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EQUATIONALLY COMPACT ARTINIAN RINGS

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By a Noetherian (Artinian) ring $\mathscr{R} = \langle R; +, -, 0, \cdot \rangle$ we mean an associative ring satisfying the ascending (descending) chain condition on left ideals. An arbitrary ring $\mathscr{R}$ is said to be *equationally compact* if every system of ring polynomial equations with constants in $\mathscr{R}$ is simultaneously solvable in $\mathscr{R}$ provided every finite subset is. (The reader is referred to [**2**; **8**; **13**; **14**] for terminology and relevant results on equational compactness, and to [**4**] for unreferenced ring-theoretical results.) In this report a characterization of equationally compact Artinian rings is given – roughly speaking, these are the finite direct sums of finite rings and Prüfer groups; as consequences it is shown that an equationally compact ring satisfying both chain conditions is always finite, as is any Artinian ring which is a compact topological ring. Further, using a result of S. Warner [**11**], we give a necessary and sufficient condition for an equationally compact Noetherian ring with identity to be a compact topological ring. A few remarks on the embedding of certain rings into equationally compact rings are made, and we obtain also here generalizations of known results on compact topological rings.

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Preliminary results. We begin by deriving a few useful tools. Let $\mathscr{R}$ be a ring and A an ideal of $\mathscr{R}$ ("ideal" always means two-sided ideal), and let Σ be a system of equations with constants in A. If $(x_0, x_1, \ldots, x_\gamma, \ldots)_{\gamma<\alpha}$ are the variables occurring in Σ, then the solution set of Σ in $\mathscr{R}$ is a certain subset S of R^α. If such a system Σ exists such that the projection of S onto the first component is the ideal A, then we shall say that A *is expressible by equations*. For example, if $\mathscr{R}$ has an identity and A is finitely generated as a left ideal, then A is expressible by the equation $x_0 = x_1a_1 + \ldots + x_na_n$, where $a_1, \ldots, a_n$ generate A.

If x is a variable and A is an ideal of $\mathscr{R}$, then "$x \in A$" will denote, quite naturally, the relational predicate $A(x)$ for the unary relation A on R.

We will make recurrent use of the following observation:

Remark. Let $\mathscr{R}$ be an equationally compact ring. Suppose $(A_i|i \in I)$ is a family of ideals of $\mathscr{R}$, each of which is expressible by equations, and suppose

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$(x_i | i \in I)$ is a family of variables. Let Σ be a set of equations with constants in $\mathscr{R}$. Then the system of formulas

$$\Omega := \Sigma \cup \{x_i \in A_i; i \in I\}$$

is solvable in $\mathscr{R}$ provided it is finitely solvable in $\mathscr{R}$.

Proof. Let A_i be expressible by the system Σ_i, $i \in I$; let

$$(x_{0i}, x_{1i}, \ldots, x_{\gamma i}, \ldots)_{\gamma < \alpha_i}$$

denote the variables appearing in Σ_i, whereby it is assumed that the variables $x_{\gamma i}$ and $x_{\delta j}$ are distinct if $i \neq j$ or $\gamma \neq \delta$, and that no $x_{\gamma i}$ occurs in Σ. Now the finite solvability of Ω implies the finite solvability of the system of equations

$$\bigcup (\Sigma_i | i \in I) \cup \{x_i = x_{0i}; i \in I\} \cup \Sigma,$$

which is then solvable by the equational compactness of $\mathscr{R}$, and a solution obviously yields a solution of Ω in $\mathscr{R}$.

PROPOSITION 1. *Let $\mathscr{R}$ be a ring and A an ideal of $\mathscr{R}$ such that A is expressible by equations and $\mathscr{R}$ is equationally compact. Then $\mathscr{R}/A$ and A are equationally compact rings.*

Proof. Suppose $\Sigma = \{\Phi_i = 0; i \in I\}$ is a system of equations with constants in $\mathscr{R}/A$ and finitely solvable in $\mathscr{R}/A$. Now each Φ_i induces a polynomial in $\mathscr{R}$, say Φ_i', by replacing the constants by arbitrary representatives in R. If z_i, $i \in I$, are variables not occurring in Σ, then the system

$$\{\Phi_i' = z_i; i \in I\} \cup \{z_i \in A; i \in I\}$$

is clearly finitely solvable in $\mathscr{R}$, hence (by the last Remark) solvable in $\mathscr{R}$, and any solution taken modulo A yields a solution for Σ in $\mathscr{R}/A$. Thus $\mathscr{R}/A$ is equationally compact, and a similar argument shows that A is equationally compact.

