THE FLOW OF ICE, TREATED AS A NEWTONIAN VISCOUS LIQUID, AROUND A CYLINDRICAL OBSTACLE NEAR THE BED OF A GLACIER

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ABSTRACT. This paper describes an analytical solution of the equations of motion and heat conduction for ice flowing around a cylindrical solid inclusion and over a solid plane boundary. This is intended to be a simplified representation of the flow of clean glacier ice around a stone and over a rigid rock bed. The ice is treated as a Newtonian viscous liquid and the equations are solved in two dimensions. Regelation boundary conditions are applied at both ice–rock interfaces. It is found that finite solutions for the temperature and stream function only exist for the special cases in which two dimensionless critical wavelengths are zero. That is, unless the stone is very far from the glacial bed, the classical regelation boundary conditions cannot be obeyed over the whole of its surface.

RÉSUMÉ. *L'écoulement de la glace, traitée comme un liquide visqueux Newtonien, autour d'un obstacle cylindrique au voisinage du fond d'un glacier.* Ce papier expose une solution analytique du mouvement de conduction de chaleur pour de la glace s'écoulant autour d'un obstacle solide cylindrique au dessus d'un support plan solide. Le but est d'obtenir une représentation simplifiée de l'écoulement d'une glace propre de glacier autour d'un bloc sur un lit plan rigide. La glace est traitée comme un liquide visqueux Newtonien et les équations sont résolues dans deux dimensions. Les conditions aux limites de regel sont appliquées aux deux interfaces glace–rocher. On trouve qu'il n'existe de solutions finies pour la température et la fonction de courant que pour les cas spéciaux où deux longueurs d'ondes critiques sans dimensions sont nulles. A moins que le bloc soit très éloigné du lit rocheux, la condition aux limites classique de regel ne peut être respectée sur la totalité de la surface.

ZUSAMMENFASSUNG. *Der Fluss von Eis, betrachtet als Newtonsche viskose Flüssigkeit, um ein zylindrisches Hindernis nahe dem Gletscherbett.* Dieser Beitrag beschreibt eine analytische Lösung der Bewegungs- und Wärmeleitungsgleichungen für Eis, das um einen festen zylindrischen Einschluss und über eine feste ebene Fläche fliesst. Sie soll den Fluss von reinem Gletschereis um einen Felsblock und über ein starres Felsbett vereinfacht darstellen. Das Eis wird als Newtonsche viskose Flüssigkeit betrachtet; die Lösungen sind zweidimensional. Die Randbedingungen für Regelation werden auf die Grenzflächen von Eis und Fels angewandt. Es zeigt sich, dass finite Lösungen für die Temperatur- und Strömungsfunktion nur für jene Spezialfälle existieren, in denen die zwei dimensionslosen kritischen Wellenlängen verschwinden. Die klassischen Randbedingungen für Regelation können nur dann über die ganze Oberfläche des Felsblocks eingehalten werden, wenn dieser sehr weit vom Gletscherbett entfernt ist.

INTRODUCTION

The problem of describing the motion of a clast in the basal ice of a glacier lies at the heart of physically-based models of erosion and deposition. Order-of-magnitude calculations have been made by Lewis (1960), who discussed the maximum force that could be exerted by a rock held in ice, and by Boulton ([ᶜ1975]), who adapted Weertman's (1957) sliding theory to produce a condition for the onset of abrasion. Röthlisberger (1968) identified and described the processes which tend to bring clasts into contact with the bed. Full analytical solutions for the motion of isolated clasts in ice unbounded by other solid surfaces have been obtained by Glen and others (1957), to explain the preferred orientation of the long axes of clasts in the direction of flow of a glacier, and by Watts (unpublished), in order to derive equivalent viscosities for dilute suspensions of particles in ice. This paper describes an analytical solution for the flow of ice past a solid object and over an external solid boundary. The analysis indicates that interaction between processes occurring on the glacier bed and on the surface of a clast in the basal ice will produce effects which have not been considered in previous calculations.

Figure 1 shows the simple two-dimensional problem that will be analysed in this paper. A long cylindrical stone with circular cross-section, radius R, lies in the ice at the point $x = 0$, $y = \Delta$. The stone lies in an area zone (i) where the glacier bed and the surface of the ice are horizontal planes, $y = 0$ and $y = H$. Up- and down-stream, in zones (ii) and (iii), the
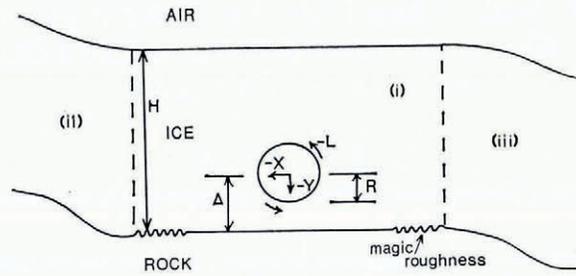
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Fig. 1. A cylindrical stone radius R lying a distance Δ from a plane rock surface in ice of thickness H.

glacier bed is undulating and has an overall slope. We shall suppose that the boundaries of zone (i) are far enough from the origin for the ice flow not to be affected by the presence of the stone.

