ON THE GEOMETRY OF LINEAL ELEMENTS ON A SPHERE, EUCLIDEAN KINEMATICS, AND ELLIPTIC GEOMETRY

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1. Introduction. The geometry of *slides* and *turns* of oriented lineal elements in the plane was first studied by Kasner [**10**]. Slides and turns generate *whirls*, which constitute a three-parameter group W_3. The product of W_3 and M_3, the three-parameter group of Euclidean displacements in the plane, yields a six-parameter group of *whirl-motions*[1] G_6. The geometry of *turbines*[2], and also of general series of lineal elements, under G_6 was investigated by Kasner in [**10**] and, in subsequent papers, by Kasner and DeCicco, particularly in [**3**], [**4**], [**11**], [**12**]. The author investigated the geometry of series of lineal elements under the seven-parameter group of *whirl-similitudes* G_7 (of which G_6 is a subgroup) in [**6**], [**7**], [**8**]. Among other things, the author showed that G_7 is isomorphic to the group of collineations of the points in *quasi-elliptic* three-space, the geometry of which had been previously studied by Blaschke [**1**], [**2**] and Grünwald [**9**]; he also showed how the geometry of W_3, G_6, and G_7 can be interpreted kinematically as the displacement of one plane over another.

In this paper we investigate the geometry of *spherical whirls* and *whirl-rotations* of oriented lineal elements on a sphere. Some results in this field have already been obtained by Strubecker [**15**], who mapped the points of elliptic three-space E_3 one-to-one upon the oriented lineal elements of a unit sphere. Using synthetic methods, Strubecker deduced, from the geometry of lines in E_3, theorems on *spherical turbines* and families of curves on a sphere, analogous to others found by Kasner for the plane [**10**]. We pursue the geometry of whirls and whirl-rotations on a sphere in other directions and by means of other methods. With the aid of quaternions we shall investigate the differential geometry of series of lineal elements on a sphere subject to two groups, $\mathfrak{W}_3$ and $\mathfrak{G}_6$—analogous respectively to W_3 and G_6 in the plane—determining their fundamental differential invariants and "Serret-Frenet formulae." Our principal objective is to present a characterization of the geometry of whirls and whirl-rotations on a sphere in terms of the kinematic geometry of continuous

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[1]Slides and turns of *non-oriented* lineal elements in the plane had been previously used by Scheffers [**14**] in an investigation of certain groups of contact transformations. Whirls are not contact transformations.

[2]A *turbine* is a series of oriented lineal elements the points of which lie on a circle (which may be a point circle), and the (oriented) lines of which are tangent to a concentric oriented circle.

Turbines in space were studied by A. Narasinga Rao [**13**] and Feld [**5**].

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displacements of one unit sphere over another, similar to the kinematic interpretation we gave in [8] of whirls and whirl-motions in the plane in terms of continuous displacements of one plane over another. The use of quaternions has the advantage of making it particularly easy to map oriented lineal elements on a sphere into the points of E_3. We indicate by means of this mapping how the differential geometry under $\mathfrak{G}_6$ of series on a sphere can serve as a model for the geometry of curves in E_3.

2. Whirl-rotations and turbines. Let the unit sphere S have its centre at the origin O of a right-hand orthogonal coordinate frame $\mathfrak{f}_0$. If an oriented lineal element $\mathfrak{e}$ is tangent to S at the point P, we shall call the great circle through P tangent to $\mathfrak{e}$ and oriented like $\mathfrak{e}$ the *great cycle* of $\mathfrak{e}$. Let the lineal element $\mathfrak{e}_0$ have its point at (1,0,0) and let it be directed so that its great cycle passes through (0,1,0) and is oriented in the counter-clockwise sense, when viewed from the point (0,0,1). We shall call $\mathfrak{e}_0$ the primitive lineal element on S, and $\mathfrak{f}_0$ its *associated frame*.

Let e_0, e_1, e_2, e_3 be the quaternion units such that

$$e_0 e_i = e_i e_0 = e_i, \qquad e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1;$$

and let

$$x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3, \qquad \bar{x} = x_0 e_0 - x_1 e_1 - x_2 e_2 - x_3 e_3.$$

Then a rotation of S around an oriented diameter is given by Hamilton's formula

$$N(x) u^* = \bar{x} u x, \qquad N(x) = x\bar{x},$$

where u and u^* are unit vectors emanating from O. The components x_i of x are the homogeneous Euler parameters of the rotation. If $\mathfrak{e}$ is any lineal element on S and x is the quaternion of the rotation $\mathfrak{e}_0 \rightarrow \mathfrak{e}$, we shall call the components of x the homogeneous coordinates of $\mathfrak{e}$. For convenience, when no confusion will result, we shall let the quaternion x designate both the rotation $\mathfrak{e}_0 \rightarrow \mathfrak{e}$ and the lineal element $\mathfrak{e}$. Evidently, if x designates $\mathfrak{e}$, so does kx, where k is a non-zero scalar. If $N(x) = 1$, we shall call x a normalized quaternion and represent it in bold type: $\mathbf{x}$. To any quaternion x, $N(x) \neq 0$, correspond two normalized ones, namely $\pm x/[N(x)]^{\frac{1}{2}}$. If the rotation x rotates $\mathfrak{e}_0 \rightarrow \mathfrak{e}$ around the unit vector v through the angle 2θ, we can let

$$\mathbf{x} = -\cos\theta + v \sin\theta$$

and

$$-\mathbf{x} = -\cos(\pi - \theta) + \bar{v}\sin(\pi - \theta) \qquad (\bar{v} = -v).$$

The rotation x also rotates $\mathfrak{f}_0$ into another Cartesian frame $\mathfrak{f}$, situated relative to $\mathfrak{e}$ as $\mathfrak{f}_0$ is situated relative to $\mathfrak{e}_0$; we shall call $\mathfrak{f}$ the frame associated with $\mathfrak{e}$.