Next we derive a useful remark on matrix rings.

PROPOSITION 2. *Let $\mathscr{R}$ be a ring with identity, let $\mathfrak{m}$ be a nonzero cardinal and let $\mathscr{S} = \mathscr{M}_{\mathfrak{m} \times \mathfrak{m}}(\mathscr{R})$ (i.e., $\mathscr{S}$ is the ring of linear transformations on the free $\mathscr{R}$-module $\mathscr{F}$ on $\mathfrak{m}$ generators). Then $\mathscr{S}$ is equationally compact if and only if $\mathscr{R}$ is equationally compact and $\mathfrak{m}$ is finite.*

Proof. Sufficiency. If Σ is a finitely solvable system of equations with constants in $\mathscr{S}$ then by replacing each variable x by the variable matrix $(x_{ij} | 1 \leqq i, j \leqq \mathfrak{m})$, Σ reduces in the obvious fashion to a system over $\mathscr{R}$, finitely solvable in $\mathscr{R}$, hence solvable in $\mathscr{R}$; such a solution yields a solution for Σ in $\mathscr{S}$.

Necessity. Let I be a set with cardinality $\mathfrak{m}$ and let $\{e_i; i \in I\}$ be a basis for $\mathscr{F}$. Fix $i_0 \in I$. For each $i \in I$ define $\pi_i \in \mathscr{S}$ as follows: $\pi_i(e_j) = \delta_{j i_0} e_i$, for all $j \in I$. Let p_i be the retraction of $\mathscr{F}$ onto $\mathscr{R}e_i$. Then the system

$$\Sigma = \{p_i x = \pi_i; i \in I\}$$

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is finitely solvable (for a finite subset $J \subseteq I$ of indices appearing, take x as follows: $x(e_{i_0}) = \sum_{i \in J} e_i$, and $x(e_j) = 0$ for $j \neq i_0$). However Σ forces x to be such that $x(e_{i_0}) = \sum_{i \in I} e_i$ which is impossible unless $\mathfrak{m}$ is finite. To see that $\mathscr{R}$ is equationally compact, consider a system Σ of equations with constants in $\mathscr{R}$ and finitely solvable in $\mathscr{R}$. For $r \in R$ let $e(r)$ denote the matrix (a_{ij}) where $a_{11} = r$ and $a_{ij} = 0$ otherwise. Replace every constant $r \in R$ appearing in Σ by $e(r)$ and every variable x by $e(1) \cdot x \cdot e(1)$. Then the system of equations thus obtained from Σ is finitely solvable in $\mathscr{S}$, hence solvable in $\mathscr{S}$. Taking the upper left hand entries from a solution in $\mathscr{S}$ yields obviously a solution of Σ in $\mathscr{R}$.

If $\mathscr{R} = \langle R; +, -, 0, \cdot \rangle$ is a ring, we denote by $\mathscr{R}^+$ the underlying additive abelian group $\langle R; +, -, 0 \rangle$.

PROPOSITION 3. *Let $\mathscr{R}$ be an equationally compact ring and let $\mathscr{D} = \langle D; +, -, 0 \rangle$ be the largest divisible subgroup of $\mathscr{R}^+$. Then $R \cdot D = D \cdot R = \{0\}$. In particular, $\mathscr{D}$ is an ideal of $\mathscr{R}$. Moreover, the ring $\mathscr{R}/\mathscr{D}$ is equationally compact.*

Proof. Let $d \in D$ and $r \in R$. Consider the system of equations

$$\Sigma = \{ (x_i - x_j)x_{ij} = r \cdot d; i, j \in I, i \neq j\}$$

where I is a set with cardinality larger than $|R|$. Σ is finitely solvable in $\mathscr{R}$, since for any finite subset of indices $J \subseteq I$, choose n_i, $i \in J$, to be distinct natural numbers, set $c_i = n_i r$, and pick d_{ij} such that $(n_i - n_j)d_{ij} = d$ for $i \neq j$. Then clearly $(c_i - c_j)d_{ij} = r \cdot d$ for all $i, j \in J, i \neq j$. Thus Σ must be solvable in $\mathscr{R}$. However Σ implies $x_i = x_j$ for some $i \neq j$, because of the cardinality of I, hence $r \cdot d = 0$. An almost identical argument shows that $d \cdot r = 0$.