In basal-sliding theories dealing with clean ice (Weertman, 1957; Lliboutry, 1968; Nye, 1969, 1970; Kamb, 1970) it is usual to assume that a lubricating "regelation layer" of water exists between ice and bed-rock. Thus, locally, the shear stress at the ice–rock interface is zero. However, if the bed undulates about a given base plane the ice exerts a net force on the rock which can be interpreted as the result of an average shear stress acting over the base plane. In zones (ii) and (iii) there is an average shear stress between the ice and the un-dulating bed. In zone (i) we will suppose that the bed can be given an effective or "magic" roughness for part or all of its area, despite the fact that geometrically it is treated as a flat plane. This means that the shear stress on this part of the boundary $y = 0$ can take non-zero values. Thus, the transition between the region of flow over an effectively rough bed and the region of flow around the smooth stone and over a smooth bed is independent of the choice of the extent of zone (i). This allows a physically reasonable description of the boundary conditions on $y = 0$ to be combined with simple boundary conditions describing the velocity distribution in the ice at the boundaries between zones (i), (ii), and (iii). The geometry of this problem has been specified in such a way that the two boundaries at which the physical processes are most complicated, the ice–rock interfaces, are coordinate curves in the bipolar coordinate system.

Suppose that in zones (ii) and (iii) the ice moves under gravity with a steady-state flow pattern which depends on the shape of the bed and the depth of the ice. In general there will not be a steady-state situation in zone (i), for, unless the stone is at rest with respect to the bed or moves in such a way that the separation Δ is maintained, the pattern of flow must change with the change in the relative positions of the ice–rock interfaces. This problem does not arise in the analysis of the motion of isolated clasts in ice.

For the sake of making a relatively simple first analysis, it is assumed in this paper that forces $-X$, $-Y$ and a couple L oppose the forces X, Y, and couple L exerted by the ice so that a steady-state situation is maintained. Thus, only a small sub-set of the possible solutions for the motion of the ice and stone is being considered. The set of solutions with $X = 0$, $L = 0$, and $-Y$ equal to the weight of the stone is physically possible. Other solutions are artificial, but the direction of the resultant force that would have to be applied to the stone, in order to maintain its position with respect to the bed, gives an indication of the direction in which the stone would tend to move in the real situation where gravity is the only force acting at a distance on it.

Boulton ([^{c}1975]) derives a condition for the lodgement of a stone on a glacier bed by estimating the horizontal force exerted on the stone by the ice and equating this to the retarding friction between stone and bed. Strictly speaking, the analysis in this paper cannot

be applied to a stone in contact with the bed. However, if the stone is extremely close to the bed, i.e. when $\Delta \approx R$, we might postulate the existence of a very small irregularity on the stone which allows normal and frictional forces to be transmitted without disturbing the pattern of flow of the ice or altering the physical processes at the boundaries. Then we might hope to derive a more accurate estimate of the lodgement condition, based on an analytical solution for the forces of the ice on the stone rather than an order-of-magnitude calculation.

THE NAVIER–STOKES EQUATIONS IN BIPOLAR COORDINATES

The bipolar coordinates α, β are defined in terms of the Cartesian coordinates x, y by the equations

$$\left.\begin{array}{l} x = \dfrac{a \sin \beta}{\cosh \alpha - \cos \beta}, \\[3mm] y = \dfrac{a \sinh \alpha}{\cosh \alpha - \cos \beta}. \end{array}\right\} \tag{1}$$
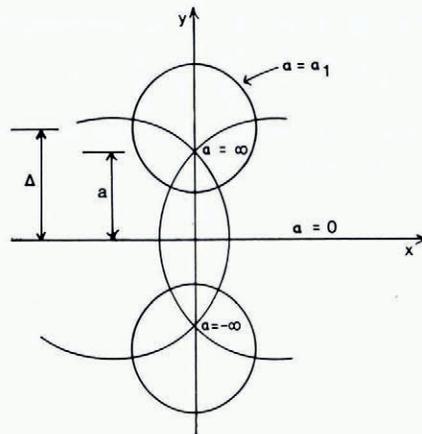

Fig. 2. *The bipolar coordinate system.*

The coordinate curves are two sets of coaxial circles (Fig. 2) which intersect orthogonally. The scale factor a is the distance between the point $\alpha = \infty$ and the line $\alpha = 0$ (the x-axis). The metric coefficient is

$$h = \frac{(\cosh \alpha - \cos \beta)}{a}. \tag{2}$$

Elements of arc have lengths $\delta\alpha/h$, $\delta\beta/h$. The radius of the circle $\alpha = \alpha_1$ is $R = a/\sinh \alpha_1$ and the distance of its centre from the line $\alpha = 0$ is $\Delta = a|\coth \alpha_1|$. In this coordinate system, the Navier–Stokes equations for the slow steady flow of a linearly viscous incompressible fluid under gravity are

$$F_\alpha + h\frac{\partial \sigma_{\alpha\alpha}}{\partial \alpha} - \sigma_{\alpha\alpha}\frac{\partial h}{\partial \alpha} + \sigma_{\beta\beta}\frac{\partial h}{\partial \alpha} - 2\sigma_{\alpha\beta}\frac{\partial h}{\partial \beta} + h\frac{\partial \sigma_{\alpha\beta}}{\partial \beta} = 0, \tag{3}$$

and

$$F_\beta + h\frac{\partial \sigma_{\beta\beta}}{\partial \beta} - \sigma_{\beta\beta}\frac{\partial h}{\partial \beta} + \sigma_{\alpha\alpha}\frac{\partial h}{\partial \beta} - 2\sigma_{\beta\alpha}\frac{\partial h}{\partial \alpha} + h\frac{\partial \sigma_{\beta\alpha}}{\partial \beta} = 0. \tag{4}$$

F_α and F_β are the components of the body force which can be derived from the hydrostatic pressure p_1

$$\left.\begin{aligned} F_\alpha &= h\frac{\partial p_1}{\partial \alpha}, \\ F_\beta &= h\frac{\partial p_1}{\partial \beta}. \end{aligned}\right\} \tag{5}$$

If the line $\alpha = 0$ is horizontal, then

$$p_1 = p_b - \rho g a \frac{\sinh \alpha}{\cosh \alpha - \cos \beta}, \tag{6}$$

where $p_1 = p_b$ is constant on $\alpha = 0$, ρ is the density of the ice, and g the acceleration due to gravity.

