Bull. Aust. Math. Soc. **102** (2020), 162–171
doi:10.1017/S0004972720000179

ANCIENT SOLUTIONS OF CODIMENSION TWO SURFACES WITH CURVATURE PINCHING - RETRACTED

ZHENGCHAO JI

(Received 4 August 2019; accepted 26 January 2020; first published online 6 March 2020)

Abstract

We prove rigidity theorems for ancient solutions of geometric flows of immersed submanifolds. Specifically, we find conditions on the second fundamental form that characterise the shrinking sphere among compact ancient solutions for the mean curvature flow in codimension two surfaces.

2010 *Mathematics subject classification*: primary 53C44; secondary 35K55.

Keywords and phrases: ancient solutions, mean curvature flow.

1. Introduction

We study ancient solutions of the mean curvature flow of immersed submanifolds. The mean curvature flow is described by

$$\frac{\partial \varphi}{\partial t}(x,t) = \mathbf{H}(x,t), \tag{1.1}$$

where $\varphi(\cdot, t)$ is a family of immersions of a closed n-dimensional manifold M^n into $\mathbb{R}^n$ and H is the mean curvature. A solution is called *ancient* if it is defined on a time interval from $(-\infty, T)$. These solutions typically arise as limits of rescalings and model the asymptotic profile of the flow near a singularity (see, for example, [13]). Ancient solutions have also been considered in theoretical physics, where they appear as steady-state solutions of the renormalisation-group flow in the boundary sigma model [6, 21].

There are many investigations of ancient solutions of the mean curvature flow in codimension one. In particular, attention has focused on ancient solutions which are convex, since this property is enjoyed by the blow-up limits of general mean convex solutions (see [16, 17]). The easiest example is provided by the shrinking sphere, which is the only compact convex homothetically contracting solution. An example of an ancient solution which is not self-similar is the so-called *Angenent oval* [3]. This is a convex solution of the curve-shortening flow in the plane which has larger and larger eccentricity as $t \to -\infty$. Compact convex ancient solutions in the plane have been completely classified by Daskalopoulos *et al.* [12] and are either shrinking round circles or Angenent ovals.

In higher dimensions, other examples of nonhomothetical ancient solutions of the mean curvature flow have been constructed and analysed in [4, 5, 8, 14, 23]. These examples suggest that the class of convex compact ancient solutions of the mean curvature flow is wide and that a complete classification is difficult to obtain. However, it is possible to prove rigidity results which provide a partial structural description of this class. A typical result of this kind is the following theorem, stating that an ancient solution with uniformly pinched principal curvature is necessarily a shrinking sphere.

Theorem 1.1 ([17]; see also [14]). *Let $\{M_t\}_{t\in(-\infty,0)}$ be a family of closed convex hypersurfaces of $\mathbb{R}^{n+1}$ evolving by mean curvature flow, with $n > 1$. Let λ_1 denote the smallest principal curvature of M_t and H the mean curvature. Suppose that there exists $\epsilon > 0$ such that $\lambda_1 > \epsilon H$ on M_t for every t. Then M_t is a family of shrinking spheres.*

Several other characterisations of the sphere have been obtained in terms of a control on the diameter growth as $t \to -\infty$, or on the ratio between the outer and inner radius (see [14, 17]). An equivalence similar to Theorem 1.1 involving pinching properties of the intrinsic curvature was obtained in [9] for the ancient solution of the Ricci flow. Another interesting rigidity result, proved by Haslhofer and Kleiner [15], is that a closed mean convex ancient solution which is uniformly noncollapsed in the sense of Andrews [1] is necessarily convex. A similar statement has been obtained by Langford [18], replacing the noncollapsing property by the assumption that λ_1/H is bounded below. Langford and Lynch [19] generalised these results to a large class of flows with speed given by a function of the curvatures homogeneous of degree one. Very recently, Bourni *et al.* [8] have shown uniqueness of rotationally symmetric collapsed ancient solutions in all dimensions.