A lineal element transformation $x \rightarrow x^*$ given by the equation (in which we suppress the factor of proportionality)

$$x^* = xa, \tag{2.1}$$

where

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$$\mathbf{a} = -\cos\alpha + u\sin\alpha \qquad (u^2 = -1),$$

represents a rotation of all the lineal elements on S around the unit vector u through the angle 2α. We shall call such transformations *lineal element rotations*, and we shall let the quaternion a represent the lineal element rotation (2.1). The lineal element rotations constitute a three-parameter group $\mathfrak{M}_3$.

A lineal element transformation $x \to x^*$ whereby every lineal element x is rotated through the same angle 2β around a unit vector u_x, situated relative to the frame $\mathfrak{f}$ associated with x as an arbitrarily given unit vector u_0 is situated relative to $\mathfrak{f}_0$, shall be called a *(spherical) whirl*.

Let the rotation around u_0 through the angle 2β be denoted by the quaternion b where

$$\mathbf{b} = -\cos\beta + u_0\sin\beta.$$

Then $x^*\bar{x} = x\bar{x}b$, so that the whirl $x \to x^*$ is given within a factor of proportionality by the equation

$$(2.2) \qquad x^* = bx.$$

The whirls constitute a three-parameter group of lineal element transformations $\mathfrak{W}_3$, skew-isomorphic to $\mathfrak{M}_3$.

The product of a whirl and a lineal element rotation is commutative. A transformation which is the product of a whirl and a lineal element rotation shall be called a *whirl-rotation*. Whirl-rotations $x \to x^*$ are given by the equation (the factor of proportionality being suppressed)

$$(2.3) \qquad x^* = bxa.$$

The whirl-rotations constitute a six-parameter group $\mathfrak{G}_6$.

Let the symbol (x,y) represent the *scalar product* of two lineal elements x and y, defined as follows:

$$(x, y) = \tfrac{1}{2}(x\bar{y} + y\bar{x}).$$

Since $x\bar{y} + y\bar{x} = \bar{x}y + \bar{y}x$, we also have

$$(x, y) = \tfrac{1}{2}(\bar{x}y + \bar{y}x).$$

With the aid of this definition we obtain the following useful equalities:

$$(2.4) \qquad \begin{aligned} &(x, y) = (y, x), \qquad (x, x) = N(x) = x\bar{x}, \\ &(\alpha x, y) = (x, \alpha y) = \alpha(x, y) \qquad (\alpha \text{ a scalar}), \\ &(x, y + z) = (x, y) + (x, z), \\ &(bxa, bxy) = (a, a)(b, b)(x, y). \end{aligned}$$

Since $N(\mathbf{x}\bar{\mathbf{y}}) = 1$, we can let $\mathbf{x}\bar{\mathbf{y}}$ equal either $-\cos\delta + v\sin\delta$ or $-\cos(\pi - \delta) + \bar{v}\sin(\pi - \delta)$, $v^2 = -1$. Therefore $(\mathbf{x},\mathbf{y}) = -\cos\delta$ in the former case and $-\cos(\pi - \delta)$ in the latter. Thus $\cos\delta = \pm(\mathbf{x},\mathbf{y})$. We shall call δ and $\pi - \delta$ $(0 \leqslant \delta \leqslant \pi)$, the *distances* between x and y: x and y coincide only when $\delta = 0$ or π.

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Evidently

$$\cos^2 \delta = \frac{(x, y)^2}{(x, x)(y, y)}. \tag{2.5}$$

If $(x,y) = 0$, in which case $\delta = \frac{1}{2}\pi$, we shall say that x and y are *orthogonal.*

If we subject x and y to the whirl-rotation (2.3) we obtain, by virtue of the last equation in (2.4),

$$(x^*, y^*) = (a, a)\ (b, b)(x, y).$$

This yields

THEOREM 2.1. *Under the group of whirl-rotations a pair of elements x and y have an invariant* $\cos^2 \delta$, *given by* (2.5), δ *and* $\pi - \delta$ *being the distances between x and y.*

Let the lineal elements x and y be distinct, that is, $\cos^2 \delta \neq 1$. The ∞^1 lineal elements z defined by the equation

$$z = \alpha x + \beta y \qquad (\alpha, \beta \text{ real scalars, } \alpha^2 + \beta^2 \neq 0), \tag{2.6}$$

shall be called a *linear series* of lineal elements. From (2.6) we obtain with the aid of (2.4)

$$(x, z) = \alpha(x, x) + \beta(x, y),$$

$$(y, z) = \alpha(x, y) + \beta(y, y),$$

$$(z, z) = \alpha(x, z) + \beta(y, z).$$

Eliminating α and β, we obtain

$$D = \begin{vmatrix} (x, x) & (x, y) & (x, z) \\ (y, x) & (y, y) & (y, z) \\ (z, x) & (z, y) & (z, z) \end{vmatrix} = 0. \tag{2.7}$$

The following theorems can now be easily established.