The components of the stress tensor σ_{ij} are related to the components of the strain-rate tensor $\dot{e}_{ij}$ by the equations

$$\left.\begin{aligned} \sigma_{\alpha\alpha} &= -p_1 - p + 2\eta\dot{e}_{\alpha\alpha}, \\ \sigma_{\beta\beta} &= -p_1 - p + 2\eta\dot{e}_{\beta\beta}, \\ \sigma_{\alpha\beta} &= 2\eta\dot{e}_{\alpha\beta}, \end{aligned}\right\} \tag{7}$$

where η is the viscosity and p is a pressure term. The continuity equation is

$$\frac{\partial u_\alpha}{\partial \alpha} - \frac{u_\alpha}{h}\frac{\partial h}{\partial \alpha} + \frac{\partial u_\beta}{\partial \beta} - \frac{u_\beta}{h}\frac{\partial h}{\partial \beta} = 0, \tag{8}$$

where u_α is the velocity in the α direction and u_β the velocity in the β direction. Thus, a stream function ψ may be defined by the equations

$$\left.\begin{aligned} u_\alpha &= -h\frac{\partial \psi}{\partial \beta}, \\ u_\beta &= h\frac{\partial \psi}{\partial \alpha}. \end{aligned}\right\} \tag{9}$$

The Navier–Stokes and continuity equations reduce to the biharmonic equation for the stream function

$$\nabla^4\psi = 0, \tag{10}$$

and the Laplace equation for the pressure p

$$\nabla^2 p = 0. \tag{11}$$

The general solution for the biharmonic equation in bipolar coordinates has been given by Jeffrey (1920, 1922)

$$h\psi = \phi_0 + \sum_{n=1}^{\infty} (\phi_n \cos n\beta + \chi_n \sin n\beta), \tag{12}$$

where the Fourier coefficients ϕ_n, χ_n are functions of α:

$$\left.\begin{aligned} \phi_0 &= A_0 \cosh \alpha + B_0\alpha \cosh \alpha + C_0 \sinh \alpha + D_0\alpha \sinh \alpha, \\ \phi_1 &= A_1 \cosh 2\alpha + B_1 + C_1 \sinh 2\alpha + D_1\alpha, \\ \phi_n &= A_n \cosh(n+1)\alpha + B_n \cosh(n-1)\alpha + C_n \sinh(n+1)\alpha + D_n \sinh(n-1)\alpha, \\ \chi_1 &= A_1' \cosh 2\alpha + B_1' + C_1' \sinh 2\alpha + D_1'\alpha, \\ \chi_n &= A_n' \cosh(n+1)\alpha + B_n' \cosh(n-1)\alpha + C_n' \sinh(n+1)\alpha + D_n' \sinh(n-1)\alpha. \end{aligned}\right\} \tag{13}$$

Since p and $\eta\nabla^2\psi$ are conjugate functions, an expression for p in terms of the constants A_n, A_n', etc., can be derived. The leading term is $2\eta(B_0 + D_1)\beta/a$. Since p must on physical grounds be single-valued with β,

$$B_0 = -D_1. \tag{14}$$

The components of the strain-rate tensor can be expressed in terms of ψ

$$
\left.
\begin{aligned}
\dot{e}_{\alpha\alpha} &= -h\frac{\partial}{\partial\beta}\left(h\frac{\partial\psi}{\partial\alpha}\right) - h\frac{\partial h}{\partial\alpha}\frac{\partial\psi}{\partial\beta}, \\[2mm]
\dot{e}_{\beta\beta} &= h\frac{\partial}{\partial\beta}\left(h\frac{\partial\psi}{\partial\alpha}\right) + h\frac{\partial h}{\partial\alpha}\frac{\partial\psi}{\partial\beta}, \\[2mm]
\dot{e}_{\alpha\beta} &= \frac{1}{2}\left\{\frac{\partial}{\partial\alpha}\left(h^2\frac{\partial\psi}{\partial\alpha}\right) - \frac{\partial}{\partial\beta}\left(h^2\frac{\partial\psi}{\partial\beta}\right)\right\},
\end{aligned}
\right\}
\tag{15}
$$

and hence, using Equations (12) and (13), may also be written as Fourier series in β with coefficients that are functions of the $h\psi$ coefficients A_n, $A_n{}'$, etc. From Equation (6), p_1 may also be expanded in a Fourier series

$$
p_1 = p_b - \rho g a - 2\rho g a \sum_1^\infty \exp\left(-n\alpha\right)\cos n\beta.
\tag{16}
$$

Hence, from Equation (7), the components of the stress tensor may be expanded in Fourier series with coefficients that are functions of the $h\psi$ coefficients.