In the case of the mean curvature flow in the sphere $\mathbb{S}^{n+1}$, there is an ancient solution consisting of shrinking geodesic n-dimensional spheres, converging to an equator as $t \to -\infty$ and to a point as $t \to 0$. In [17], some characterisations of the shrinking round solution in terms of curvature pinching are given. Compared to the case of Euclidean ambient space, it can be observed that the positive curvature of the ambient space increases rigidity and weaker pinching conditions suffice to obtain the result. A strong result in this context was obtained by Bryan *et al.* [10] for curvature flows on the sphere $\mathbb{S}^{n+1}$ for very general speed functions. They proved that the shrinking geodesic sphere is the only closed convex ancient solution with bounded curvature for large negative times. In dimension one (that is, for curve-shortening flow), Bryan and Louie [11] proved that embeddedness is sufficient to conclude that a closed ancient curve in $\mathbb{S}^2$ is either a shrinking circle or an equator.

Recently, Risa and Sinestrari [22] derived rigidity results for ancient solutions of more general curvature flows, by considering either mean curvature flow in higher codimension, or the hypersurface case with more general speeds than mean curvature. They consider various kinds of flows and their results are in a similar spirit to

https://doi.org/10.1017/S0004972720000179 Published online by Cambridge University Press

Theorem 1.1, showing that a suitable uniform pinching condition characterises the shrinking sphere among convex ancient solutions.

We consider ancient solutions on codimension two surfaces with some different pinching conditions from [22]. Our pinching conditions are inspired by [7] and our result is given in the following theorem.

THEOREM 1.2. *Let $\{M_t\}_t = \varphi(M, t)$ be a closed ancient solution of (1.1) in $\mathbb{R}^4$, with $n = 2, k = 2$. Suppose that, for all $t \in (-\infty, 0)$, we have $|H|^2 > 0$ and*

$$|A|^2 + 2\gamma|K^{\perp}| \leq k|H|^2,$$

where $\gamma = 1 - \frac{4}{3}k$ and $k \leq \frac{29}{40}$. Suppose, furthermore, that the norm of the second fundamental form is uniformly bounded away from the singularity, so there exists $A_0 > 0$ such that $|A|^2 \leq A_0$ in $(-\infty, -1)$. Then M_t is a family of shrinking spheres.

2. Notation and preliminary results

We adhere to the notation of [2] and, in particular, use the canonical space-time connections introduced in that paper. A fundamental ingredient in the derivation of the evolution equations is Simons' identity:

$$\triangle h_{ij} = \nabla_i\nabla_j H + H \cdot h_{ip}h_{pj} - h_{ij}h_{pq}h_{pq} + 2h_{jq}h_{ip}h_{pq} - h_{iq}h_{pq}h_{pj} - h_{jq}h_{qp}h_{pi}.$$

The timelike Codazzi equation, combined with Simons' identity, produces the evolution equation for the second fundamental form:

$$\nabla_{\partial t}h_{ij} = \triangle h_{ij} + h_{ij}h_{pq}h_{pq} + h_{iq}h_{pq}h_{pj} - h_{ip}h_{jq}h_{pq}. \tag{2.1}$$

The evolution equation for the mean curvature vector is found by taking the trace with g_{ij}:

$$\nabla_{\partial t}H = \triangle H + Hh_{pq}h_{pq}.$$

The evolution equations of the squared lengths of the second fundamental form and the mean curvature vector are

$$\frac{\partial}{\partial t}|A|^2 = \triangle|A|^2 - 2|\nabla A|^2 + 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij\alpha}h_{ij\beta}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big(\sum_{p} h_{ip\alpha}h_{jp\beta} - h_{jp\alpha}h_{ip\beta}\Big)^2, \tag{2.2}$$

$$\frac{\partial}{\partial t}|H|^2 = \triangle|H|^2 - 2|\overset{\perp}{\nabla} A|^2 + 2\sum_{i,j}\Big(\sum_{\alpha} h_{\alpha}h_{ij\beta}\Big)^2.$$