THEOREM 2.2. *Two distinct lineal elements determine a linear series.*

THEOREM 2.3. *A necessary and sufficient condition that three lineal elements x,y and z lie on a linear series is that $D = 0$.*

Let q, $N(q) = 1$, be a given lineal element, and let $a = -\cos\theta + r \sin\theta$, where r is a constant unit vector and θ is variable. The lineal element

$$x = qa, \quad N(q) = 1, \tag{2.8}$$

is obtained by rotating q around the vector r through the angle 2θ. It will be convenient hereafter to let a unit vector v designate also the point on S that has for its Cartesian coordinates the components of v. As θ varies from 0 to π, x describes a series of lineal elements, the points of which lie on a circle c (which may be a point circle) having its centre at r, and the great cycles of which make the same angle with c. Such a series shall be called a *spherical turbine;* c shall be called the circle of the turbine, and the points r and $-r$ the centres of the turbine. If we select three lineal elements (2.8) by assigning three arbitrary values to θ, we find that their quaternions satisfy (2.7); consequently, spherical turbines are linear series.

Let us define

$$l = qr\bar{q}.$$

Evidently l is a constant unit vector. Since

$$xr\bar{x} = (qa)r(\overline{qa}) = q(ar\bar{a})\bar{q} = qr\bar{q} = l,$$

the turbine $\mathfrak{T}$ defined parametrically by means of (2.8) has the non-parametric equation

$$\bar{x}lx = r, \qquad (x\bar{x} = 1). \tag{2.9}$$

But the equation of $\mathfrak{T}$ can also take the form $\bar{x}(-l)x = -r;$ therefore, the lineal elements of $\mathfrak{T}$ are represented by those quaternions x which correspond to the rotations of the unit sphere S that carry point l to r, and point $-l$ to $-r;$ that is to say, the quaternions x correspond to the rotations of S that carry the oriented diameter $-l \rightarrow l$ into $-r \rightarrow r$.

3. The kinematic representation of turbines. Let us consider two concentric unit spheres S_l (the left sphere) and S_r (the right sphere). Let the pair of diametrically opposite points l and $-l$ lie on S_l, and let the pair of points r and $-r$ lie on S_r. We can now map the turbine $\mathfrak{T}$ upon two ordered pairs of points on S_l and S_r, namely, l,r and the diametrically opposite pair $-l, -r$. We shall call l,r (or, alternatively, $-l, -r$) respectively the *left* and *right coordinates* of $\mathfrak{T}$, and let either of the symbols $[l, r]$ or $[-l, -r]$ represent $\mathfrak{T}$. Let this mapping whereby every turbine $\mathfrak{T}$ on S corresponds to two pairs of image points on S_l and S_r be called the *kinematic representation* $\mathscr{K}$. We can make $\mathscr{K}$ one-to-one by orienting the turbines on S. With every turbine $\mathfrak{T}$ we associate two oriented turbines $\mathfrak{T}^+$ and $\mathfrak{T}^-$ by assigning to $\mathfrak{T}^+$ the centre r and to $\mathfrak{T}^-$ the centre $-r$. A one-to-one kinematic representation of *oriented turbines* is brought about by choosing the pair of points l,r as the image and $[l, r]$ as the symbol of $\mathfrak{T}^+$, and the pair $-l, -r$ as the image and $[-l, -r]$ as the symbol of $\mathfrak{T}^-$. The simultaneous reflection of the points on S_l and S_r in their common centre corresponds to a reversal of the orientation of the turbines on S.

Let $\mathfrak{T}$: $[l, r]$ be the turbine determined by the two lineal elements x and y. The parametric equation (2.8) of $\mathfrak{T}$ yields

$$\mathbf{y} = \mathbf{x}(-\cos\theta + r\sin\theta).$$

Since

$$\bar{\mathbf{x}}\mathbf{y} + \bar{\mathbf{y}}\mathbf{x} = -2\cos\theta,$$

θ is a distance between x and $y;$ moreover, since

$$\bar{\mathbf{x}}\mathbf{y} - \bar{\mathbf{y}}\mathbf{x} = 2r \sin\theta = 2r\sin\delta,$$

where δ is either distance between x and y, we obtain

THEOREM 3.1. *The turbine determined by the lineal elements x, y ($\mathbf{x} \neq \pm \mathbf{y}$), has turbine coordinates l, r given by the formulae*

$$l = \frac{y\bar{x} - x\bar{y}}{2\{(x,x)(y,y)\}^{\frac{1}{2}}}\csc\delta, \qquad r = \frac{\bar{x}y - \bar{y}x}{2\{(x,x)(y,y)\}^{\frac{1}{2}}}\csc\delta,$$

where δ is a distance between x and y.

Evidently all turbines have the same "length" π.

Because equation (2.9) can be regarded as a necessary and sufficient condition for the incidence of a turbine $[l,r]$ and a lineal element $\mathbf{x}$, we obtain

THEOREM 3.2. *To the ∞^2 oriented turbines $\mathfrak{T}$ incident to a given lineal element x on S correspond, by virtue of the one-to-one mapping $\mathscr{K}$, ∞^2 left image points l on S_l and ∞^2 right image points r on S_r, so that the rotation of S_l that corresponds to the quaternion x brings the ∞^2 left image points into coincidence with their ∞^2 associated right image points.*

The whirl-rotation (2.3) transforms $[l, r]$ into $[l^*, r^*]$ where $l^* = \mathbf{b}\, l\, \bar{\mathbf{b}}$, $r^* = \bar{\mathbf{a}}\, r\, \mathbf{a}$.

The set of ∞^2 lineal elements orthogonal to a given lineal element u shall be called a *planar field* The elements x of this planar field $\mathfrak{F}$ are given by the parametric equation

$$x = ua, \quad a + \bar{a} = 0, \quad a\bar{a} = 1. \tag{3.1}$$

Eliminating a we obtain the non-parametric equation of $\mathfrak{F}$:

$$\bar{u}x + \bar{x}u = 0. \tag{3.2}$$

The four components of u determine $\mathfrak{F}$ and shall be called the homogeneous coordinates of $\mathfrak{F}$.

