ward identity (1.82), satisfied by the PT vertex $\widehat{\Gamma}^a_\alpha$, suggests a connection between the terms removed from Γ^a_α during the process of pinching and the ghost-related quantities D^{ab} and H^a_{ij}. As we will see in detail in Chapter 4, such a connection indeed exists and is, in fact, of central importance for the generalization of the pinch technique to all orders.

1.5.2 The one-loop, three-gluon vertex and its Ward identity

The S-matrix construction The same general principles used for the propagator also apply to the proper three-gluon (or four-gluon) vertex: choose a convenient S-matrix element in which the vertex is embedded, in our case, at one loop. This S-matrix element has not only the one-loop vertex that we want but many other graphs. Extract the pinch graphs from these and add them to the conventional vertex. The resulting proper vertex $\widehat{\Gamma}_{\alpha\mu\nu}$ is completely gauge invariant (independent of ξ in an R_ξ gauge) and satisfies ghost-free Ward identities involving the gluon PT inverse propagator. It also satisfies all other properties that we could demand of a three-point vertex: complete Bose symmetry, conventional analytic properties with physical threshholds only, and independence of the S-matrix process used to create the vertex.

Figure 1.9 shows a three-quark S-matrix element, with all quark momenta p_i, $(p-q)_i$ on shell, from which we will find the PT proper three-gluon vertex. We denote the one-loop corrections by $\widehat{\Lambda}_{\mu\nu\alpha}$ and find the full one-loop PT vertex by adding the bare vertex. There are two ways to do this: one is to add the conventional fully symmetric bare vertex $\Gamma^{(0)}_{\alpha\mu\nu}$ and the other is to add the free vertex with one line singled out, as in Eq. (1.42). This is immaterial because the only difference is

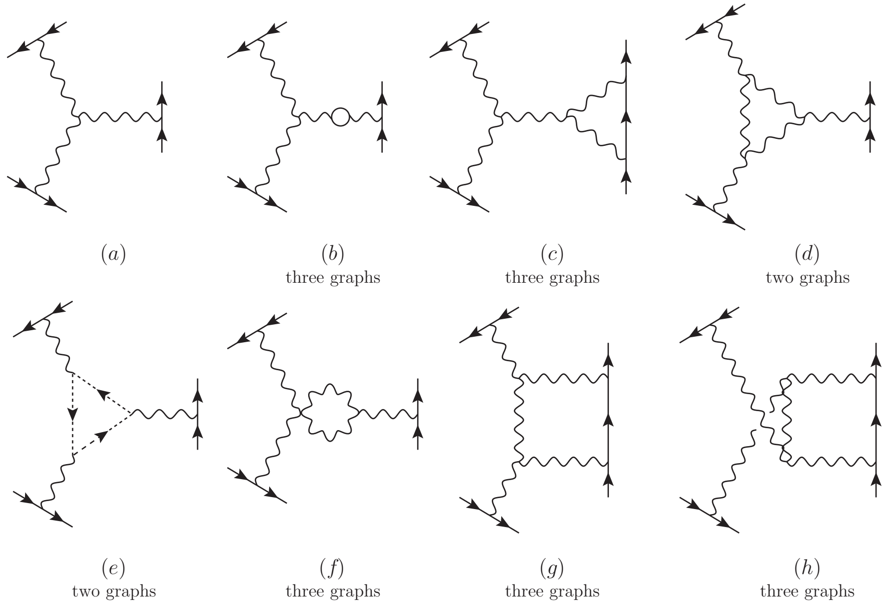

Figure 1.9. S-matrix graphs from which the one-loop pinch technique vertex is derived.

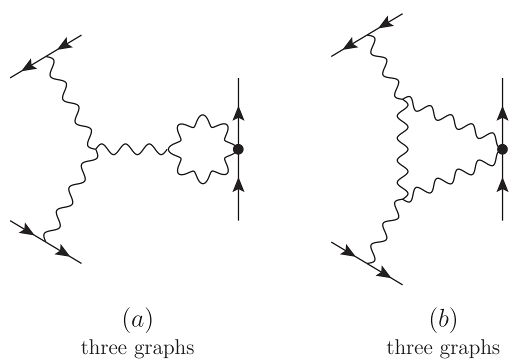

Figure 1.10. Pinched graphs for the vertex.

in gauge-dependent terms, receiving no radiative corrections. The unique gauge-invariant, ghost-free Ward identities relate the radiative correction term $\widehat{\Lambda}_{\alpha\mu\nu}$ to the PT proper self-energy; we give these subsequently.

The pinch parts from these graphs are shown in Figure 1.10. Actually, what we construct by pinching is an *improper vertex* that has PT propagators hooked

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on[6]:

$$\widehat{F}_{\alpha\mu\nu}(q_1,q_2,q_3) = \widehat{\Delta}^{\lambda}_{\alpha}(q_1)\widehat{\Delta}^{\rho}_{\mu}(q_2)\widehat{\Delta}^{\sigma}_{\nu}(q_3)\widehat{\Gamma}_{\lambda\rho\sigma}(q_1,q_2,q_3). \tag{1.83}$$

As with the PT propagator, we will work in the R_ξ Feynman gauge ($\xi = 1$). For any ξ, the longitudinal terms in the propagators of Eq. (1.83) strike the external quark lines and give no contribution, so we recover $\widehat{\Gamma}$ from $\widehat{F}$ by truncating, for example, all propagators to the form $g^{\lambda}_{\mu}d^{-1}(q_1)$. It is not necessary to calculate $\widehat{F}_{\mu\nu\alpha}$ and then truncate it; instead, the truncation is done by omitting the normal propagator graphs of Figure 1.4(b) and subtracting the pinch parts shown in Figure 1.10(a) rather than adding them. Subtraction rather than omission follows from a straightforward evaluation of combinatoric factors.

After a very lengthy computation, using the decomposition of the bare vertex into Γ^{F} and pinch parts and a recombination of three vertex terms analogous to the term used for the proper self-energy in Eq. (1.57) (see Eq. (1.95)), we find (with the momenta assignment shown in Figure 1.12)

$$\widehat{\Gamma}_{\alpha\mu\nu}(q_1,q_2,q_3) = -\frac{\mathrm{i}}{2}g^2C_A\int_{k_1}\frac{1}{k_1^2k_2^2k_3^2}\widehat{N}_{\alpha\mu\nu} + 8\widehat{B}_{\alpha\mu\nu}, \tag{1.84}$$

where

$$\begin{aligned}\widehat{N}_{\alpha\mu\nu} &= \Gamma^{\mathrm{F}}_{\alpha\lambda\rho}(q_1,k_3,-k_1)\Gamma^{\mathrm{F}}_{\mu\sigma\lambda}(q_2,k_2,-k_3)\Gamma^{\mathrm{F}}_{\nu\rho\sigma}(q_3,k_1,-k_2)\\ &\quad - 2(k_1+k_3)_\alpha(k_2+k_3)_\mu(k_1+k_2)_\nu \end{aligned} \tag{1.85}$$

$$\begin{aligned}\widehat{B}_{\alpha\mu\nu} &= \left(g_{\alpha\nu}q_{1\mu} - g_{\alpha\mu}q_{1\nu}\right)I(q_1) + \left(g_{\mu\nu}q_{2\alpha} - g_{\alpha\mu}q_{2\nu}\right)I(q_2)\\ &\quad + \left(g_{\alpha\nu}q_{3\mu} - g_{\mu\nu}q_{3\alpha}\right)I(q_3),\end{aligned} \tag{1.86}$$

and the integrals $I(q_i)$ are given by

$$I(q_i) = \frac{\mathrm{i}}{2}g^2C_A\int_k\frac{1}{k^2(k+q_i)^2}. \tag{1.87}$$

The first term in Eq. (1.85) is the vertex analog of the $\Gamma^{\mathrm{F}}\Gamma^{\mathrm{F}}$ term in the numerator of the proper self-energy (Eq. (1.63)); the other term comes from ghosts and pinches.

It is now interesting to compare the Slavnov–Taylor identity satisfied by the conventional and PT vertex when contracted with one of its momenta (say, q_1; the other identities are obtained by cyclic permutation of the indices and momenta). At tree level, the identity satisfied by the conventional R_ξ vertex has been derived in

[6] Of greater physical significance is a half-proper vertex function $\widehat{G}(q_1,q_2,q_3)$, defined [4] by $\widehat{G}_{\lambda\rho\sigma}(q_1,q_2,q_3) = g^{-2}\bar{g}(q_1)\bar{g}(q_2)\bar{g}(q_2)\widehat{\Gamma}_{\lambda\rho\sigma}(q_1,q_2,q_3)$, where $\bar{g}(q)$ is the pinch technique running charge. This half-proper vertex is not only gauge invariant but also renormalization group invariant.