We recall that an abelian group $\mathscr{G}$ is *algebraically compact* (in the sense of Kaplansky [**6**]) if

$$\mathscr{G} \cong \mathscr{C} \oplus (\Pi(\mathscr{G}_p | p = \text{prime}))$$

where $\mathscr{C}$ is divisible and each $\mathscr{G}_p$ is a module over the p-adic integers complete in its p-adic topology and containing no nonzero element which is divisible by all powers of p. The group $\mathscr{R}^+$ is equationally compact and therefore algebraically compact as was shown by S. Balcerzyk in [**1**]; thus in view of the latter condition on the G_p's the subgroup $\mathscr{D}$ under discussion equals $\mathscr{C}$ and is expressible by the equations

$$\{x_0 = n \cdot x_n; n \in \mathbf{N}\}.$$

Thus, $\mathscr{R}/\mathscr{D}$ is equationally compact by Proposition 1.

PROPOSITION 4. *If $\mathscr{A}$ is a finite ring and $\mathscr{R}$ is an equationally compact $\mathscr{A}$-algebra, then $\mathscr{A} * \mathscr{R}$ is an equationally compact ring, where the carrier set of*

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$\mathscr{A} * \mathscr{R}$ is $A \times R$, addition is defined componentwise, and multiplication is defined by

$$(a, x)(b, y) = (ab, ay + bx + xy).$$

Proof. Let Σ be a system of equations with constants in $\mathscr{A} * \mathscr{R}$, finitely solvable in $\mathscr{A} * \mathscr{R}$. Let $(x_0, x_1, \ldots, x_\gamma, \ldots)_{\gamma<\alpha}$ be the variables appearing in Σ. Replace each variable x_γ by (z_γ, y_γ), inducing the system Σ_0 with the obvious interpretation of solvability (i.e., y_γ must be replaced by an element of R and z_γ by an element of A). We construct by transfinite induction a sequence $(a_0, a_1, \ldots, a_\gamma, \ldots)_{\gamma<\alpha} \in A^\alpha$, such that $\Sigma_0((z_\gamma \to a_\gamma)_{\gamma<\alpha})$ is finitely solvable ("$z_\gamma \to a_\gamma$" means that the variable z_γ is replaced by a_γ). Let β be an ordinal and let a_γ, $\gamma < \beta$, be already constructed such that

$$\Sigma_\beta : = \Sigma_0((z_\gamma \to a_\gamma)_{\gamma<\beta})$$

is finitely solvable. (For $\beta = 0$ the construction is trivial.) Suppose for each $a \in A$ the system $\Sigma_\beta(z_\beta \to a)$ is not finitely solvable; i.e., for each $a \in A$ there exists a finite subset $\Sigma_{\beta,a}$ of Σ_β such that $\Sigma_{\beta,a}(z_\beta \to a)$ is not solvable. But then the finite system

$$\bigcup_{a \in A} \Sigma_{\beta,a} \subseteq \Sigma_\beta$$

is clearly not solvable. This is a contradiction, so there exists $a_\beta \in A$ such that $\Sigma_\beta(z_\beta \to a_\beta)$ is finitely solvable, and the induction step is complete. Thus $\Sigma_1 : = \Sigma_0((z_\gamma \to a_\gamma)_{\gamma<\alpha})$ is a finitely solvable system involving only the variables $(y_\gamma)_{\gamma<\alpha}$. Now any $\Phi \in \Sigma_1$ is equivalent to a pair of equations (Φ_1, Φ_2), where Φ_2 is an equation with constants in $\mathscr{R}$ and involving the variables $(y_\gamma)_{\gamma<\alpha}$, and Φ_1 involves *only* constants (from A). Therefore Σ_1 is solvable because $\mathscr{R}$ is equationally compact.

COROLLARY. *Let $\mathscr{R}$ be an equationally compact ring such that $\mathscr{R}^+$ is a bounded torsion group. Then there exists an equationally compact ring $\mathscr{S}$ with identity such that $\mathscr{R}$ is an ideal in $\mathscr{S}$ of finite index.*

Proof. Let n be a natural number such that $n \cdot R = (0)$, and let $\mathbf{Z}_n$ denote the ring of integers modulo n. Then clearly $\mathscr{R}$ is a $\mathbf{Z}_n$-algebra and therefore $\mathscr{S} : = \mathbf{Z}_n * \mathscr{R}$ is equationally compact by Proposition 4. But $\mathscr{S}$ is a ring with identity $(1, 0)$ and the map $r \mapsto (0, r)$ is a ring-embedding of $\mathscr{R}$ into $\mathscr{S}$, making $\mathscr{R}$ clearly an ideal of $\mathscr{S}$ of finite index.