If the lineal elements y and z lie in the field u, $x = \alpha y + \beta z$ (α and β real scalars) satisfies (3.2) identically. Therefore the turbine determined by y and z lies in u.

By means of the whirl-rotation $x \rightarrow bxa$ the planar field u is transformed into the planar field u^* where

$$u^* = bua. \tag{3.3}$$

Hence we obtain

THEOREM 3.3. *Whirl-rotations transform planar fields into planar fields.*

We can regard (3.3) as the equation of $\mathfrak{G}_6$ in planar field coordinates.

If the lineal elements y and z determine the turbine $[l, r]$, then, in order that $[l, r]$ lie in the field u, it is necessary and sufficient that y and z satisfy (3.2). Hence

$$\bar{\mathbf{u}}y = -\bar{y}\mathbf{u}, \qquad \bar{\mathbf{u}}z = -\bar{z}\mathbf{u},$$

and therefore

$$\bar{z}y = \bar{\mathbf{u}}z\bar{y}\mathbf{u}, \qquad \bar{y}z = \bar{\mathbf{u}}y\bar{z}\mathbf{u};$$

consequently

$$\bar{y}z - \bar{z}y = \bar{\mathbf{u}}(y\bar{z} - z\bar{y})\mathbf{u}.$$

Using the formulae for l and r given in Theorem 3.1 we obtain

$$\bar{\mathbf{u}}l\mathbf{u} = -r \tag{3.4}$$

as a necessary and sufficient condition that the turbine $[l, r]$ lie in the field u. We now have

THEOREM 3.4. *By means of $\mathscr{K}$ the ∞^2 oriented turbines that lie in a planar field u are mapped upon pairs of points l, r on S_l and S_r respectively, so that a symmetry (that is, an improper orthogonal transformation) will transform the left image points into their corresponding right image points; the homogeneous Euler parameters of this symmetry are the coordinates of the planar field u.*

The companion Theorems 3.2 and 3.4 justify calling $\mathscr{K}$ a kinematic representation.

Inasmuch as planar fields are transformed like lineal elements by whirl-rotations, we define the angles ϕ and $\pi - \phi$, $0 \leqslant \phi < \pi$, between the two planar fields u and v by an expression dual to that used for the distances between two lineal elements, namely,

$$\cos^2 \phi = \frac{(u, v)^2}{(u, u)(v, v)}. \tag{3.5}$$

Let the lineal element u be called the *pole* of the planar field u. The following theorems are now easily established.

THEOREM 3.5. *The angle between two planar fields is equal to the distance between their poles.*

THEOREM 3.6. *Two planar fields u and v intersect in a turbine $[l,r]$ where*

$$l = \frac{(u\bar{v} - v\bar{u}) \csc \phi}{2\{(u, u)(v, v)\}^{\frac{1}{2}}}, \qquad r = \frac{(\bar{u}v - \bar{v}u) \csc \phi}{2\{(u, u)(v, v)\}^{\frac{1}{2}}}$$

and ϕ is either one of the angles between u and v.

If x, y and z are three linearly independent lineal elements, there exists a unique lineal element u orthogonal to all of them. Since u must satisfy the equations

$$\bar{u}x + \bar{x}u = \bar{u}y + \bar{y}u = \bar{u}z + \bar{z}u = 0,$$

the components of u are given by

$$u_0{:}u_1{:}u_2{:}u_3 = |x_1y_2z_3| : -|x_0y_2z_3| : |x_0y_1z_3| : -|x_0y_1z_2|.$$

This yields

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THEOREM 3.7. *A planar field is determined by three linearly independent lineal elements.*

THEOREM 3.8. *If x, y, and z are linearly independent lineal elements, the ∞^2 lineal elements*

$$w = \alpha x + \beta y + \gamma z \qquad (\alpha, \beta, \gamma \text{ real numbers})$$

constitute the planar field determined by x, y, and z.

4. Differential invariants of series of lineal elements under $\mathfrak{W}_3$ and $\mathfrak{M}_3$. A series of lineal elements on S is a one-dimensional extent of lineal elements defined by

$$x = x(t) \tag{4.1}$$

where t is a real parameter. We assume that $dx/dt \neq 0$ in the interval $t_1 \leqslant t \leqslant t_2$ and that $x(t)$ has a continuous second derivative. We can, without loss of generality, also assume that $x(t)$ is normalized, that is, that

$$x(t)\bar{x}(t) = 1. \tag{4.2}$$

In addition we assume, as we may, that the quaternions a and b that appear in the lineal element rotation $x \to xa$ and whirl $x \to bx$ are normalized, so that normalized series are transformed by these transformations into normalized series.

Let the whirl b transform the series $\mathfrak{S}$: (4.1) into the series $\mathfrak{S}^*$: $x^*(t)$. Then

$$d\sigma^2 = d\bar{x}dx = d\bar{x}^*dx^*$$

is invariant. We shall call

$$\sigma = \int_{t_0}^{t} \left(\frac{dx}{dt}\frac{d\bar{x}}{dt}\right)^{\frac{1}{2}} dt$$

the $\mathfrak{W}_3$-arc length of $\mathfrak{S}$ measured from t_0 to t. Let the equation of $\mathfrak{S}$ be expressed in terms of the invariant parameter σ. Then, letting $x' = dx/d\sigma$, we have

$$\bar{x}(\sigma)x(\sigma) = 1, \qquad \bar{x}'x' = 1. \tag{4.3}$$

Evidently

$$Z = \bar{x}x' \tag{4.4}$$

is a differential invariant under $\mathfrak{W}_3$. Equations (4.3) yield

$$\bar{x}x' + \bar{x}'x = 0, \qquad N(x) = N(x') = 1. \tag{4.5}$$

Therefore $Z(\sigma)$ is a unit vector. The three components of Z, namely Z_i (i = 1,2,3) where $\sum Z_i^2 = 1$, are differential scalar invariants of $\mathfrak{S}$ under $\mathfrak{W}_3$.