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$$H_{\mu\nu}(k,q) \;=\; \mathrm{i}g_{\mu\nu} \;+$$

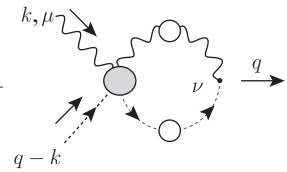

Figure 1.11. The auxiliary function H appearing in the three-gluon vertex Slavnov–Taylor identity. Shaded blobs represent the (connected) Schwinger–Dyson kernel corresponding to the ghost-gluon kernel appearing in the usual QCD skeleton expansion.

Eq. (1.34). At higher orders, the derivation of this identity is a textbook exercise: one starts from the trivial identity

$$\langle T\left[\bar{c}^a(x)A^m_\mu(y)A^n_\nu(z)\right]\rangle = 0 \tag{1.88}$$

and re-expresses it in terms of the BRST-transformed fields, making use of the equal-time commutation relations. The result is [21]

$$\begin{aligned} q_1^\alpha \Gamma^{amn}_{\alpha\mu\nu}(q_1,q_2,q_3) = \mathrm{i}gf^{amn}\left[q_1^2 D(q_1)\right]\big\{&\Delta^{-1}(q_3^2)P^\gamma_\nu(q_3)H_{\mu\gamma}(q_2,q_3) \\ &- \Delta^{-1}(q_2^2)P^\gamma_\mu(q_2)H_{\nu\gamma}(q_3,q_2)\big\}\,. \end{aligned} \tag{1.89}$$

The function H, shown in Figure 1.11, is the (amputated) one-particle irreducible (1PI) part[7] of the function ($q_1+q_2+q_3=0$):

$$L^{amn}_{\mu\nu}(q_2,q_3) = gf^{nrs}\int \mathrm{d}^4y\int \mathrm{d}^4z\, \mathrm{e}^{-\mathrm{i}q_2\cdot y}\mathrm{e}^{-\mathrm{i}q_3\cdot z}\left\langle T\left[\bar{c}^a(0)A^m_\mu(y)A^r_\nu(z)c^s(z)\right]\right\rangle, \tag{1.90}$$

which naturally appears when following the described procedure. The kernel appearing in this function is the conventional connected ghost-gluon kernel appearing in the usual QCD skeleton expansion; in addition, the function $H_{\mu\nu}(k,q)$ is related to the full gluon-ghost vertex $\mathbf{\Gamma}_\mu(k,q)$ (with k being the gluon and q being the antighost momentum) by the identity

$$q^\nu H_{\mu\nu}(k,q) = -\mathrm{i}\mathbf{\Gamma}_\mu(k,q), \tag{1.91}$$

where, at tree level, $H^{(0)}_{\mu\nu} = \mathrm{i}g_{\mu\nu}$ and $\mathbf{\Gamma}^{(0)}_\mu(k,q) = \Gamma_\mu(k,q) = -q_\mu$.

For the PT vertex, by contracting Eq. (1.84) with q_1, we instead immediately get the result

$$q_1^\alpha \widehat{\Gamma}^{amn}_{\alpha\mu\nu}(q_1,q_2,q_3) = gf^{amn}\left\{\widehat{\Delta}^{-1}(q_2)P_{\mu\nu}(q_2) - \widehat{\Delta}^{-1}(q_3)P_{\mu\nu}(q_3)\right\}, \tag{1.92}$$

[7] Let us recall that a diagram is called 1PI if it cannot be split into two disjoined pieces by cutting a single internal line; when this is not the case, it is called one-particle reducible (1PR).

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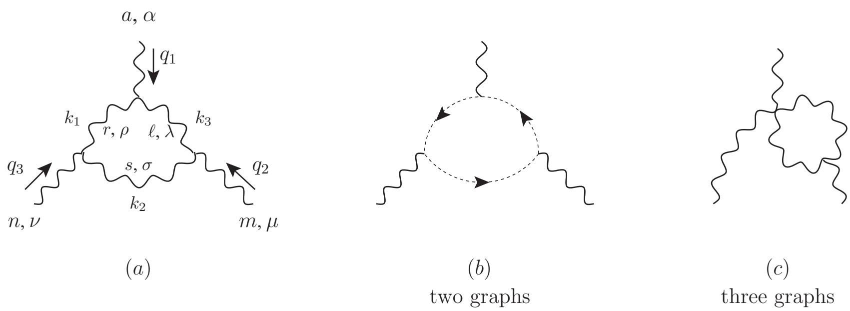

Figure 1.12. R_ξ diagrams contributing to the one-loop three-gluon vertex. Diagrams (c) carry a $1/2$ symmetry factor. Fermion diagrams are not shown.

with $\widehat{\Delta}^{-1}(q) = q^2 + \mathrm{i}\widehat{\Pi}(q^2)$; thus we find the naive one-loop generalization of the tree-level identity of Eq. (1.44). Notice that (except in ghost-free gauges) the rhs of the preceding equation is not the difference of two inverse gluon propagators because the projection operator P has no inverse; also notice that there is no longer reference to auxiliary ghost Green's functions so that Eq. (1.92) is completely gauge invariant.

The intrinsic construction As an application of the intrinsic PT algorithm described in the previous section, let us see in detail how one can construct the one-loop PT three-gluon vertex. The conventional R_ξ diagrams are shown in Figure 1.12 and read

$$\Lambda_{\alpha\mu\nu}(q_1, q_2, q_3) = -\frac{\mathrm{i}}{2} g^2 C_A \int_{k_1} \frac{1}{k_1^2 k_2^2 k_3^2} N_{\alpha\mu\nu} + \frac{9}{2} \widehat{B}_{\alpha\mu\nu}, \tag{1.93}$$

with

$$\begin{aligned} N_{\alpha\mu\nu} = {} & \Gamma_{\alpha\lambda\rho}(q_1, k_3, -k_1)\Gamma_{\mu\sigma\lambda}(q_2, k_2, -k_3)\Gamma_{\nu\rho\sigma}(q_3, k_1, -k_2) \\ & - k_{1\alpha}k_{2\nu}k_{3\mu} - k_{1\alpha}k_{2\nu}k_{3\mu}. \end{aligned} \tag{1.94}$$

Let us then introduce the shorthand notation $\Gamma_1\Gamma_2\Gamma_3$ for the combination of three-level three-gluon vertices appearing in Eq. (1.94). In this notation, all Lorentz indices are suppressed, and the number appearing in each vertex is the number corresponding to the vertex's external momentum q_i. Then, decomposing each of the Γ_i into $\Gamma_i^{\mathrm{F}} + \Gamma_i^{\mathrm{P}}$, we obtain

$$\begin{aligned} \Gamma_1\Gamma_2\Gamma_3 = {} & \Gamma_1^{\mathrm{F}}\Gamma_2^{\mathrm{F}}\Gamma_3^{\mathrm{F}} + \Gamma_1^{\mathrm{P}}\Gamma_2\Gamma_3 + \Gamma_1\Gamma_2^{\mathrm{P}}\Gamma_3 + \Gamma_1\Gamma_2\Gamma_3^{\mathrm{P}} - \Gamma_1^{\mathrm{P}}\Gamma_2^{\mathrm{P}}\Gamma_3 - \Gamma_1^{\mathrm{P}}\Gamma_2\Gamma_3^{\mathrm{P}} \\ & - \Gamma_1\Gamma_2^{\mathrm{P}}\Gamma_3^{\mathrm{P}} + \Gamma_1^{\mathrm{P}}\Gamma_2^{\mathrm{P}}\Gamma_3^{\mathrm{P}}. \end{aligned} \tag{1.95}$$

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The first term contains no pinching momenta and therefore will be kept in the PT answer, giving rise to the term

$$(\widehat{a}) = -\frac{\mathrm{i}}{2} g^2 C_A \int_{k_1} \frac{1}{k_1^2 k_2^2 k_3^2} \Gamma^{\mathrm{F}}_{\alpha\lambda\rho}(q_1, k_3, -k_1) \Gamma^{\mathrm{F}}_{\mu\sigma\lambda}(q_2, k_2, -k_3) \Gamma^{\mathrm{F}}_{\nu\rho\sigma}(q_3, k_1, -k_2). \tag{1.96}$$