Semisimplicity. A ring $\mathscr{R}$ is *semisimple* if its Jacobson radical $J(\mathscr{R})$ is zero. We consider now the impact of this condition on equationally compact Artinian and Noetherian rings.

PROPOSITION 5. *The Jacobson radical of a ring is expressible by equations.*

Proof. Recall that an element r of a ring $\mathscr{R}$ is *left quasi-regular* if there exists an element $y \in R$ with $r + y + y \cdot r = 0$. It is well-known that $J(\mathscr{R})$ is the largest left quasi-regular left ideal in $\mathscr{R}$; that is, $r \in J(\mathscr{R})$ if and only if

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the left ideal generated by r is left quasi-regular. Hence $J(\mathscr{R})$ is expressible by the set of equations

$$\{s \cdot x_0 + z \cdot x_0 + y_{s,z} + y_{s,z} \cdot (s \cdot x_0 + z \cdot x_0) = 0; s \in R, z \in \mathbf{Z}\}.$$

COROLLARY. *If the ring $\mathscr{R}$ is equationally compact, then so are the rings $\mathscr{R}/J(\mathscr{R})$ and $J(\mathscr{R})$.*

Proof. The proof follows from Propositions 1 and 5.

LEMMA 1. *A semisimple Artinian ring $\mathscr{R}$ is equationally compact if and only if it is finite.*

Proof. Sufficiency. It is perhaps appropriate at this point to remark that an arbitrary universal algebra $\mathscr{A} = \langle A; F\rangle$ which is also a compact topological algebra (i.e., A can be endowed with a compact Hausdorff topology compatible with the algebraic structure) is equationally compact (see [**8**]). Indeed, the solution set of any equation is a closed subset of an appropriate power of A endowed with the Tychonov product topology.

As a special case, any finite algebra, hence any finite ring, is equationally compact.

Necessity. It is easily seen that a finite direct sum of rings is equationally compact if and only if every summand is. By Wedderburn's theorem $\mathscr{R}$ is a finite direct sum of matrix rings over division rings, each of which, therefore, is equationally compact. By Proposition 2 the respective divisions rings are equationally compact. However, equationally compact division rings are known to be finite (consider, for example, the system $\Sigma = \{(x_i - x_j)y_{ij} = 1; i, j \in I, i \neq j\}$ for suitably large I). Thus $\mathscr{R}$ is finite.

PROPOSITION 6. *Let $\mathscr{R}$ be an equationally compact semisimple Noetherian ring with identity. Then $\mathscr{R}$ is finite.*

In view of the fact that equationally compact Noetherian rings with identity are necessarily linearly compact for the discrete topology, Proposition 6 follows from D. Zelinsky's decomposition of linearly compact semisimple rings [**15**, Proposition 11] and Lemma 1. For completeness' sake we give a proof, which is in the spirit of an argument of S. Warner [**12**, p. 55].

LEMMA 2. *Let $\mathscr{R}$ be as above but, in addition, a primitive ring. Then $\mathscr{R}$ is finite (and hence simple Artinian).*

Proof. By the Jacobson-Chevalley Density Theorem $\mathscr{R}$ is a dense ring of linear transformations on a vector space V with basis, say, $\{e_i; i \in I\}$. For each $i \in I$, let

$$A_i = \{\phi \in R; \phi(e_i) = 0\}.$$

A_i is a left ideal, hence finitely generated, and therefore expressible by equations. Let $(v_i)_{i \in I} \in V^I$ be chosen arbitrarily. By denseness there exists

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for each $i \in I$ $\phi_i \in R$ such that $\phi_i(e_i) = v_i$. Thus the system of equations

$$\Sigma = \{x = \phi_i + z_i; i \in I\} \cup \{z_i \in A_i; i \in I\}$$

is finitely solvable (again by denseness) and hence solvable. However Σ implies that x must map each e_i to v_i. Thus $\mathscr{R}$ is the complete transformation ring, and therefore by Proposition 2 and Lemma 1 a finite matrix ring over a division ring.