The last term of (2.2) is the squared length of the normal curvature, which we denote by $|Rm^{\perp}|^2$. For convenience we label the reaction terms of these evolution equations by

$$R_1 = 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij\alpha}h_{ij\beta}\Big)^2 + |Rm^{\perp}|^2,$$

$$R_2 = \sum_{i,j}\Big(\sum_{\alpha} h_{\alpha}h_{ij\beta}\Big)^2.$$

https://doi.org/10.1017/S0004972720000179 Published online by Cambridge University Press

3. Evolution of normal curvature

In this section we compute the evolution equation for the normal curvature. The normal curvature tensor in local orthonormal frames for the tangent $\{e_i : i = 1, 2\}$ and normal $\{\nu_\alpha : \alpha = 1, 2\}$ bundles is given by

$$R^\perp_{ij\alpha\beta} = h_{ip\alpha}h_{jp\beta} - h_{jp\alpha}h_{ip\beta}. \tag{3.1}$$

We will often compute in a local orthonormal normal frame $\{\nu_\alpha : \alpha = 1, 2\}$ where $\nu_1 = H/|H|$. As the normal bundle is two-dimensional, ν_2 is determined by ν_1 up to sign. With this choice of frame the second fundamental form becomes

$$\begin{cases} \overset{\circ}{A}_1 = A_1 - \dfrac{|H|}{n} Id, \\ \overset{\circ}{A}_2 = A_2 \end{cases}$$

and

$$\begin{cases} \operatorname{tr} A_1 = |H|, \\ \operatorname{tr} A_2 = 0. \end{cases}$$

It is also always possible to choose the tangent frame $\{e_i : i = 1, 2\}$ to diagonalise A_1. We often refer to the orthonormal frame $\{e_1, e_2, e_3, e_4\} = \{e_1, e_2, \nu_1, \nu_2\}$, where the $\{e_i\}$ diagonalise A_1 and $\nu_1 = H/|H|$, as the 'special orthonormal frame'. Codimension two surfaces have four independent components of the second fundamental form, which still makes it tractable to work with individual components, similar to the role of principal curvatures in hypersurface theory. Working in the special orthonormal frame, we often find it convenient to represent the second fundamental form by

$$h_{ij} = \begin{bmatrix} |H|/2 + a & 0 \\ 0 & |H|/2 - a \end{bmatrix} \nu_1 + \begin{bmatrix} b & c \\ c & -b \end{bmatrix} \nu_2,$$

so that $h_{111} = |H|/2 + a$, $h_{221} = b$, $h_{122} = c$ and so on. Note that $|\overset{\circ}{A}|^2 = 2a^2 + 2b^2 + 2c^2$.

Just as a surface has only one sectional curvature K, a codimension two surface also has only one normal curvature, which we denote by $K^\perp$. In the special orthonormal frame the normal curvature is

$$\begin{aligned} K^\perp = R^\perp_{1234} &= \sum_p (h_{1p1}h_{2p2} - h_{2p1}h_{1p2}) \\ &= h_{111}h_{212} - h_{211}h_{112} + h_{121}h_{222} - h_{221}h_{122} \\ &= 2ac. \end{aligned}$$

Note also that $|Rm^\perp|^2 = 16a^2c^2$. Differentiating (3.1) and using Equation (2.1),

$$\begin{aligned} \frac{\partial}{\partial_t} R^\perp_{ij\alpha\beta} &= \triangle R^\perp_{ij\alpha\beta} - 2\sum_{p,r} (\nabla_p h_{ip\alpha}\nabla_q h_{jp\beta} - \nabla_q h_{jp\alpha}\nabla_q h_{ip\beta}) \\ &\quad + \sum_p \Big(\frac{d}{dt}h_{ip\alpha}h_{jp\beta} + h_{ip\alpha}\frac{d}{dt}h_{jp\beta} - \frac{d}{dt}h_{jp\alpha}h_{ip\beta} + h_{jp\alpha}\frac{d}{dt}h_{ip\beta}\Big). \end{aligned} \tag{3.2}$$