Each of the other six terms has pinching terms generated when Γ_i^{P} acts on the full Γs that trigger the corresponding Ward identities; according to the rules of intrinsic pinching, we will then discard all the terms that are proportional to $d^{-1}(q_i^2)$. However, these d^{-1} terms can also refer to a virtual momentum k_i, in which case, they give rise to an integral with only two propagators. The last term on the rhs of Eq. (1.95) yields terms of this sort as well as a contribution that adds to the ghost graph

$$\begin{aligned} \Gamma_1^{\mathrm{P}} \Gamma_2^{\mathrm{P}} \Gamma_3^{\mathrm{P}} = &-d^{-1}(k_1^2) \left(g_{\mu\nu} k_{3\alpha} + g_{\alpha\mu} k_{1\alpha} \right) - d^{-1}(k_2^2) \left(g_{\alpha\mu} k_{1\nu} + g_{\alpha\nu} k_{3\mu} \right) \\ &- d^{-1}(k_3^2) \left(g_{\alpha\nu} k_{2\mu} + g_{\mu\nu} k_{1\alpha} \right) - k_{1\alpha} k_{2\mu} k_{3\nu} - k_{1\nu} k_{2\mu} k_{3\alpha}. \end{aligned} \tag{1.97}$$

The rest of the terms instead have external pinches, which we drop, keeping only the relevant terms:

$$\begin{aligned} \Gamma_1^{\mathrm{P}} \Gamma_2 \Gamma_3 = &\, d^{-1}(k_3^2) \left[\Gamma_{\nu\alpha\mu}(k_1, -k_2) + \Gamma_{\mu\nu\alpha}(k_2, -k_3) \right] \\ &+ k_{2\mu} \left[d^{-1}(k_1^2) g_{\alpha\nu} - k_{1\alpha} k_{1\nu} \right] + k_{2\nu} \left[d^{-1}(k_3^2) g_{\alpha\mu} - k_{3\alpha} k_{3\mu} \right] \end{aligned} \tag{1.98}$$

$$\begin{aligned} \Gamma_1^{\mathrm{P}} \Gamma_2^{\mathrm{P}} \Gamma_3 = &\, d^{-1}(k_3^2) \Gamma_{\nu\alpha\mu}(k_1, -k_2) - k_{3\alpha} \left[d^{-1}(k_2^2) g_{\mu\nu} - k_{2\mu} k_{2\nu} \right] \\ &- k_{3\mu} \left[d^{-1}(k_1^2) g_{\nu\alpha} - k_{1\nu} k_{1\alpha} \right]. \end{aligned} \tag{1.99}$$

Similar expressions can be found for all the other terms appearing on the rhs of Eq. (1.95). Isolating all the pinching terms that do not pinch out any (internal) propagator and adding them to the conventional ghost graph of Figure 1.12(b), we get the result

$$(\widehat{b}) = 2 \frac{\mathrm{i}}{2} g^3 C_A \int_{k_1} \frac{1}{k_1^2 k_2^2 k_3^2} 2(k_1 + k_3)_\alpha (k_2 + k_3)_\mu (k_1 + k_2)_\nu, \tag{1.100}$$

with the remaining pinching contribution giving

$$\begin{aligned} (c)^{\mathrm{P}} = &-\frac{\mathrm{i}}{2} g^2 C_A \int_{k_2} \frac{1}{k_2^2 k_3^2} \left[g_{\alpha\mu}(k_1 - q_3)_\nu + 2 g_{\alpha\nu}(q_3 - q_1)_\mu + g_{\mu\nu}(k_1 + q_1)_\alpha \right] \\ &- \frac{\mathrm{i}}{2} g^2 C_A \int_{k_1} \frac{1}{k_1^2 k_3^2} \left[g_{\alpha\mu}(k_2 + q_3)_\nu + g_{\alpha\nu}(k_2 - q_2)_\mu + 2 g_{\mu\nu}(q_2 - q_3)_\alpha \right] \\ &- \frac{\mathrm{i}}{2} g^2 C_A \int_{k_1} \frac{1}{k_1^2 k_2^2} \left[2 g_{\alpha\mu}(q_1 - q_2)_\nu + g_{\alpha\nu}(k_3 + q_2)_\mu + g_{\mu\nu}(k_3 - q_1)_\alpha \right]. \end{aligned} \tag{1.101}$$

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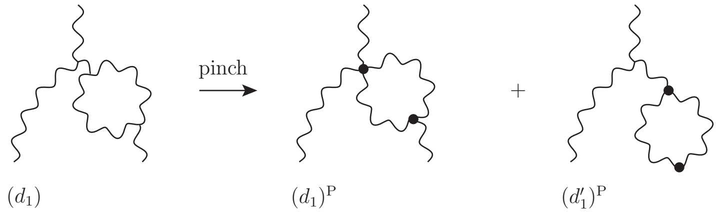

Figure 1.13. 1PR diagram giving rise to effectively 1PI pinching contributions (diagram $(d_1)^{\mathrm{P}}$). Two more diagrams (corresponding to having the gluon self-energy correction on the remaining legs) that give rise to similar terms are not shown.

As we have seen, in the presence of longitudinal momenta, the topology of a Feynman diagram is not a well-defined property because longitudinal momenta will pinch out internal propagators, turning t-channel diagrams into s-channel diagrams. This same caveat applies also to the notion of one-particle reducibility. It turns out that by pinching out internal propagators, one can effectively convert 1PR diagrams into 1PI diagrams (see Figure 1.13); of course, the opposite cannot happen. Evidently, one-particle reducibility is a gauge-dependent concept. Thus, when constructing the purely (1PI) gauge-invariant three-gluon vertex at one loop, one has to take into account possible 1PI pinching contributions coming from apparently 1PR diagrams, like those coming from the graphs shown in Figure 1.13. Notice also that not all the pinching terms coming from the diagrams of Figure 1.13 will be producing 1PI terms but only those that will remove the *internal* gluon propagator (diagram $(d_1)^{\mathrm{P}}$). Those removing the *external* gluon propagator (diagram $(d_1')^{\mathrm{P}}$) ought to be discarded, in full accordance with the rules of the intrinsic pinch technique, because they will inevitably cancel against analogous contributions coming (in the S-matrix PT implementation) from non-Abelian vertices attached to the external test-quark.

We show in detail what happens in the case shown in Figure 1.13. One has

$$(d_1) = -\frac{\mathrm{i}}{2} g^2 C_A \Gamma_{\alpha\mu'\nu}(q_1, q_2, q_3) d(q_2^2) g^{\mu'\nu'} \times \int_k \frac{1}{k^2(k+q_2)^2} \Gamma_{\nu'\rho\sigma}(-q_2, k+q_2, -k) \Gamma_\mu^{\rho\sigma}(-q_2, k+q_2, -k). \quad (1.102)$$

As explained, of all the possible pinching contributions appearing after the splitting of the two self-energy three-gluon vertices, shown in Eqs (1.59) and (1.60), one needs to retain only half of the first term appearing on the rhs of Eq. (1.59); the other half removes instead the external propagator, thus generating diagram $(d_1')^P$ of Figure 1.13. Therefore one has

$$(d_1)^{\mathrm{P}} = \mathrm{i} g^2 C_A \Gamma_{\alpha\mu\nu}(q_1, q_2, q_3) I(q_2), \quad (1.103)$$

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where we kept only the $g^{\mu\sigma}$ part of the $P^{\mu\sigma}$ appearing in the pinching term because the $q_2^\mu q_2^\sigma$ term will remove the external propagator and thus ought to be discarded. All that is left to do is to add this term to the first term appearing in Eq. (1.101), denoted by $(c_1)^{\mathrm{P}}$; a straightforward (setting $k_2 = k$) calculation shows then that

$$(c_1)^{\mathrm{P}} + (d_1)^{\mathrm{P}} = \mathrm{i}\frac{7}{2}g^2 C_A(g_{\mu\nu}q_{2\alpha} - g_{\alpha\mu}q_{2\nu})I(q_2). \tag{1.104}$$

The same procedure can be repeated for the diagrams (d_2) and (d_3), which would show the gluon self-energy on the ν and α leg, respectively; after adding them to the corresponding contributions $(c_2)^{\mathrm{P}}$ and $(c_3)^{\mathrm{P}}$, we get

$$(c_2)^{\mathrm{P}} + (d_2)^{\mathrm{P}} = \mathrm{i}\frac{7}{2}g^2 C_A(g_{\alpha\nu}q_{3\mu} - g_{\mu\nu}q_{3\alpha})I(q_3) \tag{1.105}$$

$$(c_3)^{\mathrm{P}} + (d_3)^{\mathrm{P}} = \mathrm{i}\frac{7}{2}g^2 C_A(g_{\alpha\nu}q_{1\mu} - g_{\alpha\nu}q_{1\nu})I(q_1). \tag{1.106}$$

Because these terms have exactly the same structure as the conventional (c) diagrams of Figure 1.12, they can be combined with them (viz. with the last term in Eq. (1.93)). Then, putting everything together, we recover exactly the same result found in Eq. (1.84).