Proof of Proposition 6. As is well-known $\mathscr{R}$ is a subdirect product of a family of primitive rings $\{\mathscr{R}/A_i; i \in I\}$ where the A_i's are ideals of $\mathscr{R}$. Since $\mathscr{R}$ is Noetherian with identity, each A_i is expressible by equations, so $\mathscr{R}/A_i$ is equationally compact by Proposition 1 and Noetherian. Hence by Lemma 2 $\mathscr{R}/A_i$ is finite, simple, and Artinian. Hence the A_i's are maximal ideals. Let $r = (r_i + A_i)_{i \in I} \in \Pi(\mathscr{R}/A_i | i \in I)$. The system

$$\Sigma = \{x = r_i + z_i; i \in I\} \cup \{z_i \in A_i; i \in I\}$$

is finitely solvable by the Chinese Remainder Theorem, hence solvable in $\mathscr{R}$. But Σ implies $x = r$, so $r \in \mathscr{R}$. Hence $\mathscr{R}$ is the full direct product and so I must be finite because $\mathscr{R}$ is Noetherian.

We summarize these results in the following

THEOREM 1. *For an equationally compact semisimple ring $\mathscr{R}$ the following are equivalent:*

(i) *$\mathscr{R}$ is finite.*
(ii) *$\mathscr{R}$ is Artinian.*
(iii) *$\mathscr{R}$ is Noetherian with identity.*

Noetherian rings. Although we are not able to characterize structurally those Noetherian rings with identity which are equationally compact, Theorem 1 and a crucial result of Warner yield a pleasant criterium relating equational compactness and topological compactness in this class of rings. We paraphrase the relevant result:

PROPOSITION 7 [**11**, Theorem 2]. *Let $\mathscr{R}$ be a topological Noetherian ring with identity. Then $\mathscr{R}$ is topologically compact if and only if the topology of $\mathscr{R}$ is the radical topology $\mathscr{T}$, $\mathscr{R}$ is complete for that topology and $\mathscr{R}/J(\mathscr{R})$ is a finite ring.*

Now let $\mathscr{R}$ be an equationally compact Noetherian ring with identity. By Theorem 1, $\mathscr{R}/J(\mathscr{R})$ is finite. Now the topology $\mathscr{T}$ defined by taking the powers of $J(\mathscr{R})$ as a neighbourhood base of 0 is not necessarily Hausdorff. However, we shall show that the space $(R, \mathscr{T})$ is complete. To see this, consider a Cauchy sequence $(r_i)_{i=1,2,\ldots}$ in R. For each natural number n choose i_n such that the subsequence $(r_i | i \geqq i_n)$ is $J(\mathscr{R})^n$-close. Since $\mathscr{R}$ is Noetherian with identity, the ideal $J(\mathscr{R})^n$ is expressible by equations, so we have the system of equations

$$\Sigma = \{x = r_{i_n} + z_n; n \in \mathbf{N}\} \cup \{z_n \in J(\mathscr{R})^n; n \in \mathbf{N}\}$$

which is finitely solvable (if m is the largest index appearing in a finite subset, set $x = r_{i_m}$ and $z_n = r_{i_m} - r_{i_n}$ for all $n \leqq m$). Hence Σ is solvable and obviously any solution is a limit of $(r_i)_{i=1,2,\ldots}$. As a matter of fact, $\mathscr{T}$ is compact. To see this we quote the following

LEMMA 3. *Let $\mathscr{R}$ be a ring with identity, A and B two ideals such that B is finitely generated as a left ideal and both $\mathscr{R}/A$ and $\mathscr{R}/B$ are finite. Then $\mathscr{R}/A \cdot B$ is finite.*

The proof is a straightforward counting of cosets as given in the proof of [**10**, Lemma 4], where the hypothesized commutativity is not used.

Now by Lemma 3 and induction, we see that $J(\mathscr{R})^n$ has finite index in $\mathscr{R}$ for each n. This means that the family of cosets

$$\mathscr{F} = \{r + J(\mathscr{R})^n; r \in R, n \in \mathbf{N}\}$$

is a subbase of closed sets for the topology $\mathscr{T}$, and by the Alexander Subbase Theorem $\mathscr{T}$ is compact if every subfamily of $\mathscr{F}$ with the finite intersection property has a nonempty intersection. The latter is however clear by equational compactness of $\mathscr{R}$ and the fact that each $J(\mathscr{R})^n$ is expressible by equations. In view of Proposition 7 we have proved