We proceed to find geometric interpretations for σ and Z. Let us consider the distance $\Delta\delta$ between the lineal elements $x(\sigma)$ and $x(\sigma + \Delta\sigma)$ on $\mathfrak{S}$. Since

$$2 \cos \Delta\delta = \bar{x}(\sigma)x(\sigma + \Delta\sigma) + \bar{x}(\sigma + \Delta\sigma)x(\sigma)$$

(see §3), we obtain

$$2\left(\frac{d\delta}{d\sigma}\right)^2 = -\ (\bar{x}x'' + \bar{x}''x).$$

But differentiation of the first equation in (4.5) yields

$$(4.6) \qquad \bar{x}x'' + \bar{x}''x = -\ 2.$$

Hence

$$d\sigma = \pm\ d\delta.$$

Next, let us consider the turbine $\mathfrak{T}:[l,r]$ with centres at r and $-\ r$, tangent to $\mathfrak{S}$ at σ_0. By Theorem 3.1

$$r = \lim_{\Delta\sigma\to 0} \tfrac{1}{2}[\bar{x}(\sigma_0)x(\sigma_0 + \Delta\sigma) - \bar{x}(\sigma_0 + \Delta\sigma)x(\sigma_0)] \csc \Delta\sigma.$$

Since $\bar{x}x'$ is a unit vector, we obtain

$$r = \tfrac{1}{2}[\bar{x}(\sigma_0)x'(\sigma_0) - \bar{x}'(\sigma_0)x(\sigma_0)] = \bar{x}(\sigma_0)x'(\sigma_0) = Z(\sigma_0).$$

Therefore $Z(\sigma)$ and $-\ Z(\sigma)$ are the loci of the centres of the turbines tangent to the series $\mathfrak{S}$.

Under the group $\mathfrak{M}_3$ we obtain the same invariant parameter σ as under $\mathfrak{W}_3$, and we assume again that $\mathfrak{S}$ is expressed in terms of this parameter. It is evident that the unit vector

$$(4.7) \qquad W = x'\bar{x}$$

is invariant under $\mathfrak{M}_3$. The turbine tangent to $\mathfrak{S}$ at σ_0 has a left image vector l which, by Theorem 3.1, is given by

$$\begin{aligned} l &= \lim \tfrac{1}{2}[x(\sigma_0 + \Delta\sigma)\bar{x}(\sigma_0) - x(\sigma_0)\bar{x}(\sigma_0 + \Delta\sigma)] \csc \Delta\sigma \\ &= x'(\sigma_0)\bar{x}(\sigma_0) = W(\sigma_0). \end{aligned}$$

Consequently we obtain a geometric interpretation of the differential invariant $W(\sigma)$ of $\mathfrak{S}$ under $\mathfrak{M}_3$, namely, the locus of the left image point l of the turbines tangent to $\mathfrak{S}$.

Differential invariants of $\mathfrak{S}$ of higher order relative to $\mathfrak{W}_3$ [$\mathfrak{M}_3$] result from differentiating $Z(\sigma)$ [$W(\sigma)$] with respect to σ.

To find kinematic interpretations for $Z(\sigma)$ and $W(\sigma)$, we proceed as follows:

Let S_f and S_m be two unit spheres concentric at O; S_f is fixed in position but S_m is mobile around O. Let $\mathfrak{e}_f$ be a primitive lineal element on S_f and let F_f be its associated rectangular Cartesian frame. Let $\mathfrak{e}_m$ be an arbitrary (primitive) lineal element on S_m and F_m its associated frame; $\mathfrak{e}_m$ and its frame F_m are mobile with S_m relative to S_f and $\mathfrak{e}_f$. Lineal elements and points on S_f will be referred to $\mathfrak{e}_f$ and F_f, but lineal elements and points on S_m will be referred both to $\mathfrak{e}_m$ and $\mathfrak{e}_f$, or, what is equivalent, to their associated frames. Let the initial position of $\mathfrak{e}_m$, and therefore also of S_m, relative to $\mathfrak{e}_f$ be given by the quaternion x_0, namely, the quaternion of the rotation $\mathfrak{e}_f \to \mathfrak{e}_m$. As S_m undergoes a continuous displacement $\mathscr{D}$ around O, $\mathfrak{e}_m$ traces on S_f a series $\mathfrak{S}$ which, referred to $\mathfrak{e}_f$, has the equation $x = x(\sigma)$, where $x_0 = x(\sigma_0)$ denotes the initial position of $\mathfrak{e}_m$. $\mathfrak{S}$ defines completely the displacement $\mathscr{D}$. But $\mathscr{D}$ can be defined as well by a series $\mathfrak{S}^*$ traced on S_f by any other lineal element $\mathfrak{e}^*_m$ on S_m. Let the quaternion that determines the position of $\mathfrak{e}^*_m$ relative to $\mathfrak{e}_f$ be x^*_0; then, if $x^*_0 = bx_0$ is the whirl $\mathfrak{e}_m \to \mathfrak{e}^*_m$, this whirl also transforms $\mathfrak{S} \to \mathfrak{S}^*$.