RETRACTED

Computing in the special orthonormal frame and denoting the reaction terms by $dK^{\perp}/dt$, the nonlinearity for codimension two surfaces simplifies to

$$\frac{d}{dt}K^{\perp} = 4ac\bigg(\Big(\frac{|H|}{2} - a\Big)^2 - \Big(\frac{|H|}{2} + a\Big)\Big(\frac{|H|}{2} - a\Big) + 2b^2\bigg) + 4ac\bigg(3c^2 + \Big(\frac{|H|}{2} + a\Big)^2\bigg)$$
$$= K^{\perp}(|A|^2 + |\mathring{A}|^2 - 2b^2).$$

For notational convenience we set

$$\nabla_{\mathrm{evol}}K^{\perp} = \sum_{p,q}(\nabla_q h_{ip\alpha}\nabla_q h_{jp\beta} - \nabla_q h_{jp\alpha}\nabla_q h_{ip\beta})$$

and

$$R_3 = K^{\perp}(|A|^2 + |\mathring{A}|^2 - 2b^2).$$

Substituting the simplified nonlinearity into (3.2), we obtain the evolution equation for the normal curvature

$$\frac{d}{dt}K^{\perp} = \triangle K^{\perp} - 2\nabla_{\mathrm{evol}}K^{\perp} + K^{\perp}(|A|^2 + |\mathring{A}|^2 - 2b^2),$$

and a little more computation shows that the length of the normal curvature evolves by

$$\frac{d}{dt}|K^{\perp}| = \triangle|K^{\perp}| - 2\frac{K^{\perp}}{|K^{\perp}|}\nabla_{\mathrm{evol}}K^{\perp} + |K^{\perp}|(|A|^2 + |\mathring{A}|^2 - 2b^2).$$

We remark that the complicated structure of the gradient terms prevents an application of the maximum principle to conclude the flat normal bundle is preserved.

With the exception of the latter estimates, the following gradient estimates are well known; the third estimate is new and was proved in [7].

Proposition 3.1. *We have*

$$|\nabla A|^2 \geq \frac{3}{n+2}|\nabla H|^2,$$
$$|\nabla A|^2 - \frac{1}{n}|\nabla H|^2 \geq \frac{2(n-1)}{3n}|\nabla A|^2,$$
$$|\nabla A|^2 \geq 2\nabla_{\mathrm{evol}}K^{\perp} \quad \textit{if } n = 2.$$

4. Flow in Euclidean space

In this section we will prove the following result.

Theorem 4.1. *Let $\{M_t\}_t = \varphi(M,t)$ be a closed ancient solution of (1.1) in $\mathbb{R}^4$, with $n = 2, k = 2$. Suppose that, for all $t \in (-\infty, 0)$, we have $|H|^2 > 0$ and*

$$|A|^2 + 2\gamma|K^{\perp}| \leq k|H|^2,$$

where $\gamma = 1 - \frac{4}{3}k$ and $k \leq \frac{29}{40}$. Suppose, furthermore, that the norm of the second fundamental form is uniformly bounded away from the singularity, so there exists $A_0 > 0$ such that $|A|^2 \leq A_0$ in $(-\infty, -1)$. Then M_t is a family of shrinking spheres.

https://doi.org/10.1017/S0004972720000179 Published online by Cambridge University Press

As in [2, 17], for fixed small $\sigma > 0$, we consider the function

$$f_\sigma = \frac{|\mathring{A}| + 2\gamma|K^\perp|}{H^{2(1-\sigma)}},$$

and we observe that, for any σ, f_σ vanishes at $x \in M$ if and only if x is an umbilical point. Therefore, if $f_\sigma = 0$ everywhere on M_t, then M_t is a totally umbilical submanifold, hence an n-dimensional sphere in $\mathbb{R}^{n+k}$.