THEOREM 2. *Let $\mathscr{R}$ be an equationally compact Noetherian ring with identity. Then the radical topology is a complete and compact topology on R, and $\mathscr{R}/J(\mathscr{R})$ is finite. Moreover, $\mathscr{R}$ is a compact topological ring if and only if*

$$\bigcap \, (J(\mathscr{R})^n | n \in \mathbf{N}) = \{0\}.$$

Remark. By [**3**], equational and topological compactness coincide when $\mathscr{R}$ is a commutative Noetherian ring with identity. In general, we do not know of an equationally compact Noetherian ring with identity which is not topologically compact. Moreover, this still leaves open the question, first posed (for universal algebras) by J. Mycielski [**8**, p. 5 P484] whether every equationally compact Noetherian ring with identity is a retract of a compact topological ring.

Artinian rings. As an immediate consequence of Theorem 2 we have the following

COROLLARY 1. *An equationally compact Artinian ring $\mathscr{R}$ with identity is finite.*

Proof. Two well-known results assert that $\mathscr{R}$ is Noetherian and $J(\mathscr{R})$ is nilpotent. Hence $J(\mathscr{R})^n = (0)$ for some n, thus the radical topology is discrete and, by Theorem 2, compact, which forces $\mathscr{R}$ to be finite.

COROLLARY 2 [**11**, Theorem 2, Corollary]. *A compact topological Artinian ring with identity is finite.*

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The case of arbitrary Artinian rings requires a closer look.

LEMMA 4. *If $\mathscr{R}$ is an equationally compact Artinian ring such that $\mathscr{R}^+$ is a bounded torsion group, then $\mathscr{R}$ is finite.*

Proof. By Proposition 4 there is an equationally compact ring with identity $\mathscr{S}$, such that $\mathscr{R}$ is an ideal of $\mathscr{S}$ and $\mathscr{S}/\mathscr{R}$ is finite. Thus $\mathscr{R}$ is an Artinian $\mathscr{S}$-module, as is the finite $\mathscr{S}$-module $\mathscr{S}/\mathscr{R}$, and so $\mathscr{S}$ is an Artinian $\mathscr{S}$-module, i.e., $\mathscr{S}$ is an Artinian ring. But then $\mathscr{S}$ is finite by Corollary 1.

LEMMA 5. *Let $\mathscr{R}$ be an equationally compact torsion-free Artinian ring. Then $\mathscr{R} = (0)$.*

Proof. A torsion-free Artinian ring has, as well-known, a left identity e and is an algebra over the rationals. But then the system of equations

$$\{(x_i - x_j)y_{ij} = e; i, j \in I, i \neq j\}$$

is finitely solvable in $\mathscr{R}$, hence solvable in $\mathscr{R}$. Taking $|I| > |R|$ forces $e = 0$, i.e., $\mathscr{R} = (0)$.

Recall that the Prüfer group $\mathbf{Z}(p^\infty)$ is the subgroup of the unit circle in the complex plane consisting of all p^nth roots of unity for all natural numbers n and fixed prime p.

THEOREM 3. *For an Artinian ring $\mathscr{R}$ the following are equivalent:*
(i) *$\mathscr{R}$ is equationally compact.*
(ii) *$\mathscr{R}^+ \cong \mathscr{B} \oplus \mathscr{P}$ where $\mathscr{B} = \langle B; +, -, 0\rangle$ is a finite group, $\mathscr{P} = \langle P; +, -, 0\rangle$ is a finite direct sum of Prüfer groups, and $R \cdot P = P \cdot R = \{0\}$.*
(iii) *$\mathscr{R}$ is a (algebraic) retract of a compact topological ring.*

Proof. (iii) $\Rightarrow$ (i) holds for arbitrary universal algebras (see [**8**]).

(i) $\Rightarrow$ (ii): By a result of F. Szász [**9**, Satz 4] every Artinian ring is the ring direct sum of its torsion ideal $\mathscr{T}$ and some torsion-free ideal $\mathscr{D}$. But $\mathscr{D}$ is then an equationally compact torsion-free Artinian ring, so must be (0) by Lemma 5. Hence $\mathscr{R} = \mathscr{T}$. Let $\mathscr{R}^+ = \mathscr{B} \oplus \mathscr{P}$ be the (group) decomposition of $\mathscr{R}^+$ into its divisible part $\mathscr{P}$ and reduced part $\mathscr{B}$ (as a torsion divisible abelian group $\mathscr{P}$ is, as well-known, a direct sum of Prüfer groups). Now by Proposition 3, $R \cdot P = P \cdot R = \{0\}$. Thus every subgroup of $\mathscr{P}$ is an ideal of $\mathscr{R}$ and therefore $\mathscr{P}$ is a finite direct sum, because $\mathscr{R}$ is Artinian.