BOUNDARY CONDITIONS

Figure 1 shows a cylindrical stone, radius R, at a distance Δ from the horizontal bed of a glacier. The surfaces of the stone and bed are defined by the curves $\alpha = \alpha_1$, $\alpha = 0$ respectively. The equation of the upper surface of the ice is

$$
y = H = \frac{p_b}{\rho g}.
\tag{17}
$$

Up- and down-stream boundaries are defined by the equation

$$
x = \pm X_1.
\tag{18}
$$

The coefficients in Equations (13) are determined by applying boundary conditions to these five curves.

(i) The ice–rock interfaces

Conditions at these interfaces will be described in terms of the "classical" regulation theory (Nye, 1967) although laboratory experiments have shown that this theory is over-simplified (Nunn and Rowell, 1967; Townsend and Vickery, 1967; Drake and Shreve, 1973; Morris, 1976). Drake and Shreve have pointed out that the effect of impurities, supercooling, and the formation of a trace must be included in a complete regelation analysis. However, we use the simple theory since it allows a simple and clear mathematical demonstration of a difficulty which I believe would arise in any case. In this geometry the classical regelation analysis proceeds as follows: Between ice and rock on the surfaces $\alpha = 0$, $\alpha = \alpha_1$ there are thin films of water, the regelation layers. On the boundary between the ice and the stone ($\alpha = \alpha_1$) there is no shear stress;

$$
\sigma_{\alpha\beta} = 0.
\tag{19}
$$

On part of the boundary between the ice and the bed we may want to allow a non-zero shear stress τ produced by the effective roughness. Thus

$$\sigma_{\alpha\beta} = \tau, \qquad \begin{array}{c} \beta_2 \leqslant \beta \leqslant \beta_1 \\ -\beta_1 \leqslant \beta \leqslant \beta_2 \end{array} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \qquad (20)$$

$$\sigma_{\alpha\beta} = 0, \qquad \begin{array}{c} \beta > \beta_1, \quad \beta < -\beta_1 \\ -\beta_2 < \beta < \beta_2 \end{array}$$

where β_1 and β_2 are arbitrary constants.

This boundary condition can be written in the form of a Fourier expansion for $\sigma_{\alpha\beta}$,

$$\sigma_{\alpha\beta} = \frac{\tau}{\pi}\left\{(\beta_1-\beta_2)+ \sum_{n=1}^{\infty}\frac{2}{n}(\sin n\beta_1 - \sin n\beta_2)\cos n\beta\right\} \qquad (21)$$

on the surface $\alpha = 0$.

In areas where the normal pressure on the rock is high, ice melts and the water travels through the regelation layer to the areas of low normal pressure. The ice–water interfaces are always at the melting-point temperature, which varies with the stress normal to the interface. Thus

$$T = C\sigma_{\alpha\alpha} \qquad \alpha = \alpha_1, \quad \alpha = 0, \qquad (22)$$

where C is a constant having the value 0.7×10^{-7} deg Pa^{-1}.

The high-pressure areas have a lower temperature than the low-pressure areas. Thus the latent heat released by the ice when it refreezes in the low-pressure areas flows through ice, stone, and bed to melt more ice in the high-pressure areas.

The velocity of the ice in the positive direction is

$$u_\alpha = -h\frac{\partial\psi}{\partial\beta}. \qquad (23)$$

The latent heat released by freezing per unit time on an element $\delta\beta/h$ is $Lu_\alpha\delta\beta/h$ on $\alpha = 0$ and $-Lu_\alpha\delta\beta/h$ on $\alpha = \alpha_1$, L is the latent heat of melting of ice per unit volume (2.8×10^8 J m^{-3}). Let T_i, T_s, T_b be the temperature distributions in the ice, stone, and bed respectively, and k_i, k_r the thermal conductivities of the ice and rock. Then, on $\alpha = 0$,

$$\left(k_r h\frac{\partial T_b}{\partial\alpha} - k_i h\frac{\partial T_i}{\partial\alpha}\right)\frac{\delta\beta}{h} = Lu_\alpha\frac{\delta\beta}{h}, \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\}$$

$$k_r\frac{\partial T_b}{\partial\alpha} - k_i\frac{\partial T_i}{\partial\alpha} = L\frac{u_\alpha}{h}, \qquad (24)$$

and on $\alpha = \alpha_1$

$$k_i\frac{\partial T_i}{\partial\alpha} - k_r\frac{\partial T_s}{\partial\alpha} = -L\frac{u_\alpha}{h}. \qquad (25)$$

In the classical regelation analysis the temperature distributions are solutions of the Poisson equation with boundary conditions given by Equations (24) and (25), and the restriction that the temperature should remain finite as $\alpha \to 0$, $\beta \to 0$ (i.e. as $x \to \infty$, $y \to \infty$) and as $\alpha \to \pm\infty$.