As spheres evolve by homothetic shrinking, f_σ will remain zero for all subsequent times. Thus, to obtain Theorem 4.1, it is enough to show that f is identically zero on some time interval $(-\infty, T_1]$, with $T_1 < 0$. For this purpose, we prove the following estimate.

Proposition 4.2. *Under the hypotheses of Theorem 4.1, there are constants $\alpha, \beta > 0$ depending only on n, k and $C = C(k, n, A_0) > 0$ such that, for all $[T_0, T_1] \in (-\infty, -1)$ and for all $p > \alpha, \sigma < \beta/\sqrt{p}, \sigma p > n$, we have*

$$\left(\int_{M_t} f_\sigma^p\right)^{1/\sigma p} \leq \frac{C}{|T_0|^{1-n/\sigma p} - |t|^{1-n/\sigma p}} \quad \text{for all } t \in (T_0, T_1]. \tag{4.1}$$

The proposition immediately implies Theorem 4.1. Indeed, sending T_0 to $-\infty$ in (4.1), we see that f_σ^p is zero for every $t < T_1$ for suitable values of σ and p. M_t is then a family of shrinking spheres.

Proof of Proposition 4.2. The first part of the proof follows the strategy of [17] together with the estimates of [7] (see also [20]). If we set

$$\epsilon_\nabla = 1 - \tfrac{4}{3}k - \gamma,$$

differentiate f_σ in time and substitute in the relevant evolution equations, then

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma = {} & \frac{\triangle|A|^2 - 2|\nabla A|^2 + 2R_1}{|H|^{2(1-\sigma)}} + \frac{2\gamma(\triangle|K|^\perp - 2(K^\perp/|K^\perp|)\nabla_{\mathrm{evol}}K^\perp + R_3)}{|H|^{2(1-\sigma)}} \\ & - \frac{(1-\sigma)(|A|^2 + 2\gamma|K^\perp| - 1/n|H|^2)}{|H|^{2(2-\sigma)}}(\triangle|H|^2 - 2|\nabla H|^2 + 2R_2) \\ & - \frac{1}{n}\frac{(\triangle|H|^2 - 2|\nabla H|^2 + 2R_2)}{|H|^{2(1-\sigma)}}.\end{aligned}$$

After some computation, we find that the Laplacian of f_σ is

$$\begin{aligned}\triangle f_\sigma = {} & \frac{(2-\sigma)(1-\sigma)(|A|^2 + 2\gamma|K^\perp| - 1/n|H|^2)}{(|H|^2)^{3-\sigma}}|\nabla|H|^2|^2 \\ & - \frac{2(1-\sigma)}{|H|^{2(2-\sigma)}}\langle\nabla_i(|A|^2 + 2\gamma|K^\perp| - 1/n|H|^2), \nabla_i|H|^2\rangle \\ & - \frac{(1-\sigma)(|A|^2 + 2\gamma|K^\perp| - 1/n|H|^2)}{|H|^{2(1-\sigma)}}\triangle|H|^2 + \frac{\triangle(|A|^2 + 2\gamma|K^\perp| - 1/n|H|^2)}{|H|^{2(1-\sigma)}},\end{aligned}$$

https://doi.org/10.1017/S0004972720000179 Published online by Cambridge University Press

and the gradients satisfy

$$
\begin{aligned}
&-\frac{2(1-\sigma)}{|H|^{2(2-\sigma)}}\langle \nabla_i(|A|^2+2\gamma|K^\perp|-1/n|H|^2), \nabla_i|H|^2\rangle \\
&\quad = -\frac{2(1-\sigma)}{|H|^2}\langle \nabla_i|H|^2, \nabla f_\sigma\rangle - \frac{(1-\sigma)^2}{(|H|^2)^2} f_\sigma |\nabla|H|^2|^2.
\end{aligned}
$$

These two formulae yield

$$
\frac{\partial}{\partial t} f_\sigma \le \triangle f_\sigma + \frac{2(1-\sigma)}{|H|^2}\langle \nabla_i|H|^2, \nabla_i f_\sigma\rangle - \frac{2\epsilon_\nabla}{|H|^{2(1-\sigma)}}|\nabla A|^2 + 2\sigma|A|^2 f_\sigma. \tag{4.2}
$$