Now the family $\mathscr{F} = \{n \cdot \mathscr{B} \oplus \mathscr{P}; n \in \mathbf{N}\}$ is easily seen to be a downward directed set of ideals of $\mathscr{R}$, hence has a smallest element $n_0 \cdot \mathscr{B} \oplus \mathscr{P}$ since $\mathscr{R}$ is Artinian. However $n_0 \cdot \mathscr{B} \oplus \mathscr{P}$ is clearly divisible, being the meet of $\mathscr{F}$, and so $n_0 \cdot \mathscr{B} = (0)$ as $\mathscr{B}$ is reduced. Thus $\mathscr{B}$ is a bounded torsion group. The quotient $\mathscr{R}/\mathscr{P}$ is Artinian and, again by Proposition 3, equationally compact; moreover, $(\mathscr{R}/\mathscr{P})^+ \cong \mathscr{B}$. Hence $\mathscr{B}$ is finite by Lemma 4, and we are done.

(ii) ⇒ (iii): Let $\mathscr{R}^+ \cong \mathscr{B} \oplus \mathscr{P}_1 \oplus \ldots \oplus \mathscr{P}_n$ where $\mathscr{B}$ is finite and $\mathscr{P}_i = \mathbf{Z}(p_i^\infty)$, $i = 1, \ldots, n$. Each $\mathscr{P}_i$ is divisible, hence injective and therefore retract of every extending abelian group, e.g., the compact topological circle group $\mathscr{C}$. Let $f_i : \mathscr{C} \to \mathscr{P}_i$ be a retraction. Endowing $\mathscr{B}$ with the discrete topology, we have then a (group) retraction

$$f : \mathscr{H} \to \mathscr{R}^+$$

where $\mathscr{H}$ is the compact topological group $\mathscr{B} \oplus (\oplus (\mathscr{C}|i = 1, \ldots, n))$ and $f = \mathrm{id} \quad \oplus f_1 \oplus \ldots \oplus f_n$.

If multiplication is defined on H by letting every element of

$$\oplus (C|i = 1, \ldots, n)$$

annihilate H and then extending by distributivity, $\mathscr{H}$ clearly becomes a ring. Moreover $\mathscr{H}$ is a topological ring under the given topology, because the inverse image under the multiplication map of any subset of H is the finite union of sets of the form $A_1 \times A_2$ where each A_j is a coset of

$$\oplus (\mathscr{C}|i = 1, \ldots, n)$$

in $\mathscr{H}$, all of which, however, are closed; thus multiplication is continuous. By a straightforward calculation one sees that f is a ring homomorphism, and the proof is complete.

It should be noted that the equivalence of (i) and (iii) answers a special case of the previously mentioned problem in Mycielski [**8**, P484].

Remark. It is not possible, in general, to obtain a ring-direct sum in the decomposition given in condition (ii). Consider, for example, the ring $\mathscr{R}$, where $\mathscr{R}^+ = \mathbf{Z}_2 \oplus \mathbf{Z}(2^\infty)$, $R \cdot \mathbf{Z}(2^\infty) = \mathbf{Z}(2^\infty) \cdot R = \{0\}$, and $(1, 0) \cdot (1, 0)$ is defined to be the primitive square root of unity in $\mathbf{Z}(2^\infty)$. Here we have a nonzero divisible element appearing as a product of two nondivisible elements.

The following improves Corollary 2 of Theorem 2:

COROLLARY 1. *A compact topological Artinian ring $\mathscr{R}$ is finite.*

Proof. By Theorem 3 we have $\mathscr{R}^+ \cong \mathscr{B} \oplus \mathscr{P}_1 \oplus \ldots \oplus \mathscr{P}_n$, where $\mathscr{B}$ is finite and $\mathscr{P}_i = \mathbf{Z}(p_i^\infty)$. Let $\mathscr{P}_i^k$ be the subgroup of $\mathscr{P}_i$ consisting of all p_i^kth roots of unity, and let

$$\mathscr{R}^k = \mathscr{B} \oplus \mathscr{P}_1^k \oplus \ldots \oplus \mathscr{P}_n^k.$$

Now $R = \bigcup (R^k|k = 1, 2, 3, \ldots)$, that is, the intersection of the complements $R \backslash R^k$ is empty. By the Baire Category Theorem [**7**, p. 200] at least one of the sets $R \backslash R^k$ is not dense in R, i.e., for some k_0 the finite subgroup $\mathscr{R}^{k_0}$ contains a nonempty open set; this forces the topology to be discrete and therefore by compactness $\mathscr{R}$ must be finite.