1.5.3 The four-gluon vertex

Clearly, there should be a generalization of the three-gluon pinch technique to the four-gluon proper vertex; it has been given at one-loop order in [22] with the by-now standard technique of forming an S-matrix element with, in this case, eight on-shell quark legs and then finding the pinch graphs in the Feynman gauge. We will state here only the Ward identity that this vertex satisfies, which is the naive ghost-free generalization that we have learned to expect. This one-loop ghost-free Ward identity has exactly the structure of the tree-level Ward identity. One of four Ward identities, one for each momentum q_i, reads

$$\begin{aligned} q_1^\alpha \Gamma^{amnr}_{\alpha\mu\nu\rho}(q_1, q_2, q_3, q_4) = {} & g f^{adm}\widehat{\Gamma}^{dnr}_{\mu\nu\rho}(q_1 + q_2, q_3, q_4) \\ & + g f^{adr}\widehat{\Gamma}^{drm}_{\nu\rho\mu}(q_1 + q_3, q_4, q_2) \\ & + g f^{adn}\widehat{\Gamma}^{dmr}_{\nu\mu\rho}(q_1 + q_4, q_2, q_3), \end{aligned} \tag{1.107}$$

where the $\widehat{\Gamma}$ with three indices are the PT three-gluon vertices we found earlier. Note that a possible new renormalization constant $\widehat{Z}_4$ for the four vertex is, by virtue

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of the Ward identity, equal to $\widehat{Z}_3$. We now have four ghost-free Ward identities (at least at the one-loop level), as given by Eqs. (1.62), (1.82), (1.92), and (1.107).

1.6 The pinch technique in the light-cone gauge

The light-cone gauge is one of a class of gauges that is ghost free, which simplifies our conceptual tasks in understanding the pinch technique (in fact, it was the first gauge used [1, 2, 3] in the development of the pinch technique). We recount here the pinch technique as explained in [3]. It is not necessarily any easier to compute in this gauge – in fact, in some respects, it is harder. But it is easier, as we will see, than the axial gauge, which is similar but also definable in Euclidean space.

The light-cone gauge can only be defined in Minkowski space, but that will not prevent us from using it in Euclidean space *after* cancellation of all gauge-dependent terms has taken place. In fact, this cancellation takes place *before* any momentum-space integrations are done, so we can convert easily to Euclidean integrals in stating PT results derived from the light cone; this is useful for applications to finite-temperature gauge theory [23].

The light-cone gauge introduces a lightlike vector n_μ, with $n^2 = 0$, for the gauge-fixing:

$$n^\mu A_\mu = 0. \tag{1.108}$$

To fix the gauge completely, some other lower-dimensional constraints are needed that give a precise meaning to operators like $(n \cdot \partial)^{-1}$. But we do not even need to know the constraints because all such inverse operators will disappear from the PT propagator before it is necessary to define them.

Although it is not required to implement light-cone gauge fixing as we do here, it is convenient; the alternative is canonical quantization in a gauge such as $A_0 = A_3$. We replace the gauge-fixing term of the R_ξ gauges by

$$\frac{1}{2\eta}\mathrm{Tr}\left(n^\mu A_\mu\right)^2; \tag{1.109}$$

later, we will take the limit $\eta = 0$ to enforce the light-cone gauge. This limit can only be taken after all calculations are done. The free propagator and inverse propagator are as follows:

$$\mathrm{i}\Delta^{(0)}_{\mu\nu}(q) = \frac{1}{q^2}Q_{\mu\nu}(q) + \eta\frac{q_\mu q_\nu}{(n\cdot q)^2}, \tag{1.110}$$
$$-\mathrm{i}[\Delta^{(0)}]^{-1}_{\mu\nu}(q) = q^2 P_{\mu\nu}(q) + \frac{n_\mu n_\nu}{\eta},$$

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where[8]

$$Q_{\mu\nu}(q) = g_{\mu\nu} - \frac{n_\mu q_\nu + n_\nu q_\mu}{n \cdot q}. \tag{1.111}$$

The propagator should be annihilated by n_μ and indeed $n^\mu Q_{\mu\nu} = 0$. The term multiplying η does not vanish on multiplication by n^μ, but this is no surprise because it comes from the gauge-dependent gauge-fixing term. Of course, it vanishes in the physical limit $\eta \to 0$.

The virtue of the light-cone gauge is that (except for the unphysical η-dependent term) the propagator is *homogeneous* of degree zero in the vector n_μ, as a moment's thought shows it must be. Any scalar function of a single momentum constructed from the light-cone gauge propagator can *only* depend on q^2 and not in any way on n_μ itself because of this homogeneity requirement. It would seem to follow that the only way in which the gauge choice can be manifested in the propagator is through kinematic factors such as $n_\mu n_\nu/(n \cdot q)^2$. This is to be contrasted with the explicit dependence of the R_ξ-gauge propagators on ξ.[9] So results for the propagator in the light-cone gauge must be very close to those of the pinch technique, even without taking pinching into account.

The conventional one-loop Feynman propagator calculated in the light-cone gauge is [3]

$$\mathrm{i}\Delta_{\mu\nu}(q) = \mathrm{i}\Delta^{(0)}_{\mu\nu}(q) + Q_{\mu\nu}(q)\left[\frac{22}{3}I(q) + 8I'(q)\right] + \frac{n_\mu n_\nu}{(n\cdot q)^2}\left[8I(q) + 8I'(q)\right], \tag{1.112}$$

where $I(q)$ has already been defined in Eq. (1.87), while

$$I'(q) = \frac{\mathrm{i}}{2} g^2 C_A (n\cdot q)\int_k \frac{1}{k^2(k-q)^2(n\cdot k)}. \tag{1.113}$$

The radiative corrections have the most general form allowed in the light-cone gauge, where the propagator must be annihilated (except for the gauge-fixing term) by n_μ or n_ν. The integral $I(q) = 1/(16\pi^2)\ln(-q^2/\Lambda^2) + \cdots$ is the one appearing in the R_ξ pinch technique, and if the first term on the rhs were the only contribution, we would again recover exactly the same PT propagator involving the gauge-invariant running charge, except for the kinematics such as the factor $Q_{\mu\nu}(q)$. As for the integral $I'(q)$, it is, as advertised, homogeneously degree zero in n_μ, but it is not clear how to evaluate it. In fact, many learned papers have been written on how to regulate the $1/(n\cdot k)$ singularity and to evaluate integrals such as $I'(q)$, but we do

8 The notation used here differs slightly from that of [3]; in particular $Q_{\mu\nu}$ is defined with the opposite sign.
9 Certain integrals arise, such as I', that have n_μ in their definition, and their value is not clear. Fortunately, the pinch technique cancels all such terms before the integrations need to be done.

not need them. The reason is that when we add the pinch terms to the conventional light-cone propagator, now stemming from the longitudinal terms in $Q_{\mu\nu}(q)$, the terms in $I'(q)$ and the terms multiplying $n_\mu n_\nu$ are all canceled *before* one needs to face up to the task of doing the strange integral in $I'(q)$. The only term remaining is the one giving the gauge-invariant running charge. We can therefore write the PT propagator in the light-cone gauge as

$$\mathrm{i}\widehat{\Delta}_{\mu\nu}(q) = Q_{\mu\nu}(q)\frac{1}{q^2}\left[1 - bg^2 \ln\left(\frac{-q^2}{\Lambda^2}\right) + \dots\right] + \eta\frac{q_\mu q_\nu}{(n\cdot q)^2}, \quad (1.114)$$

where b is the by-now familiar one-loop coefficient in the running charge (see Eq. (1.69)).

The absence of the $n_\mu n_\nu$ term in the PT propagator persists to all orders. Such a term is kinematically allowed because it is annihilated by n_μ or n_ν, and its absence to all orders is not a trivial matter.

The light-cone version of the PT propagator differs from the earlier PT propagator not only through the gauge-dependent term multiplying η but also in the kinematical factor $Q_{\mu\nu}$, which depends on the gauge. The first gauge dependence is expected, but perhaps not the second. We can exhibit a more exact correspondence between PT propagators in various gauges by looking not at the propagator but at its inverse (or more to the point, at the PT proper self-energy).