Using the above evolution inequality, we can derive the evolution inequality for the L^p norm of f_σ^p. Firstly,

$$
\partial_t \int f_\sigma^p \, d\mu = p\int (f_\sigma^{p-1}\partial_t f_\sigma - H^2 f_\sigma^p)\, d\mu.
$$

We insert (4.2) into the above equation to get

$$
\begin{aligned}
\partial_t \int f_\sigma^p \, d\mu \le& -p(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2 - \int \frac{2p\epsilon_\nabla}{H^{2(1-\sigma)}}|\nabla A|^2 f_\sigma^{p-1}\, d\mu \\
&+ \int \frac{2p(1-\sigma)H}{|H|^2}|\nabla H||\nabla f_\sigma f_\sigma^{p-1}\, d\mu + 2\sigma \int |A|^2 f_\sigma^p \, d\mu.
\end{aligned}
$$

By the well-known inequality

$$
|\nabla A|^2 \ge \frac{3}{n+2}|\nabla H|^2
$$

and the pinching condition

$$
|A|^2 + 2\gamma|K^\perp| \le k|H|^2,
$$

where $\gamma = 1 - \frac{4}{3}k$ and $k \le \frac{29}{40}$, we get

$$
\begin{aligned}
\partial_t \int f_\sigma^p \, d\mu \le& -p(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2 - \int \frac{3p\epsilon_\nabla}{2|H|^{2(1-\sigma)}}|\nabla H|^2 f_\sigma^{p-1}\, d\mu \\
&+ \int \frac{2p(1-\sigma)H}{|H|^2}|\nabla H||\nabla f_\sigma|f_\sigma^{p-1}\, d\mu + \frac{29}{20}\sigma \int |H|^2 f_\sigma^p \, d\mu.
\end{aligned} \tag{4.3}
$$

Using Young's inequality,

$$
\begin{aligned}
&2p(1-\sigma)\int_{M_t} \frac{H}{|H|^2}|\nabla H||\nabla f_{\sigma,\eta}|f_{\sigma,\eta}^{p-1}\, d\mu \\
&\quad \le p\int_{M_t} \frac{1}{|H|^2}\Big(\frac{1}{\beta} f_{\sigma,\eta}^{p-2}|H|^2|\nabla f_{\sigma,\eta}|^2 + \beta f_{\sigma,\eta}^p |\nabla H|^2|\Big)\, d\mu \\
&\quad \le \frac{p}{\beta}\int_{M_t} f_{\sigma,\eta}^{p-2}|\nabla f_{\sigma,\eta}|^2 + p\beta \int_{M_t} \frac{f_{\sigma,\eta}^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\, d\mu.
\end{aligned}
$$

https://doi.org/10.1017/S0004972720000179 Published online by Cambridge University Press

If we take $\beta = 1/(p-1)$ and $p > (2/\epsilon_\nabla) + 1$, and use the above inequality in (4.3), then

$$\partial_t \int f_\sigma^p \, d\mu \le -\frac{p(p-1)}{2} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 - \int \frac{p\epsilon_\nabla}{|H|^{2(1-\sigma)}} |\nabla H|^2 f_\sigma^{p-1} \, d\mu + \frac{29}{20}\sigma \int |H|^2 f_\sigma^p \, d\mu. \tag{4.4}$$