The distributions have the form

$$T_i = \sum_{k=1}^{\infty}\left\{(E_n \exp(n\alpha)+G_n \exp(-n\alpha))\frac{\cos n\beta}{n}+ \right.$$
$$\left. +(F_n \exp(n\alpha)+H_n \exp(-n\alpha))\frac{\sin n\beta}{n}\right\}+I\alpha+J, \qquad 0 \leqslant \alpha \leqslant \alpha_1, \qquad (26)$$

$$T_s = \sum_{n=1}^{\infty} \left\{ (E_n \exp(2n\alpha_1) + G_n) \exp(-n\alpha) \frac{\cos n\beta}{n} + \right.$$
$$\left. + (F_n \exp(2n\alpha_1) + H_n) \exp(-n\alpha) \frac{\sin n\beta}{n} \right\} + I\alpha_1 + J, \qquad \alpha \geqslant \alpha_1, \qquad (27)$$

and

$$T_b = \sum_{n=1}^{\infty} \left\{ (E_n + G_n) \exp(n\alpha) \frac{\cos n\beta}{n} + \right.$$
$$\left. + (F_n + H_n) \exp(n\alpha) \frac{\sin n\beta}{n} \right\} + J, \qquad \alpha \leqslant 0. \qquad (28)$$

Using Equation (22) the T coefficients, E_n, G_n, etc., can be defined in terms of the $h\psi$ coefficients. Then Equations (24) and (25) give sets of equations relating groups of the $h\psi$ coefficients. Further equations are derived from the boundary conditions (19) and (21) for the shear stress. We arrive finally at sets of linear equations from which all the $h\psi$ coefficients may be determined given A_0, B_0, C_0, D_0, B_1, A_2, C_2, A_3, C_3, A_1', B_1', C_1', D_1', A_2', and C_2'. The further equations which determine the values of these last coefficients are derived from other boundary conditions.

(ii) *The ice–air.interface*

Since $H \gg a$ we have $\alpha \to 0$, $\beta \to 0$ on the upper surface of the glacier. The expansions for p and $\dot{e}_{\alpha\alpha}$ reduce to

$$p = -\frac{2\eta}{a} \sum_{n=1}^{\infty} C_n' + D_n', \qquad \begin{matrix} \alpha \to 0, \\ \beta \to 0, \end{matrix}$$

$$\dot{e}_{\alpha\alpha} = 0.$$

Therefore, from Equation (7) the boundary condition

$$\sigma_{\alpha\alpha} = 0; \qquad y = H,$$

reduces to

$$\sum_{n=1}^{\infty} (C_n' + D_n') = 0. \qquad (29)$$

Now $h \to 0$ as α and $\beta \to 0$ so $hu_\alpha \to 0$, $h\psi \to 0$ if u_α and ψ remain finite on $y = H$. We may therefore specify two more conditions

$$\left. \begin{matrix} B_0 = 0, \\[2mm] A_0 + \sum_{1}^{n} (A_n + B_n) = 0. \end{matrix} \right\} \qquad (30)$$

(iii) *The up-stream and down-stream boundaries*

In a complete analysis of glacier flow the gravitational forces which produce motion in the ice are determined from the slope of the bed and/or the upper surface of the ice (e.g. Morland, 1976). We have focussed attention on a small horizontal area of the bed over which the ice has a uniform thickness. We have to suppose that up-stream and down-stream of this area gravitational forces act so as to produce a certain pattern of flow of the ice as it enters and leaves the area of interest around the stone. This pattern is defined by boundary conditions

on $x = \pm X_1$. For example, we might suppose that far away from the stone there is simple shear flow parallel to the bed. Thus, for example,

$$\psi \to \frac{-\tau}{2\eta} y^2 - u_b y, \qquad x = \pm x_1 \to \pm \infty, \tag{31}$$

where u_b is the sliding velocity at the bed.

FORCES ON THE STONE

Let the forces applied to the surface $\alpha = \alpha_1$ be statically equivalent to forces X and Y acting at the centre of the stone and a couple L. X acts in the direction parallel to the bed and Y in the perpendicular direction

$$
\left.
\begin{aligned}
X &= \int_0^{2\pi} \left(\sigma_{\alpha\alpha} \frac{\partial x}{\partial \alpha} - \sigma_{\alpha\beta} \frac{\partial y}{\partial \alpha} \right) d\beta, \\[2em]
Y &= \int_0^{2\pi} \left(\sigma_{\alpha\alpha} \frac{\partial y}{\partial \alpha} + \sigma_{\alpha\beta} \frac{\partial x}{\partial \alpha} \right) d\beta, \\[2em]
L &= -R \int_0^{\pi} \sigma_{\alpha\beta} \frac{d\beta}{h}.
\end{aligned}
\right\} \tag{32}
$$

On the surface of the stone $\sigma_{\alpha\beta} = 0$. From Equations (22) and (26), using the series expansions

$$
\left.
\begin{aligned}
\frac{\partial x}{\partial \alpha} &= -2a \sum_1^\infty n \exp(-n\alpha_1) \sin n\beta, \\[2em]
\frac{\partial y}{\partial \alpha} &= -2a \sum_1^\infty n \exp(-n\alpha_1) \cos n\beta.
\end{aligned}
\right\} \tag{33}
$$

Expressions for the forces may be obtained in terms of the T coefficients which are themselves known in terms of the $h\psi$ coefficients. Thus,

$$
\left.
\begin{aligned}
X &= \frac{2a\pi}{C} \sum_{n=1}^\infty \exp(-n\alpha_1)(F_n \exp(n\alpha_1) + H_n \exp(-n\alpha_1)), \\[2em]
Y &= \frac{2a\pi}{C} \sum_{n=1}^\infty \exp(-n\alpha_1)(E_n \exp(n\alpha_1) + G_n \exp(-n\alpha_1)), \\[2em]
L &= 0.
\end{aligned}
\right\} \tag{34}
$$

DISCUSSION

The solution of the equations relating the coefficients of the series expansion for $h\psi$ (Equation (12)) is discussed in the Appendix. It is found that the series of coefficients $\{\phi_n\}$ and $\{\psi_n\}$ diverge, unless both $\lambda_\star$ and $\lambda_\star'$ (two dimensionless critical wavelengths) are zero.