It is straightforward to calculate the inverse of the pinch propagator given in Eq. (1.114); the renormalized version follows:

$$-\,\mathrm{i}\widehat{\Delta}^{-1}_{\mu\nu}(q) = P_{\mu\nu}(q)\left\{q^2\left[1 + bg^2\ln\left(\frac{-q^2}{\mu^2}\right)\right]\right\} + \frac{n_\mu n_\nu}{\eta}. \quad (1.115)$$

We see that the inverse of the light-cone PT propagator is exactly what we found earlier, except for the η-dependent terms. These gauge-dependent terms never receive radiative corrections, except for a multiplicative renormalization of η. In effect, they are only associated with *free* Green's functions.

The importance of the inverse PT propagator, or equivalently, the proper self-energy as well as proper vertices, is that they are the natural ingredients of Ward identities. No matter what gauge is used, the proper self-energy has the simple transverse form (cf. Eq. (1.55))

$$\widehat{\Pi}_{\mu\nu}(q) = P_{\mu\nu}(q)\widehat{\Pi}(q), \quad (1.116)$$

where $\widehat{\Pi}(q)$ is independent of any gauge choice. There are, as we have seen, PT three-point vertices $\widehat{\Gamma}_{\mu\nu\alpha}$ that obey, in any gauge, certain ghost-free Ward identities (cf. Eq. (1.92)). In the light-cone gauge, this Ward identity actually has true inverse

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propagators on the rhs:

$$q^{\alpha}\widehat{\Gamma}_{\mu\nu\alpha}(k, q-k, -q) = \widehat{\Delta}^{-1}_{\mu\nu}(k) - \widehat{\Delta}^{-1}_{\mu\nu}(q-k). \tag{1.117}$$

Note that although the inverse of the PT propagator (Eq. (1.115)) depends on the gauge parameter η, the difference of two inverse propagators is independent of η so that all terms in the Ward identity are strictly gauge invariant. This is just the naive Ward identity that one would expect if NAGTs behaved like QED, with no ghosts to worry about. Of course, this Ward identity is not the Slavnov–Taylor identity satisfied by the conventional full vertex Γ, which has ghost contributions and is gauge dependent.

Just as in QED, this Ward identity ensures that $\widehat{Z}_1 = \widehat{Z}_{3g}$, as we found earlier in the covariant gauge.

1.7 The absorptive pinch technique construction

Here we show how unitarity is defined for the pinch technique, with one-loop examples. In one respect, unitarity for PT Green's functions is the same as for the S-matrix; in another respect, it differs – and it must differ, or it is impossible to reconcile with asymptotic freedom.

The two aspects of PT unitarity are as follows:

1. Off-shell PT Green's functions obey dispersion relations of conventional type, having only physical threshholds (i.e., Goldstone and ghost masses, which are generically gauge dependent, cannot occur in the set of allowed threshholds). The Feynman (background-field) gauge is singled out here because the ghost and Goldstone masses are the same as the gauge-boson mass in this gauge, and so a threshhold criterion cannot rule out that a propagator ultimately stemmed from a ghost or Goldstone particle.
2. The absorptive parts of PT Green's functions are calculated from the PT Green's functions with the standard (Cutkosky rules) construction. However, because the PT Green's functions differ from the conventional ones by terms that subtract out gauge dependence and ghost lines, it happens for a NAGT – but not for QED – that there can be absorptive parts with a negative sign. This is in no way inconsistent with physical unitarity for the S-matrix, which contains not only PT parts but other parts that restore positivity for the absorptive parts of the S-matrix.

The difference between QED and NAGTs is the following: the PT is empty for QED, which is a situation realized by an exact cancellation of terms that would

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contribute to the pinch parts. But for QCD, the cancellation is not exact, as we have seen. The study of unitarity properties for the pinch technique reveals a similar situation: unitarity has its familiar form for QED, which means, among other things, that the imaginary part of the photon propagator is positive definite (i.e., essential for the propagator to obey the Källen–Lehmann representation) and the beta function of QED is also positive definite. But as we show here, the construction of the absorptive part of a PT Green's function, such as the propagator, usually involves an incomplete cancellation between positive and negative terms that can leave negative terms in absorptive parts where a positive term would normally be expected. This is essential for a negative beta function in an asymptotically free theory. The appearance of negative absorptive terms in a PT propagator is certainly not an indication that the PT fails to respect normal unitarity for the S-matrix, any more than the need for ghosts means a violation of unitarity.

Such cancellations have appeared in a related context, even before there was a pinch technique. Long ago, people asked whether it was possible to derive the fundamental structure of a renormalizable theory of multiple massive vector mesons from some straightforward assumption such as high-energy unitarity, or whether one simply had to adopt by fiat theories of the NAGT-Higgs type. Several authors [24, 25, 26] gave the answer: starting from the most general Lagrangian of spin 0, $1/2$, and 1 particles that is renormalizable by power counting for massless vector bosons, one can, by imposing the requirements of high-energy unitarity, derive uniquely the structure of a NAGT–Higgs theory coupled minimally to spin 0 and $1/2$ matter fields. The issue is that the massive vector mesons of a general Lagrangian have longitudinal modes that, if their effects were uncanceled because of some relations among couplings that give the Lagrangian a very specific structure, would lead through the usual optical theorem to unbridled growth in energy E of perturbative amplitudes through a power of E/M for every longitudinal mode. Another way to say this is that, in the unitary gauge, a massive vector propagator has a longitudinal numerator part $\sim k_\mu k_\nu/M^2$ that is unrenormalizable unless the theory has a special form – that of an NAGT. These authors showed that requiring the longitudinal modes to be canceled led uniquely to an NAGT–Higgs theory (at least in perturbation theory). But they did not cancel completely; they simply were tamed to the point where total amplitudes behaved like positive powers of M rather than negative powers. Studies of PT unitarity show similar incomplete cancellations for NAGTs [27, 28], as we review here.

Because positivity is often an important physical constraint on absorptive parts, one might question whether PT unitarity, with some negative terms, can be physically useful. In [29], it was conjectured that the product of the PT propagator and the coupling g^2 factors into two terms, both with positive absorptive parts, and the product fails to have a positive absorptive part only because of asymptotic freedom. This factorization allows a re-organization of terms in the Schwinger–Dyson equations for higher-point PT Green's functions such that ordinary positivity requirements still hold.

1.7.1 The strong version of the optical theorem

The T-matrix element of a reaction $i \to f$ is defined via the relation

$$\langle f|S|i\rangle = \delta_{fi} + \mathrm{i}(2\pi)^4\delta^{(4)}(P_f - P_i)\langle f|T|i\rangle, \tag{1.118}$$

where P_i (P_f) is the sum of all initial (final) momenta of the $|i\rangle$ ($|f\rangle$) state. Furthermore, imposing the unitarity relation $S^\dagger S = 1$ leads to the generalized optical theorem

$$\langle f|T|i\rangle - \langle i|T|f\rangle^* = \mathrm{i}\sum_j (2\pi)^4\delta^{(4)}(P_j - P_i)\langle j|T|f\rangle^*\langle j|T|i\rangle. \tag{1.119}$$

In Eq. (1.119), the sum $\sum_j$ is over the entire phase space and spins of all possible on-shell intermediate particles j.

An important corollary of this theorem is obtained if $f = i$, corresponding to the case of so-called forward scattering. Then, Eq. (1.119) reduces to

$$\Im m\langle i|T|i\rangle = \frac{1}{2}\sum_j (2\pi)^4\delta^{(4)}(P_j - P_i)|\langle j|T|i\rangle|^2. \tag{1.120}$$

In what follows, we will refer to the relation given in Eq. (1.120) as the optical theorem.

The rhs of the optical theorem consists of the sum of the (squared) amplitudes, $\mathcal{M}^{ij}$, of all kinematically allowed elementary processes connecting the initial and final states. Note in particular that only physical particles may appear as intermediate $|j\rangle$ states. If the particles involved are fermions, gauge bosons, or both when calculating $\mathcal{M}^{ij}$, one averages over the initial-state polarizations and sums over the final-state polarizations. In addition, the integration over all available phase space, implicit in the sum $\sum_j$, must be carried out. The left-hand side (lhs) of the optical theorem is given by the imaginary part of the entire amplitude, i.e., including all Feynman diagrams contributing to it. For example, in the case of NAGTs, to obtain the lhs of the optical theorem, one must calculate the imaginary part of all diagrams, regardless of whether they contain physical (gluons, quarks) or unphysical (ghosts or would-be Goldstone bosons) fields inside their loops. To do that, one usually uses the Cutkosky rules or cutting rules, whereby in all diagrams, the propagators of physical and unphysical particles are put simultaneously on shell.