In addition, [1, Proposition 12] shows that there exists a constant ϵ_0 such that

$$\int_{M_t} |H|^2 f_\sigma^p \, d\mu \le \frac{p\eta}{\epsilon_0} \int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} |\nabla H|^2 \, d\mu + \frac{p-1}{\epsilon_0 \eta} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu$$

for all $p \ge 2, \eta > 0$. If we fix $\eta = 8\sigma/\epsilon_0$ and we take any p, σ such that $p > 16/\epsilon_\nabla$, $\sigma \le (\epsilon_0/8)\sqrt{\epsilon_\nabla/p}$, we obtain

$$\begin{aligned} 4p\sigma \int_{M_t} |H|^2 f_\sigma^p \, d\mu &\le \Big(\frac{32\sigma^2 p}{\epsilon_0{}^2} + \frac{16\sigma}{\epsilon_0}\Big) p \int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} |\nabla H|^2 \, d\mu \\ &\quad + \frac{p(p-1)}{2} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \\ &\le \Big(\frac{\epsilon_0}{2} + \frac{\epsilon_0}{2}\Big) p \int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} |\nabla H|^2 \, d\mu + \frac{p(p-1)}{2} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2, \end{aligned}$$

so that (4.4) implies

$$\begin{aligned} \partial_t \int f_\sigma^p \, d\mu &\le -\frac{p(p-1)}{2} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \\ &\quad - \int \frac{p\epsilon_\nabla}{|H|^{2(1-\sigma)}} |\nabla H|^2 f_\sigma^{p-1} \, d\mu + \frac{29}{20}\sigma \int |H|^2 f_\sigma^p \, d\mu. \\ &\le -\frac{p(p-1)}{2} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \\ &\quad - \int \frac{p\epsilon_\nabla}{|H|^{2(1-\sigma)}} |\nabla H|^2 f_\sigma^{p-1} \, d\mu + 4\sigma \int |H|^2 f_\sigma^p \, d\mu. \\ &\le -2p\sigma \int_{M_t} |H|^2 f_\sigma^p \, d\mu \end{aligned}$$

for $p > 16/\epsilon_\nabla$, $\sigma \le (\epsilon_0/8)\sqrt{\epsilon_\nabla/p}$.

Thanks to the definition and our pinching assumption,

$$0 \le f_\sigma \le |H|^{2\sigma},$$

so we obtain

$$\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu \le -2p\sigma \int_{M_t} f_\sigma^{p+1/\sigma} \, d\mu \le \Big(\int_{M_t} f_\sigma^p \, d\mu\Big)^{1+1/\sigma} \cdot |M_t|^{-1/(\sigma p)} \tag{4.5}$$

using Hölder's inequality, where $|M_t|$ is the volume of M_t.

RETRACTED

https://doi.org/10.1017/S0004972720000179 Published online by Cambridge University Press

From [22], there exists a constant $C = C(n, A_0)$ such that

$$|M_t| \leq C|t|^n,$$

for all $t \leq -1$. Using this fact, the statement of the proposition follows easily. Setting

$$\psi(t) = \int_{M_t} f_\sigma^p \, d\mu,$$

and using (4.5),

$$\frac{d}{dt}\psi^{-1/(\sigma p)} = -\frac{-1}{\sigma p}\psi^{((1/\sigma p)+1)}\frac{d}{dt}\psi \geq C(|t|)^{n/(\sigma p)}.$$

As $\psi(t) \neq 0$ implies $\psi(s) \neq 0$ for $s < t$, we obtain, by integrating on a time interval $(T_0, t]$ with $t \leq T_1$,

$$\psi^{-1/(\sigma p)}(t) \geq \psi^{-1/(\sigma p)}(T_0) + C\int_{|t|}^{|T_0|} \tau^{-n/(\sigma p)} > -\frac{1}{\sigma p} > C(|T_0|^{1-1/(\sigma p)} - |t|^{1-1/(\sigma p)})$$

as $\sigma p > n$. □

Acknowledgements

The author would like to thank Professor Kefeng Liu and Professor Hongwei Xu for their continued support, advice and encouragement. Thanks also to Professor En-Tao Zhao for helpful discussions.