Let P be a point on S_m, and let its coordinates, when referred to $\mathfrak{e}_m$, be the components of the unit vector v. Then, referred to $\mathfrak{e}_f$ on S_f, P has for its coordinates the components of the unit vector

$$V = \bar{x}vx, \tag{4.8}$$

because the rotation that transports $\mathfrak{e}_f \to \mathfrak{e}_m$ transforms $v \to V$. During the motion $x(\sigma)$ of S_m the vector V describes a cone, the intersection of which with S_f is the trajectory that the point P of S_m traces on S_f. To find the *poles* (instantaneous centres of rotation) of the motion, we seek those points V on S_f for which $V'(\sigma) = O$. From (4.8) we get $xV' + x'V = vx'$. Consequently $V = \bar{x}'vx'$, and therefore $x'V\bar{x}' = v = xV\bar{x}$. Hence

$$V = \bar{x}'xV\bar{x}x',$$

which implies that V is collinear with the vector $\bar{x}x'$. Therefore the locus of the pole on the fixed sphere (the *fixed* or *space centrode*) is $\pm Z(\sigma)$.

The locus of the pole on the mobile sphere (the *mobile* or *body centrode*), referred to $\mathfrak{e}_m$, is

$$v = xV\bar{x} = x(\pm Z)\bar{x} = \pm x\bar{x}x'\bar{x} = \pm x'\bar{x} = \pm W(\sigma). \tag{4.9}$$

During the displacement $\mathscr{D}$ defined by the series $\mathfrak{S}$, the curve $W(\sigma)$ on S_m rolls without slipping on the curve $Z(\sigma)$ on S_f, while, of course, $-W(\sigma)$, diametrically opposite to $W(\sigma)$ on S_m, rolls on $-Z(\sigma)$. The motion $\mathscr{D}$ is completely determined by the centrodes $Z(\sigma)$ and $W(\sigma)$, which, in turn, are determined by $\mathscr{D}$. However, it should be observed that the equation of the mobile centrode is referred not to $\mathfrak{e}_f$ but to any lineal element of $\mathfrak{S}$, say to $\mathfrak{e}_m: x(\sigma_0)$. Therefore, if we replaced on S_m the primitive element $\mathfrak{e}_m$ by another primitive element $\mathfrak{e}^*_m: x^*$, the motion $\mathscr{D}$ that was defined by the series $\mathfrak{S}$ traced out on S_f by $\mathfrak{e}_m$ would instead be defined by a series $\mathfrak{S}^*$ traced out on S_f by $\mathfrak{e}^*_m$. But now the mobile centrode would be referred to $\mathfrak{e}^*_m$ and, according to (4.9), would be given by $W = \pm x^*Z\bar{x}^*$. Consequently the motion $\mathscr{D}$ is determined by the fixed centrode $\pm Z(\sigma)$ and an arbitrary primitive lineal element. If w is the whirl $\mathfrak{e}_m \to \mathfrak{e}^*_m$, w transforms the series $\mathfrak{S}$ generated by $\mathfrak{e}_m$ into the series $\mathfrak{S}^*$ generated by $\mathfrak{e}^*_m$. Since $\mathfrak{S}$ and $\mathfrak{S}^*$ define the same motion $\mathscr{D}$, and $\mathscr{D}$ is defined by $\pm Z(\sigma)$ and an arbitrary lineal element on S_m, $Z(\sigma)$ determines a series within a whirl. We can therefore regard $Z = Z(\sigma)$ as the intrinsic equation of a series relative to $\mathfrak{W}_3$.

The motion defined by a turbine $[l, r]$ is a continuous rotation of S_m around the diameter $(-l \to l)$, and therefore has for its fixed [mobile] centrode the pair of diametrically opposite points $\pm r[\pm l]$. If $\mathscr{D}$ is a displacement defined by a series $\mathfrak{S}$ other than a turbine, the fixed [mobile] centrode of $\mathscr{D}$ is the locus of the right [left] image points $\pm r(\sigma)$ [$\pm l(\sigma)$] of the turbines tangent to $\mathfrak{S}$.

5. Differential invariants of series under $\mathfrak{G}_6$. Let the series $\mathfrak{S}$ have the equation

$$x = x(t), \qquad x(t)\bar{x}(t) = 1, \tag{5.1}$$

where $x(t)$ has a continuous third derivative in the interval $t_1 \leqslant t \leqslant t_2$ in which $dx/dt \neq 0$. Subjecting $\mathfrak{S}$ to the whirl-rotation

$$x^* = bxa, \qquad N(a) = N(b) = 1,$$

we obtain

$$dx^* d\bar{x}^* = b\, dx\, a\bar{a}\, d\bar{x}\, \bar{b} = dx\, d\bar{x}.$$

We let the invariant $dx\, d\bar{x} = d\sigma^2$ as before, but now we designate

$$\sigma = \int_{t_0}^{t} \left(\frac{dx}{dt}\frac{d\bar{x}}{dt}\right)^{\frac{1}{2}} dt$$

as the $\mathfrak{G}_6$-arc length of $\mathfrak{S}$ measured from t_0 to t. Let the parameter t in (5.1) be expressed in terms of σ; then the equation of $\mathfrak{S}$ becomes $x = x(\sigma)$ where

$$x(\sigma)\bar{x}(\sigma) = 1, \qquad x'(\sigma)\bar{x}'(\sigma) = 1. \tag{5.2}$$

We shall consider only series $x(\sigma)$ for which

$$(x'', x'') - 1 \neq 0$$

in the interval $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. The significance of this restriction will be explained later.