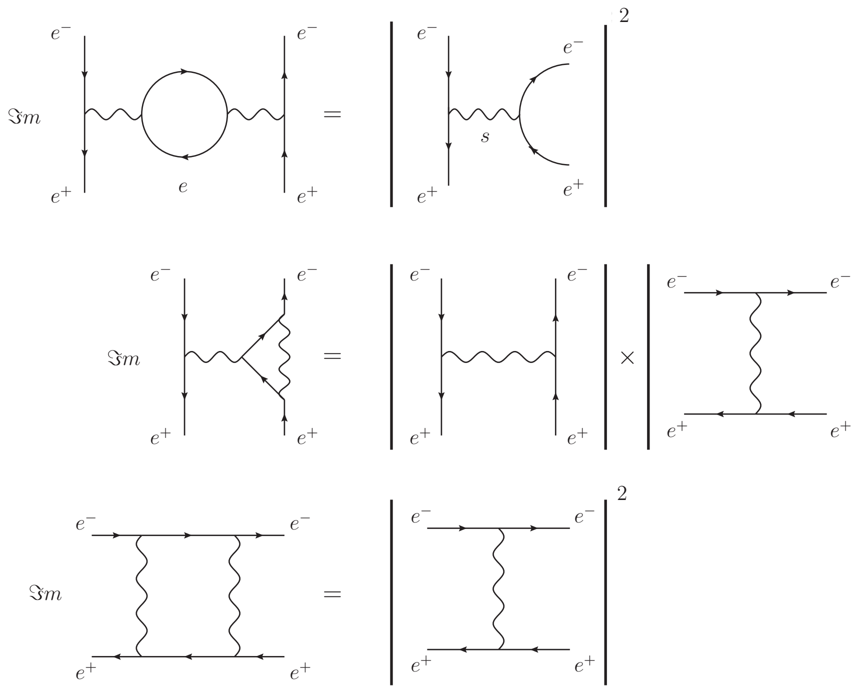

Figure 1.14. The strong version of the optical theorem in QED.

An issue of central importance for what follows is the way the optical theorem is realized at the level of the conventional diagrammatic expansion, or equivalently, at the level of the individual propagator-, vertex-, and boxlike amplitudes. Specifically, in its general formulation of Eq. (1.120), the optical theorem is a statement at the level of entire amplitudes and not individual Feynman graphs or Green's functions. Thus, the imaginary part of a given diagram appearing on the rhs does not necessarily correspond to an easily identifiable diagrammatic (or kinematic) piece on the rhs. Of course, there are theories in which the optical theorem holds also at the level of individual graphs and kinematic subamplitudes. This strong version of the optical theorem is realized in scalar theories and QED (see Figure 1.14) but fails in NAGTs (see Figure 1.15). This is so because, with the exception of certain gauges, in the NAGTs, the propagator-, vertex-, and box-like subamplitudes of each side of the optical theorem are totally different. For example, in the case of the forward QCD process $q(p_1)\bar{q}(p_2) \to q(p_1)\bar{q}(p_2)$, the propagator-like part of the lhs, computed in the renormalizable gauges, is determined by cutting through one-loop graphs containing ξ-dependent gluon propagators and unphysical ghosts (omit quark loops), whereas the propagator-like part of the rhs contains the polarization tensors corresponding to physical massless particles of spin 1 (two physical

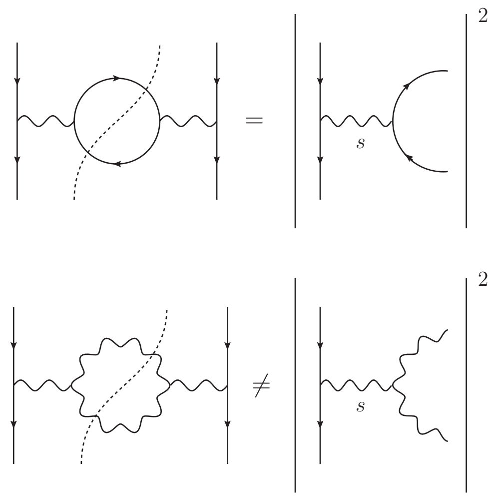

Figure 1.15. The strong version of the optical theorem in QCD, which holds for the quark loop but fails for the gluon loop.

polarizations). This profound difference complicates the diagrammatic verification of the optical theorem and invalidates, at the same time, its strong version. A crucial advantage of the PT is that it permits the realization of the optical theorem at the level of kinematically distinct, well-defined subamplitudes, even in the context of non-Abelian gauge theories; these privileged subamplitudes are, of course, none other than the PT Green's functions. In other words, the strong version of the optical theorem holds if and only if the identification of the subamplitudes on each side occurs after the application of the PT.

1.7.2 The fundamental s-t cancellation

As we demonstrate in this section, the application of the PT on the rhs (the physical side) of the optical theorem is tantamount to the explicit use of an underlying fundamental cancellation between s-channel and t-channel graphs [27, 28]. This cancellation results in a nontrivial reshuffling of terms, which, in turn, allows for the definition of kinematically distinct contributions; interestingly enough, they correspond to the imaginary parts of the one-loop PT subamplitudes constructed in the previous section.

To see all this in detail, we consider the forward-scattering process $q(p_1)\bar{q}(p_2) \to q(p_1)\bar{q}(p_2)$ and concentrate on the optical theorem to lowest order. Obviously, the intermediate states appearing on the rhs may involve quarks or gluons. The quarks can be treated essentially as in QED and are, in that sense, completely straightforward. We will therefore focus on the part of the optical theorem where

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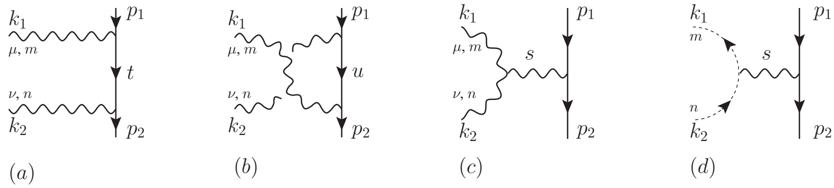

Figure 1.16. Diagrams defining (a and b) the amplitudes $\mathcal{T}_t$ and (c) $\mathcal{T}_s$. Diagram (d) is related to the amplitude $\mathcal{S}$ defined in Eq. (1.126).

the intermediate states are two gluons; we have that

$$\Im m\langle q\bar{q}|T|q\bar{q}\rangle = \frac{1}{2}\times\frac{1}{2}\int_{\mathrm{PS}_{gg}}\langle q\bar{q}|T|gg\rangle\langle gg|T|q\bar{q}\rangle^*, \tag{1.121}$$

where $\int_{\mathrm{PS}_{gg}}$ is the two gluon phase space measure integral.[10] The extra $\frac{1}{2}$ factor is statistical and arises from the fact that the final on-shell gluons should be considered identical particles in the total rate. Let us now focus on the rhs of Eq. (1.121) and set $\mathcal{T}\equiv\langle q\bar{q}|T|gg\rangle$. Diagrammatically, the tree-level amplitude $\mathcal{T}$ is the sum of two distinct parts: t- and u-channel graphs that contain an internal quark propagator, $\mathcal{T}^{mn}_{t\,\mu\nu}$, as shown in of Figure 1.16(a) and 1.16(b), and an s-channel amplitude, $\mathcal{T}^{mn}_{s\,\mu\nu}$, given in Figure 1.16($c$). Defining $\mathcal{V}^a_\alpha = \bar{v}(p_2)t^a\gamma_\alpha u(p_1)$, we have that

$$\mathcal{T}^{mn}_{s\,\mu\nu} = g^2 f^{mnc}\mathcal{V}^c_\alpha d(q)\Gamma^\alpha_{\mu\nu}(q,k_1,k_2) \tag{1.122}$$
$$\mathcal{T}^{mn}_{t\,\mu\nu} = \mathrm{i}g^2\bar{v}(p_2)\big[t^n\gamma_\nu S^{(0)}(p_1+k_1)t^m\gamma_\mu\ + t^m\gamma_\mu S^{(0)}(p_1+k_2)\gamma_\nu t^n\big]u(p_1).$$