COROLLARY 2. *An equationally compact ring satisfying both chain conditions is finite.*

Proof. The proof is clear.

Compactifications. We conclude with a few remarks on the question of embedding rings into equationally compact ones. Following the terminology of [**14**] we define, for a fixed universal algebra $\mathscr{A}$, a *compactification* of $\mathscr{A}$ to be an algebra $\mathscr{B}$ such that $\mathscr{B}$ is equationally compact and $\mathscr{A}$ is a subalgebra of $\mathscr{B}$. $\mathscr{B}$ is a *quasi-compactification* of $\mathscr{A}$ if $\mathscr{A}$ is a subalgebra of $\mathscr{B}$ and every system of equations with constants in $\mathscr{A}$ and finitely solvable in $\mathscr{A}$ is solvable in $\mathscr{B}$. The classes of compactifications (respectively quasi-compactifications) of $\mathscr{A}$ are denoted by $\text{Comp}(\mathscr{A})$ (respectively $c(\mathscr{A})$). Clearly $\text{Comp}(\mathscr{A}) \subseteq c(\mathscr{A})$. A *positive formula* is a formula of the first order predicate calculus which is built up from polynomial equations (of a fixed algebraic type) by application of the logical connectives $\forall$, $\exists$, $\wedge$, $\vee$ in a finite number of steps. We quote the following result of G. H. Wenzel:

PROPOSITION 8 [**14**, Theorems 8.10 and 12]. *Let $\mathscr{A}$ be an algebra and let K be one of* $\text{Comp}(\mathscr{A})$ *or* $c(\mathscr{A})$. *If K is not empty then there is an algebra $\mathscr{B}$ in K such that $\mathscr{B}$ satisfies every positive formula with constants in $\mathscr{A}$ which is satisfiable in $\mathscr{A}$.*

PROPOSITION 9. *Let $\mathscr{R}$ be a ring and Δ an infinite division ring. If $\mathscr{R}$ contains Δ as a subring, then $c(\mathscr{R}) = \emptyset$. In particular, an infinite semisimple Artinian ring cannot be quasi-compactified, and hence not (algebraically) embedded into a compact topological ring. If $\mathscr{R}$ is an algebra over Δ and $R^2 \neq \{0\}$, then $c(\mathscr{R}) = \emptyset$. If $\mathscr{D}$ denotes any divisible subgroup of $\mathscr{R}^+$ and $R \cdot D \neq \{0\}$, then $c(\mathscr{R}) = \emptyset$. In particular, if $\mathscr{R}$ is a subring of a compact topological ring, then $R \cdot D = D \cdot R = \{0\}$.*

Proof. If $c(\mathscr{R}) \neq \emptyset$, then $c(\mathscr{R})$ contains a ring by Proposition 8; the proofs are then implicit in Proposition 3.

PROPOSITION 10. *Let $\mathscr{R}$ be an infinite Artinian ring with identity. Then* $\text{Comp}(\mathscr{R}) = \emptyset$. *In particular, $\mathscr{R}$ cannot be (algebraically) embedded in a compact topological ring.*

Proof. $\mathscr{R}$ is Noetherian by a well-known result; hence $\mathscr{R}$ has finite length. If n is the (unique!) length of a maximal chain of left ideals then as is easily checked, the property of "maximal length of at most n" is characterized by the positive formula

$$\Psi = (\forall x_1) \ldots (\forall x_{n+2})(\exists y_1) \ldots (\exists y_{n+2})\Big(\bigvee_{1<k\leqq n+2} x_k = y_1x_1 + \ldots + y_{k-1}x_{k-1}\Big).$$

Thus if $\text{Comp}(\mathscr{R}) \neq \emptyset$, there is by Proposition 8 an $\mathscr{S} \in \text{Comp}(\mathscr{R})$ satisfying Ψ, i.e., of finite length. But this cannot be, since by Corollary 2 of Theorem 3, $\mathscr{S}$ would be finite.

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