Let us associate with any lineal element σ of $\mathfrak{S}$ in the interval $(\sigma_1\sigma_2)$ a frame composed of four mutually orthogonal lineal elements represented by the normalized quaternions ξ_i ($i = 1,2,3,4$) in the following manner:

According to (5.2), x and x' are normalized quaternions. Moreover,

$$x\bar{x}' + x'\bar{x} = 2(x, x') = 0. \tag{5.3}$$

Therefore x and x' are orthogonal lineal elements. Let

$$\xi_1 = x, \ \xi_2 = x'. \tag{5.4}$$

The second equations in (5.2) and (5.3) yield

$$(x', x'') = 0, \qquad (x, x'') = -1. \tag{5.5}$$

Let $y = x + \alpha x''$ where α is a scalar. We seek a value for α for which $(x, y) = 0$. Since

$$(x, y) = (x, x) + \alpha(x, x'') = 1 - \alpha,$$

we obtain $\alpha = 1$. Therefore $y = x + x''$, and y is consequently orthogonal to x and to x'. Now

$$N(y) = (x + x'', x + x'') = (x, x) + 2(x, x'') + (x'', x'') = (x'', x'') - 1 \neq 0.$$

Let

$$\xi_3 = \frac{x + x''}{[(x'', x'') - 1]^{\frac{1}{2}}}. \tag{5.6}$$

Thus ξ_3 is a normalized quaternion that represents a lineal element orthogonal to ξ_1 and to ξ_2.

Let z be the pole of the planar field determined by the linearly independent lineal elements x, x' and y. Then

$$(x, z) = (x', z) = (y, z) = 0.$$

Since $(y, z) = (x, z) + (x'', z)$, we get $(x'', z) = 0$. Therefore

$$z = \left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \\ x_0'' & x_1'' & x_2'' & x_3'' \end{matrix} \right\| \tag{5.7}$$

and

$$N(z) = \begin{vmatrix} (x, x) & (x, x') & (x, x'') \\ (x', x) & (x', x') & (x', x'') \\ (x'', x) & (x'', x') & (x'', x'') \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & (x'', x'') \end{vmatrix} = (x'', x'') - 1.$$

Let

$$\xi_4 = \frac{z}{[(x'', x'') - 1]^{\frac{1}{2}}}. \tag{5.8}$$

Thus ξ_4 is a normalized quaternion representing a lineal element orthogonal to the lineal elements ξ_1, ξ_2, and ξ_3.

Let $\mathfrak{d}$ represent the determinant of the components of the four ξ_i. Then

$$\mathfrak{d}^2 = |(\xi_i, \xi_j)| = 1.$$

Consequently the four normalized quaternions ξ_i are linearly independent. They therefore constitute a linear basis for arbitrary quaternions. Hence we can set the four quaternions $\xi_i' (= d\xi_i/d\sigma)$ equal to the linear combinations

$$\xi_i' = a_{i1}\xi_1 + a_{i2}\xi_2 + a_{i3}\xi_3 + a_{i4}\xi_4 \qquad (i = 1, 2, 3, 4). \tag{5.9}$$

Since $(\xi_i, \xi_j) = \delta_{ij}$, we obtain

$$(\xi_i, \xi_j') + (\xi_i', \xi_j) = 0. \tag{5.10}$$

Scalar multiplication of the equations (5.9) by the ξ_j yields

$$a_{ij} = (\xi_i', \xi_j) = -(\xi_i, \xi_j') = -a_{ji}.$$

Therefore the matrix $\|a_{ij}\|$ is skew-symmetric. Furthermore, since $\xi'_1 = \xi_2$, we find that

$$a_{12} = 1, \qquad a_{13} = a_{14} = 0.$$

Let

$$\frac{1}{\rho} = [(x'', x'') - 1]^{\frac{1}{2}}. \tag{5.11}$$

Then (5.4) and (5.5) yield

$$\xi_2' = -\xi_1 + \frac{1}{\rho}\xi_3.$$

Consequently

$$\alpha_{23} = (\xi_2', \xi_3) = \frac{1}{\rho} \quad \text{and} \quad \alpha_{24} = 0.$$

It remains to find the value of $\alpha_{34} = (\xi'_3, \xi_4) = -(\xi'_4, \xi_3)$. Since

$$\xi_3 = \rho(x + x''),$$
$$\xi_3' = \rho' x + \rho' x'' + \rho x' + \rho x'''.$$

Scalar multiplication of ξ'_3 by $\xi_4 = \rho z$ yields

$$(\xi_3', \xi_4) = \rho\rho'(x, z) + \rho^2(x', z) + \rho\rho'(x'', z) + \rho^2(x''', z).$$

But z is orthogonal to x, x' and x''; therefore

$$(\xi_1', \xi_4) = \rho^2(x''', z) = \frac{\Delta}{1 - (x'', x'')} \tag{5.12}$$

where

$$\Delta = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \\ x_0'' & x_1'' & x_2'' & x_3'' \\ x_0''' & x_1''' & x_2''' & x_3''' \end{vmatrix}. \tag{5.13}$$

Let

$$\frac{1}{\tau} = \frac{\Delta}{1 - (x'', x'')}. \tag{5.14}$$

Then the system of equations (5.9) reduces to the following:

$$\begin{aligned} \xi_1' &= \quad \xi_2 \\ \xi_2' &= -\xi_1 \qquad\quad + \frac{1}{\rho}\xi_3 \\ \xi_3' &= \qquad -\frac{1}{\rho}\xi_2 \qquad\quad + \frac{1}{\tau}\xi_4 \\ \xi_4' &= \qquad\qquad -\frac{1}{\tau}\xi_3 \end{aligned} \tag{5.9*}$$

This system of equations is the analogue for a series $\mathfrak{S}$ under $\mathfrak{G}_6$ of the Serret-Frenet formulae for a curve in Euclidean space. We shall call $1/\rho$ and $1/\tau$ the $\mathfrak{G}_6$-*curvature* and $\mathfrak{G}_6$-*torsion* of $\mathfrak{S}$ respectively. Given two arbitrary functions $\rho(\sigma)$ and $\tau(\sigma)$, a series is determined within a whirl-rotation by means of (5.9*). We can therefore regard $\rho = \rho(\sigma)$ and $\tau = \tau(\sigma)$ as the intrinsic equations of a series relative to $\mathfrak{G}_6$.