Given that L, C, k_r, k_l, and η are known, finite, and non-zero physical constants, this condition means that $a \to \infty$ and the ratio of the distance of the radius of the stone to the distance of its centre from the bed $R/\Delta \to 0$. In other words, a solution exists for an isolated cylinder in ice (and has indeed been given by Watts (unpublished)), but not when there is another solid boundary.

Thus, we have found that, in the steady state, the classic regelation boundary conditions cannot be obeyed both at a flat bed and at the surface of a circular stone. That is, the temperature and stress distributions cannot be matched so that both the surfaces are at the pressure-melting point. A similar problem has arisen before in theoretical work on regelation and has been discussed by Nye (1967) and Morris (1976). In their analyses, for wires and spheres moving through ice and for ice moving around a cylinder with a wavy surface respectively, there is apparently only one boundary at which normal stress and temperature must be matched. However, at any point within the ice where there is a water inclusion, for example at a three-grain intersection, a relation between the normal stress across the interface and the temperature can also be defined. Thus there are internal boundaries to be considered. Morris suggested that melting and refreezing within the ice would produce an "internal" temperature distribution to be added to the "regelation" temperature distribution which is produced by melting and refreezing on the solid boundary. The total temperature distribution would match the stress distribution so that the ice is always at the pressure-melting point. The classic method of analysis of regelation problems depends on the assumption that any "internal" component of temperature is negligible compared to the "regelation" component at the solid boundary.

I suggest that, even in the time-dependent case, when there are two solid boundaries with melting and refreezing on each there is no reason to suppose that the temperature and stress distributions can be matched so that the ice at both surfaces is at the pressure melting point. There are three ways in which this problem could be resolved:

(1) melting and refreezing within the ice could produce an "internal" temperature distribution such that the ice at the two boundaries and at the internal water inclusions was at the pressure-melting point. The magnitude of the internal heat sources and sinks would not be negligible compared to the magnitude of the sources and sinks on the boundaries;

(2) the ice could separate from the boundaries producing water-filled cavities with shapes such that the temperature at the ice–water interface was always the correct melting-point temperature for the normal stress across the interface. At the moment we do not know if there is any shape for which a steady-state solution is mathematically possible given our formulation of the problem with linearly viscous ice and classical regelation boundary conditions;

(3) the ice could separate from the boundaries producing cavities filled with air and/or water vapour. In this case, parts of the boundary where there is no melting can be well below the pressure-melting point and the requirement that stress and temperature distributions should match at all points of the boundaries is relaxed.

Field observations of clasts in basal ice (e.g. Vivian and Bocquet, 1973; Boulton and others, 1979) indicate that cavities frequently occur in the lee of large particles. Some of these cavities are full of water, others have long spicules of clear ice. The habit of these ice crystals suggests that they may have grown into a cold atmosphere which is below the pressure-melting point.

Of course, the conditions at the bed of a glacier are vastly more complicated than the simple problem that has been analysed in this paper. We have, for example, ignored the effect of geothermal heating, drainage of water at the glacier bed, and the interaction between several clasts in transport which may obscure the effects of the stress–temperature adjustment

process described here. The classical regelation theory used in this paper and in current basal sliding and lodgement theories is known to be inadequate. However, two conclusions can be drawn which are important for theories of erosion and deposition:

1. Cavities may form around clasts in transport in the basal ice because of the presence of the bed. These cavities must be distinguished from those produced by "cavitation" (Nye, 1970) which is controlled by the overburden pressure of ice.
2. The size and shape of such cavities will depend on the pattern of flow of the ice. If the stone moves to a part of the bed with a different flow pattern, or the pattern at a given point changes in time, bulk melting or refreezing must take place as the cavities adjust to the new situation.

Thus, we cannot expect that the forces on a stone due to the ice can be calculated by a simple application of classic regelation and plastic-flow theory. Even when the clast is in contact with the bed and there is only one ice–rock boundary there may not be solutions of the partial differential equations for ψ and T if regelation boundary conditions are applied along the whole of the boundary. Happel and Brenner ([c1973], p. 61) remark that it is difficult to generalize on the required conditions for the existence of unique solutions to the Navier–Stokes equations given combinations of prescribed velocities and derivatives at the boundary. We know there are solutions for the rather odd mixed boundary conditions that arise from classical regelation theory when these are applied on spherical, cylindrical, and plane surfaces (note that in Nye's solution for a perturbed plane and the Morris adaptation of this for a perturbed circular cylinder the boundary conditions are applied at the *unperturbed* surface). However, these are rather special geometries in which the orthogonal curvilinear coordinate system defined by the boundary curve has metric coefficients which do not vary with position along the boundary. Whether solutions exist for Newtonian flow of ice over rock humps or clasts of any shape resting on the bed with classic regelation conditions applied over the whole boundary is an open question.

I do not believe that the problem of the adjustment of stress and temperature distributions that has been raised in this paper is an artefact of the particular geometry, flow law, and boundary conditions chosen. The simplifications I have made merely allow a clear analytical demonstration of the need for an adjustment process. This process will almost certainly occur whenever there are two ice–rock interfaces close together and may also have a part to play in a complete description of flow over steep bedrock obstacles.