The subscripts s and t refer, as usual, to the corresponding Mandelstam variables, i.e., $s=q^2=(p_1+p_2)^2=(k_1+k_2)^2$ and $t=(p_1-k_1)^2=(p_2-k_2)^2$. We then have

$$\mathcal{M} = [\mathcal{T}_s+\mathcal{T}_t]^{mn}_{\mu\nu}\,L^{\mu\mu'}(k_1)L^{\nu\nu'}(k_2)\left[\mathcal{T}_s^*+\mathcal{T}_t^*\right]^{mn}_{\mu'\nu'}, \tag{1.123}$$

where the polarization tensor $L^{\mu\nu}(k)$ corresponding to a massless spin one particle is given by

$$L_{\mu\nu}(k) = -g_{\mu\nu}+\frac{n_\mu k_\nu+n_\nu k_\mu}{n\cdot k}+\eta^2\frac{k_\mu k_\nu}{(n\cdot k)^2}\,. \tag{1.124}$$

For on-shell gluons, i.e., for $k^2=0$, $k^\mu L_{\mu\nu}(k)=0$. By virtue of this last property, we see immediately that if we carry out the PT decomposition of Eq. (1.41) to

[10] In general the (4-dimensional) two body phase space integral $\int_{\mathrm{PS}}$ is defined as

$$\int_{\mathrm{PS}} = \frac{1}{(2\pi)^2}\int d^4k_1\int d^4k_2\delta_+(k_1^2-m_1^2)\delta_+(k_2^2-m_2^2)\delta^{(4)}(q-k_1-k_2),$$

where m_1 and m_2 are the masses of the intermediate particles produced.

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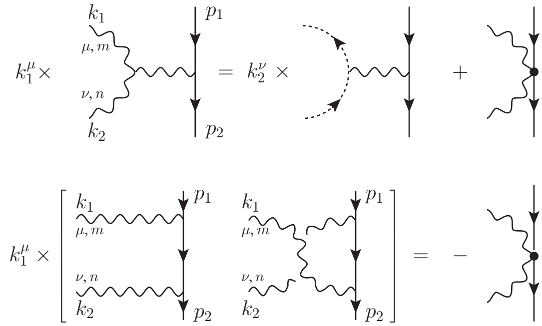

Figure 1.17. The s-t cancellation at tree level.

the three-gluon vertex Γ, the term Γ^{P} vanishes after being contracted with the polarization vectors, and only the Γ^{F} piece of the vertex survives. Thus Eq. (1.123) becomes

$$\mathcal{M} = [\mathcal{T}_s^{\mathrm{F}} + \mathcal{T}_t]^{mn}_{\mu\nu} L^{\mu\mu'}(k_1)\, L^{\nu\nu'}(k_2)[\mathcal{T}_s^{\mathrm{F}} + \mathcal{T}_t]^{mn*}_{\mu'\nu'}, \tag{1.125}$$

where $\mathcal{T}_s^{\mathrm{F}}$ is given by Figure 1.16(c) after substituting Γ by Γ^{F}.

Let us now define the quantities $\mathcal{S}^{mn}$ and $\mathcal{R}^{mn}_{\mu}$ (see Figure 1.16(d)):

$$\mathcal{S}^{mn} = \frac{1}{2} g f^{amn} d(q^2)(k_1 - k_2)^{\mu} \mathcal{V}^a_{\mu} \qquad \mathcal{R}^{mn}_{\mu} = g f^{amn} \mathcal{V}^a_{\mu}, \tag{1.126}$$

where $\mathcal{V}^a_{\rho}(p_2, p_1) = \bar{v}(p_2) g t^a \gamma_{\rho} u(p_1)$; they are related by $k_1^{\mu}\mathcal{R}^{mn}_{\mu} = -k_2^{\mu}\mathcal{R}^{mn}_{\mu} = q^2\mathcal{S}^{mn}$. Then, using the conditions $k_1^2 = k_2^2 = 0$, together with current conservation $q^{\rho}\mathcal{V}^a_{\rho} = 0$, we obtain the WI

$$k_1^{\mu}\Gamma^{\mathrm{F}}_{\alpha\mu\nu}(q, -k_1, -k_2) = -q^2 g_{\alpha\nu} + (k_1 - k_2)_{\alpha} k_{2\nu}. \tag{1.127}$$

Now the crucial point is that the q^2 term on the rhs of the preceding Ward identity will cancel against the $d(q^2)$ inside $\mathcal{T}_s^{\mathrm{F}}$, allowing the communication of this part with the (contracted) t-channel graph. Specifically,

$$k_1^{\mu}[\mathcal{T}_s^{\mathrm{F}}]^{mn}_{\mu\nu} = 2k_{2\nu}\mathcal{S}^{mn} - \mathcal{R}^{mn}_{\nu} \qquad k_1^{\mu}[\mathcal{T}_t]^{mn}_{\mu\nu} = \mathcal{R}^{mn}_{\nu} \tag{1.128}$$

so that

$$k_1^{\mu}[\mathcal{T}_s^{\mathrm{F}} + \mathcal{T}_t]^{mn}_{\mu\nu} = 2k_{2\nu}\mathcal{S}^{mn}. \tag{1.129}$$

This is the s-t cancellation: the term $\mathcal{R}$ comes with opposite sign and drops out in the sum (see Figure 1.17). An exactly analogous cancellation takes place if we contract by k_2^{ν}.

It is now easy to check that, indeed, all dependence on both n_{μ} and η^2 cancels in Eq. (1.125), as it should, and we are finally left with (omitting the fully contracted

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color and Lorentz indices)

$$\mathcal{M} = \left(\mathcal{T}_s^{\mathrm{F}}\mathcal{T}_s^{\mathrm{F}*} - 8\mathcal{S}\mathcal{S}^*\right) + \left(\mathcal{T}_s^{\mathrm{F}}\mathcal{T}_t^* + \mathcal{T}_s^{\mathrm{F}*}\mathcal{T}_t\right) + \mathcal{T}_t\mathcal{T}_t^*. \tag{1.130}$$

The reader may wonder what happens if, in Eq. (1.125), we do not eliminate Γ^{P} using the transversality of the polarization tensors and keep the full Γ instead of just Γ^{F}. In that case, the tree-level WI to use would be that of Eq. (1.35) instead of Eq. (1.127). This modification leaves the s-t cancellation unaffected but changes the terms proportional to $\mathcal{S}^{mn}$. However, the presence of the longitudinal parts in Γ^{P} triggers further s-t cancellations, exposed by using the decomposition of Eq. (1.57). As was shown in [27] and [28], the end result of this equivalent but slightly lengthier procedure is exactly that given in Eq. (1.130).

At this point, i.e., after the implementation of the s-t cancellation, we can define the *genuine* propagator-like, vertexlike, and boxlike subamplitudes, corresponding to the first, second, and third terms on the rhs of Eq. (1.130). Thus the propagator-like part of the rhs of the optical theorem, to be denoted by $(\mathrm{rhs})_1$, is given by

$$(\mathrm{rhs})_1 = \frac{1}{2} \times \frac{1}{2}\int_{\mathrm{PS}_{gg}} \left(\mathcal{T}_s^{\mathrm{F}}\mathcal{T}_s^{\mathrm{F}*} - 8\mathcal{S}\mathcal{S}^*\right). \tag{1.131}$$

It is elementary to verify that

$$\mathcal{T}_s^{\mathrm{F}}\mathcal{T}_s^{\mathrm{F}*} - 8\mathcal{S}\mathcal{S}^* = g^2 C_A \mathcal{V}_\mu^a d(q^2)\left[8q^2 P^{\mu\nu}(q) + 2(k_1-k_2)^\mu(k_1-k_2)^\nu\right] d(q^2)\mathcal{V}_\nu^a. \tag{1.132}$$

For the case of two massless gluons in the final state, the phase-space integrals give

$$\int_{\mathrm{PS}_{gg}} = \frac{1}{8\pi}, \qquad \int_{\mathrm{PS}_{gg}} (k_1-k_2)_\mu(k_1-k_2)_\nu = -\frac{1}{24\pi}q^2 P_{\mu\nu}(q), \tag{1.133}$$

and thus Eq. (1.131) becomes

$$(\mathrm{rhs})_1 = \mathcal{V}_\mu^a d(q^2)[\pi b g^2 q^2 P_{\mu\nu}(q) d(q^2)]\mathcal{V}_\nu^a. \tag{1.134}$$

On the other hand, for the propagator-like part of the lhs of the optical theorem, we have

$$(\mathrm{lhs})_1 = \mathcal{V}_\mu^a d(q^2)\Im m\widehat{\Pi}^{\mu\nu}(q) d(q^2)\mathcal{V}_\nu^a, \tag{1.135}$$

where the $\Im m\widehat{\Pi}^{\mu\nu}(q)$ should be obtained from the one-loop expression for $\widehat{\Pi}^{\mu\nu}(q)$ in Eqs (1.67) and (1.68); it is then obvious that, indeed, $(\mathrm{lhs})_1 = (\mathrm{rhs})_1$, namely, that the PT gluon self-energy satisfies the strong version of the optical theorem, as announced.