References

[1] B. Andrews, 'Noncollapsing in mean-convex mean curvature flow', *Geom. Topol.* **16** (2012), 1413–1418.
[2] B. Andrews and C. Baker, 'Mean curvature flow of pinched submanifolds to spheres', *J. Differential Geom.* **85** (2010), 357–396.
[3] S. Angenent, 'Shrinking doughnuts', *Progr. Nonlinear Differential Equations Appl.* **7** (1992), 21–38.
[4] S. Angenent, 'Formal asymptotic expansions for symmetric ancient ovals in mean curvature flow', *Netw. Heterog. Media* **8** (2013), 1–8.
[5] S. Angenent, P. Daskalopoulos and N. Sesum, 'Unique asymptotics of ancient convex mean curvature flow solutions', *J. Diff. Geom.* **3** (2019), 381–455.
[6] I. Bakas and C. Sourdis, 'Dirichlet sigma models and mean curvature flow', *J. High Energy Phys.* **06** (2007), 437–437.
[7] C. Baker and H. T. Nguyen, 'Codimension two surfaces pinched by normal curvature evolving by mean curvature flow', *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34** (2017), 1599–1610.
[8] T. Bourni, M. Langford and G. Tinaglia, 'Deforming hypersurfaces of the sphere by their mean curvature', *Math. Z.* **195** (1987), 205–219.
[9] S. Brendle, G. Huisken and C. Sinestrari, 'Ancient solutions to the Ricci flow with pinched curvature', *Duke Math. J.* **158** (2011), 537–551.
[10] P. Bryan, M. Ivaki and J. Scheuer, 'On the classification of ancient solutions to curvature flows on the sphere', Preprint, arXiv:1604.01694, 2016.
[11] P. Bryan and J. Louie, 'Classification of convex ancient solutions to curve shortening flow on the sphere', *J. Geom. Anal.* **26** (2016), 858–872.

https://doi.org/10.1017/S0004972720000179 Published online by Cambridge University Press

[12] P. Daskalopoulos, R. Hamilton and N. Sesum, 'Classification of compact ancient solutions to the curve shortening flow', *J. Differential Geom.* **84** (2010), 455–464.
[13] R. Hamilton, 'Formation of singularities in the Ricci flow', *Surv. Differ. Geom.* **2** (1996), 7–136.
[14] R. Hamilton and O. Hershkovits, 'Ancient solutions of the mean curvature flow', *Commun. Anal. Geom.* **24** (2016), 593–604.
[15] R. Haslhofer and B. Kleiner, 'Mean curvature flow of mean convex hypersurfaces', *Commun. Pure Appl. Math.* **70** (2017), 511–546.
[16] G. Huisken and C. Sinestrari, 'Convexity estimates for mean curvature flow and singularities of mean convex surfaces', *Acta. Math.* **183** (1999), 45–70.
[17] G. Huisken and C. Sinestrari, 'Convex ancient solutions of the mean curvature flow', *J. Differential Geom.* **101** (2015), 267–287.
[18] M. Langford, 'A general pinching principle for mean curvature flow and applications', *Calc. Var. Partial Differential Equations* **56** (2017), article ID 107.
[19] M. Langford and S. Lynch, 'Sharp one-sided curvature estimates for fully nonlinear curvature flows and applications to ancient solutions', Preprint, arXiv:1704.03802, 2017.
[20] K. F. Liu, H. W. Xu, F. Ye and E. T. Zhao, 'The extension and convergence of mean curvature flow in higher codimension', *Trans. Amer. Math. Soc.* **370** (2017), 223–2261.
[21] S. Lukyanov, E. Vitchev and A. Zamolodchikov, 'Integrable model of boundary interaction: the paper clip', *J. Nucl. Phys.* B **683** (2004), 423–454.
[22] S. Risa and C. Sinestrari, 'Ancient solutions of geometric flows with curvature pinching', *J. Geom. Anal.* **29** (2017), 1206–1232.
[23] B. White, 'The nature of singularities in mean curvature flow of mean-convex sets', *J. Amer. Math. Soc.* **16** (2003), 123–138.

ZHENGCHAO JI, Center of Mathematical Sciences, Zhejiang University,
Hangzhou, 310027, PR China
e-mail: jizhengchao@zju.edu.cn

https://doi.org/10.1017/S0004972720000179 Published online by Cambridge University Press