APPENDIX

This Appendix gives the derivation of the coefficients of ϕ_n for high n. Similar equations hold for the coefficients of χ_n. For low values of n some terms of the equations are lost or altered but the overall form is the same.

The shear-stress boundary conditions (Equations (21) and (19)) lead to

$$2n\{(n+1)A_n+(n-1)B_n\}-(n-1)\{nA_{n-1}+(n-2)B_{n-1}\}-(n+1)\{(n+2)A_{n+1}+nB_{n+1}\}$$

$$= \frac{2\tau a}{\pi n \eta}\{\sin(n\beta_1)-\sin(n\beta_2)\}, \qquad \text{(A-1)}$$

and

$$2n\cosh\alpha_1\{(n+1)A_n\cosh(n+1)\alpha_1+(n-1)B_n\cosh(n-1)\alpha_1+$$
$$+(n+1)C_n\sinh(n+1)\alpha_1+(n-1)D_n\sinh(n-1)\alpha_1\}-$$
$$-(n-1)\{nA_{n-1}\cosh n\alpha_1+(n-2)B_{n-1}\cosh(n-2)\alpha_1+nC_{n-1}\sinh n\alpha_1+$$
$$+(n-2)D_{n-1}\sinh(n-2)\alpha_1\}-$$
$$-(n+1)\{(n+2)A_{n+1}\cosh(n+2)\alpha_1+nB_{n+1}\cosh n\alpha_1+(n+2)C_{n+1}\sinh(n+2)\alpha_1+$$
$$+nD_{n+1}\sinh n\alpha_1\}=0. \qquad \text{(A-2)}$$

The normal stress boundary conditions lead to equations for the T coefficients

$$F_n = \frac{n\eta C}{2a \sinh n\alpha_1} \{R_1 A_{n-1} + R_2 B_{n-1} + (R_3 - S_1 \exp(-n\alpha_1)) C_{n-1} +$$
$$+ (R_4 - S_1 \exp(-n\alpha_1)) D_{n-1} + R_5 A_n + R_6 B_n +$$
$$+ (R_7 - S_2 \exp(-n\alpha_1)) C_n + (R_8 - S_2 \exp(-n\alpha_1)) D_n + R_9 A_{n+1} +$$
$$+ R_{10} B_{n+1} + (R_{11} - S_3 \exp(-n\alpha_1)) C_{n+1} + (R_{12} - S_3 \exp(-n\alpha_1)) D_{n+1}\}, \qquad \text{(A-3)}$$

$$H_n = \frac{-n\eta C}{2a \sinh n\alpha_1} \{R_1 A_{n-1} + R_2 B_{n-1} + (R_3 - S_1 \exp(n\alpha_1)) C_{n-1} +$$
$$+ (R_4 - S_1 \exp(n\alpha_1)) D_{n-1} + R_5 A_n + R_6 B_n +$$
$$+ (R_7 - S_2 \exp(n\alpha_1)) C_n + (R_8 - S_2 \exp(n\alpha_1)) D_n + R_9 A_{n+1} +$$
$$+ R_{10} B_{n+1} + (R_{11} - S_3 \exp(n\alpha_1)) C_{n+1} + (R_{12} - S_3 \exp(n\alpha_1)) D_{n+1}\}. \qquad \text{(A-4)}$$

R_i and S_i are functions of n, $\cosh n\alpha_1$ and $\sinh n\alpha_1$:

$$R_1 = -(n-1)(n-2)\sinh n\alpha_1,$$
$$R_2 = -(n-1)(n-2)\sinh(n-2)\alpha_1,$$
$$R_3 = -(n-1)(n-2)\cosh n\alpha_1,$$
$$R_4 = -(n-1)(n-2)\cosh(n-2)\alpha_1,$$
$$R_5 = 2(n+1)(n\cosh\alpha_1 \sinh(n+1)\alpha_1 - \sinh n\alpha_1),$$
$$R_6 = 2(n-1)(n\cosh\alpha_1 \sinh(n-1)\alpha_1 + \sinh n\alpha_1),$$
$$R_7 = 2(n+1)(n\cosh\alpha_1 \cosh(n+1)\alpha_1 - \cosh n\alpha_1),$$
$$R_8 = 2(n-1)(n\cosh\alpha_1 \cosh(n-1)\alpha_1 + \cosh n\alpha_1),$$
$$R_9 = -(n+1)(n+2)\sinh(n+2)\alpha_1,$$
$$R_{10} = -(n+1)(n+2)\sinh n\alpha_1,$$
$$R_{11} = -(n+1)(n+2)\cosh(n+2)\alpha_1,$$
$$R_{12} = -(n+1)(n+2)\cosh n\alpha_1,$$
$$S_1 = -(n-1)(n-2),$$
$$S_2 = 2(n+1)(n-1),$$

and

$$S_3 = -(n+1)(n+2).$$

If u_λ/h is expanded in the Fourier series

$$\frac{u_\alpha}{h} = \frac{\pi_0}{2} + \sum_{n=1}^{\infty} (\pi_n \cos n\beta + \rho_n \sin n\beta), \qquad \alpha = 0, \qquad \text{(A-5)}$$

and

$$\frac{u_\alpha}{h} = \frac{X_0}{2} + \sum_{n=1}^{\infty} (X_n \cos n\beta + Y_n \sin n\beta), \qquad \alpha = \alpha_1, \qquad \text{(A-6)}$$

the velocity boundary conditions (Equations (24) and (25)) give

$$\rho_n = \frac{1}{L} \{(k_r - k_i) F_n + (k_r + k_i) H_n\}, \qquad \text{(A-7)}$$

and

$$Y_n = \frac{1}{L} \{(k_r + k_i) \exp(n\alpha_1) F_n + (k_r - k_i) \exp(-n\alpha_1) H_n\}. \qquad \text{(A-8)}$$