1.8 Positivity and the pinch technique gluon propagator

The Ward identity of Eq. (1.92) (or Eq. (1.119), in the light-cone gauge) tells us that the renormalization constant $\widehat{Z}_V$ for the PT proper vertex is the same as the wave function renormalization constant $\widehat{Z}_G$ for the PT propagator; just as in QED, the charged vertex renormalization constant Z_1 equals the charged propagator renormalization constant Z_2. In view of this equality, the relations between the radiatively corrected but unrenormalized PT propagator $\widehat{d}_U$ and the bare coupling g_U, and their renormalized counterparts, both renormalized at momentum μ, are as follows:

$$\widehat{d}(\mu; q^2) = \widehat{Z}_G^{-1}\widehat{d}_U(q^2) \qquad g(\mu) = g_U \widehat{Z}_G^{1/2}, \tag{1.136}$$

from which it follows that the renormalized product $g^2\widehat{d}$ is not only gauge invariant but renormalization group invariant (i.e., independent of μ). The same is true of the running charge $\bar{g}^2(q^2)$, so it is natural to make the factorization

$$g^2\widehat{d}(q^2) = \bar{g}^2(q^2)\widehat{H}(q^2), \tag{1.137}$$

where $\widehat{H}$ is a propagator of conventional type; for example,

$$-\widehat{H}(q^2) = \frac{1}{m^2(q^2) - q^2 - i\epsilon}, \tag{1.138}$$

which is also gauge and renormalization group invariant. Both the running charge $\bar{g}^2(q^2)$ and the other factor $-\widehat{H}(q^2)$ should obey the Källen–Lehmann (K-L) representation, for example,

$$\bar{g}^2(q^2) = \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}\sigma\, \frac{\rho(\sigma)}{\sigma - q^2 - i\epsilon}, \tag{1.139}$$

with a positive spectral function ρ and threshholds determined by the gluon mass m. This ensures that $\bar{g}^2$ is always positive for spacelike (negative) q^2 and has no real zeroes.

Now consider the product $\bar{g}^2\widehat{H}$. Asymptotic freedom tells us that as $|q^2|$ grows, $\bar{g}^2(q^2) \to 1/[b\ln(-q^2/\Lambda^2)]$, and we certainly expect that in the same limit, $\widehat{H}$ behaves like a free propagator so that $-\widehat{H} \sim 1/q^2$. But their product decreases *faster* than $1/q^2$ and therefore cannot obey a K-L representation with a positive spectral function.[11]

It turns out [29] that the Schwinger–Dyson equations for both the PT propagator and gluonic PT proper vertices can be expressed solely in terms of the well-behaved

[11] In QED, where e^2 times the photon propagator is gauge and renormalization group invariant, it is possible to write a K-L representation for the photon propagator. But this only holds because QED is asymptotically unstable and has a positive beta function, consistent with positivity of the spectral function, as the renormalization group equation for the propagator shows.

factor $\widehat{H}$ and other pieces that are both gauge and renormalization group invariant, such as given in Eq. (1.84) for the three-gluon PT vertex and Eq. (1.137) for the PT propagator. Thus, as a matter of practice, the nonpositivity of the PT propagator spectral function is not apparent or harmful.

As one might expect, the ansatz of factorization of the PT propagator into parts obeying the K-L represesntation is feasible only if the (zero-momentum) dynamical gluon mass m is large enough; in a simple model of the PT gluon gap equation [29], it was estimated that m/Λ should exceed about 1.2. This corresponds to an upper limit on the strong coupling at zero momentum of roughly $\alpha_s(0) = \bar{g}^2(0)/(4\pi) \lesssim 0.7$.

References

[1] J. M. Cornwall, Confinement and infrared properties of Yang–Mills theory, paper presented at the U.S.–Japan Seminar on Geometric Models of Elementary Particles, Osaka, Japan (1976).
[2] J. M. Cornwall, Nonperturbative mass gap in continuum QCD, in *Proceedings of the French–American Seminar on Theoretical Aspects of Quantum Chromodynamics*, Marseille, France (1981).
[3] J. M. Cornwall, Dynamical mass generation in continuum QCD, *Phys. Rev.* **D26** (1982) 1453.
[4] J. M. Cornwall and J. Papavassiliou, Gauge-invariant three-gluon vertex in QCD, *Phys. Rev.* **D40** (1989) 3474.
[5] D. Binosi and J. Papavassiliou, The pinch technique to all orders, *Phys. Rev.* **D66** (2002) 111901(R).
[6] D. Binosi and J. Papavassiliou, Pinch technique self-energies and vertices to all orders in perturbation theory, *J. Phys.* **G30** (2004) 203.
[7] L. F. Abbott, The background field method beyond one loop, *Nucl. Phys.* **B185** (1981) 189.
[8] D. Binosi, Electroweak pinch technique to all orders, *J. Phys.* **G30** (2004) 1021.
[9] S. Hashimoto, J. Kodaira, Y. Yasui, and K. Sasaki, The background field method: Alternative way of deriving the pinch technique's results, *Phys. Rev.* **D50** (1994) 7066.
[10] A. Denner, G. Weiglein, and S. Dittmaier, Gauge invariance of Green functions: Background field method versus pinch technique, *Phys. Lett.* **B333** (1994) 420.
[11] J. Papavassiliou, On the connection between the pinch technique and the background field method, *Phys. Rev.* **D51** (1995) 856.
[12] P. Di Vecchia, L. Magnea, A. Lerda, R. Russo, and R. Marotta, String techniques for the calculation of renormalization constants in field theory, *Nucl. Phys.* **B469** (1996) 235.
[13] M. Lavelle, Gauge invariant effective gluon mass from the operator product expansion, *Phys. Rev.* **D44**, 26 (1991).
[14] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison Wesley; New York, 1994).
[15] C. Becchi, A. Rouet, and R. Stora, Renormalization of gauge theories, *Ann. Phys.* **98** (1976) 287.
[16] I. V. Tyutin, Gauge invariance in field theory and statistical physics in operator formalism, *LEBEDEV-75-39*.

[17] N. Nakanishi, Covariant quantization of the electromagnetic field in the Landau gauge, *Prog. Theor. Phys.* **35** (1966) 1111.
[18] B. Lautrup, Canonical quantum electrodynamics in covariant gauges, *Mat. Fys. Medd. Dan. Vid. Selsk.* **35** (1966) 1.
[19] G. 't Hooft, Renormalization of massless Yang-Mills fields, *Nucl. Phys.* **B33** (1971) 173.
[20] P. Pascual and R. Tarrach, QCD: Renormalization for the practitioner, *Lect. Notes Phys*. **194** (1984) 53.
[21] J. S. Ball and T. W. Chiu, Analytic properties of the vertex function in gauge theories, 2, *Phys. Rev.* **D22** (1980) 2550 [Erratum, ibid. **D23** (1981) 3085].
[22] J. Papavassiliou, The gauge invariant four gluon vertex and its ward identity, *Phys. Rev.* **D47** (1993) 4728.
[23] J. M. Cornwall, W. S. Hou, and J. E. King, Gauge invariant calculations in finite temperature QCD: Landau ghost and magnetic mass, *Phys. Lett.* **B153** (1985) 173.
[24] J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Uniqueness of spontaneously broken gauge theories, *Phys. Rev. Lett.* **30** (1973) 1268 [Erratum, ibid. **31** (1973) 572].
[25] J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Derivation of gauge invariance from high-energy unitarity bounds on the *S* matrix, *Phys. Rev.* **D10** (1974) 1145 [Erratum, ibid. **D11** (1975) 972].
[26] C. H. Llewellyn Smith, High-energy behavior and gauge symmetry, *Phys. Lett.* **B46** (1973) 233.
[27] J. Papavassiliou and A. Pilaftsis, Gauge-invariant resummation formalism for two point correlation functions, *Phys. Rev.* **D54** (1996) 5315.
[28] J. Papavassiliou, E. de Rafael, and N. J. Watson, Electroweak effective charges and their relation to physical cross sections, *Nucl. Phys.* **B503** (1997) 79.
[29] J. M. Cornwall, Positivity issues for the pinch-technique gluon propagator and their resolution, *Phys. Rev. D* **80** (2009) 096001.