Substitution for F_n and H_n from Equations (A-3) and (A-4) into Equations (A-7) and (A-8) gives

$$\rho_n = \frac{na}{4 \sinh n\alpha_1} \{(\lambda_*'^2 - \lambda_*^2)(R_1 A_{n-1} + R_2 B_{n-1} + R_3 C_{n-1} + R_4 D_{n-1} + R_5 A_n +$$
$$+ R_6 B_n + R_7 C_n + R_8 D_n + R_9 A_{n+1} + R_{10} B_{n+1} + R_{11} C_{n+1} + R_{12} D_{n+1}) +$$
$$+ (\lambda_*^2 \exp(n\alpha_1) - \lambda_*'^2 \exp(-n\alpha_1))(S_1(C_{n-1} + D_{n-1}) +$$
$$+ S_2(C_n + D_n) + S_3(C_{n+1} + D_{n+1}))\}, \qquad \text{(A-9)}$$

and

$$Y_n = \frac{-na}{4 \sinh n\alpha_1} \{(\lambda_*^2 \exp(n\alpha_1) - \lambda_*'^2 \exp(-n\alpha_1))(R_1 A_{n-1} + R_2 B_{n-1} + R_3 C_{n-1} +$$
$$+ R_4 D_{n-1} + R_5 A_n + R_6 B_n + R_7 C_n + R_8 D_n + R_9 A_{n+1} + R_{10} B_{n+1} + R_{11} C_{n+1} + R_{12} D_{n+1}) +$$
$$+ (\lambda_*'^2 - \lambda_*^2)(S_1(C_{n-1} + D_{n-1}) + S_2(C_n + D_n) + S_3(C_{n+1} + D_{n+1}))\}. \qquad \text{(A-10)}$$

$\lambda_\star$ and $\lambda_\star'$ are dimensionless critical wavelengths which may be compared to the critical wavelength given by Nye (1969):

$$\lambda_\star{}^2 = \frac{2C(k_r + k_i)\,\eta}{La^2}, \tag{A-11}$$

and

$$\lambda_\star'^2 = \frac{2C(k_r - k_i)\,\eta}{La^2}. \tag{A-12}$$

Now, from Equation (9)

$$-ahu_\alpha = ah\frac{\partial}{\partial\beta}(h\psi) - ah\psi\frac{\partial h}{\partial\beta}, \tag{A-13}$$

hence from Equations (A-5) and (A-6)

$$\tfrac{1}{4}(\rho_{n-2} + \rho_{n+2}) - (\rho_{n-1} + \rho_{n+1}) + \tfrac{3}{2}\rho_n$$
$$= na(A_n + B_n) - (n+2)\frac{a}{2}(A_{n+1} + B_{n+1}) - (n-2)\frac{a}{2}(A_{n-1} + B_{n-1}), \tag{A-14}$$

and

$$\tfrac{1}{4}(\Upsilon_{n-2} + \Upsilon_{n+2}) - (\Upsilon_{n-1} + \Upsilon_{n+1}) + \tfrac{3}{2}\Upsilon_n$$
$$= na\cosh\alpha_1\{A_n\cosh(n+1)\,\alpha_1 + B_n\cosh(n-1)\,\alpha_1 + C_n\sinh(n+1)\,\alpha_1 + D_n\sinh(n-1)\,\alpha_1\} -$$
$$- (n+2)\frac{a}{2}\{A_{n+1}\cosh(n+2)\,\alpha_1 + B_{n+1}\cosh n_1\alpha + C_{n+1}\sinh(n+2)\,\alpha_1 + D_{n+1}\sinh n\alpha_1\} -$$
$$- (n-2)\frac{a}{2}\{A_{n-1}\cosh n\alpha_1 + B_{n-1}\cosh(n-2)\,\alpha_1 + C_{n-1}\sinh n\alpha_1 + D_{n-1}\sinh(n-2)\,\alpha_1\}. \tag{A-15}$$

Substitution for ρ_i and Υ_i from Equations (A-9) and (A-10) and for B_i and D_i from Equations (A-1) and (A-2) using $i = n-3, n+3$ leads to two simultaneous equations for A_{n+3} and C_{n+3} in terms of the known coefficients A_{n-3} to A_{n+2} and C_{n-3} to C_{n+2}. Note that some coefficients are functions of $\lambda_\star$ or $\lambda_\star'$, others depend only on α_1.

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DISCUSSION

J. WEERTMAN: Do your boundary conditions take into account the surplus, unfrozen water that is left over after the regelation cycle is completed? The volume of this water produced per unit time should be equal to the work done per unit time divided by the latent heat of melting–freezing per unit volume of ice. You could not obtain a consistent set of equations that could give a steady-state solution for this problem. But if water is continuously produced by the regelation cycle—which is a Carnot cycle—the problem cannot be a steady-state one.

E. M. MORRIS: My boundary conditions are those used by other authors such as G. S. Boulton and B. Hallet, since I am trying to investigate the validity of their order-of-magnitude analysis. I agree that the regelation equations used by these authors are inadequate.

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