If $\mathfrak{S}$ is a turbine, its parametric equation (see (2.8)) may be expressed in the form

$$x = q(-\cos t + r \sin t), \qquad q\bar{q} = 1, \qquad r^2 = -1,$$

where q and r are constant quaternions. Since $(dx/dt)\,(d\bar{x}/dt) = 1$, $t = \pm\,\sigma + \text{const}$. Consequently, if $x(t)$ is a turbine,

$$(x', x') = 1 \quad \text{and} \quad (x'', x'') - 1 = 0$$

where x' and x'' denote differentiation with respect to σ. Conversely, it can be shown that a series $\mathfrak{S}: x(\sigma)$, such that

$$N(y) = (x'', x'') - 1 = 0, \tag{5.15}$$

is a turbine. For (5.15) implies that $y = x'' + x = 0$. Therefore $\bar{x}x'' + \bar{x}x = 0$. Consequently $\bar{x}x'' = -1$. But the fixed centrode of $\mathfrak{S}$, namely $\pm Z = \pm\,\bar{x}x'$. Therefore

$$\frac{dZ}{d\sigma} = \bar{x}x'' + \bar{x}'x' = 0,$$

which implies that the fixed centrode of $\mathfrak{S}$ is a pair of diametrically opposite points and consequently, that $\mathfrak{S}$ is a turbine. Thus (5.15) is a necessary and sufficient condition that a series be a turbine; or, what is equivalent, a necessary and sufficient condition that $\mathfrak{S}$ be a turbine is that $1/\rho = 0$. Torsion is not defined for turbines.

Reverting to the original parameter t, we find the following expressions for the differential invariants:

$$\frac{1}{\rho^2} = \omega \Big/ \left(\frac{dx}{dt}, \frac{dx}{dt}\right)^3 \tag{5.11*}$$

and

$$\frac{1}{\tau} = -\frac{\Delta_1}{\omega} \tag{5.14*}$$

where

$$\omega = \left(\frac{dx}{dt}, \frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}, \frac{d^2}{dt^2}\right) - \left(\frac{dx}{dt}, \frac{d^2x}{dt^2}\right)^2 - \left(\frac{dx}{dt}, \frac{dx}{dt}\right)^3,$$

$$\Delta_1 = \left| x_0, \frac{dx_1}{dt}, \frac{d^2x_2}{dt^2}, \frac{d^3x_3}{dt^3} \right|.$$

6. **Kinematic and non-Euclidean interpretations of the differential invariants ρ and τ.** Let the displacement $\mathscr{M}$, defined by a series $\mathfrak{S}: x(\sigma)$, have $\pm Z(\sigma) = \pm\,\bar{x}x'$ for its fixed centrode. We assume that $\mathfrak{S}$ is not a turbine, that is, that $(x'', x'') - 1 \neq 0$. Let the unit vector

$$\zeta_1 = \bar{x}x' \tag{6.1}$$

have its initial point at O, the centre of the fixed sphere S_f. Let s be the Euclidean arc-length of the fixed centrode $Z(\sigma)$ measured from σ_0 to σ. Then

$$ds^2 = (Z', Z')d\sigma^2 = (\zeta_1', \zeta_1')d\sigma^2.$$

But $Z' = \bar{x}x'' + \bar{x}'x' = 1 + \bar{x}x''$. Therefore

$$ds^2 = [(x'', x'') - 1]\, d\sigma^2 = \frac{1}{\rho^2} d\sigma^2.$$

We orient $Z(\sigma)$ so that $ds/d\sigma = 1/\rho > 0$. Now

$$\frac{d\zeta_1}{ds} = (\bar{x}x'' + \bar{x}'x')\frac{d\sigma}{ds} = \rho\bar{x}(x'' + x).$$

Let

(6.2) $$\zeta_2 = \frac{d\zeta_1}{ds} = \rho\bar{x}(x'' + x).$$

Evidently ζ_2 is a unit vector tangent to the fixed centrode at the point σ.

Let ζ_3 be the vector product of ζ_1 and ζ_2. Then

(6.3) $$\zeta_3 = \zeta_1 \times \zeta_2 = \zeta_1\zeta_2.$$

Therefore ζ_3 is a unit vector tangent to S_f and orthogonal to ζ_1 and ζ_2. The three orthogonal unit vectors ζ_i constitute a moving trihedral of the fixed centrode considered as a spherical curve. The vectors ζ'_i $(i = 1,2,3)$ being linearly dependent on the ζ_i, we can let

(6.4) $$\zeta'_i = \beta_{i1}\zeta_1 + \beta_{i2}\zeta_2 + \beta_{i3}\zeta_3 \qquad (i = 1, 2, 3;\ \beta_{ij} \text{ real scalars}).$$

Evidently the (Euclidean) scalar product $\zeta_i \cdot \zeta_j$ of the vectors ζ_i, ζ_j is equal to (ζ_i, ζ_j). Since $\zeta_i \cdot \zeta_j = \delta_{ij}$,

$$\zeta_i \cdot \zeta'_j + \zeta'_i \cdot \zeta_j = 0.$$

Hence $||\beta_{ij}||$ is skew-symmetric. From (6.2) we obtain

$$\beta_{12} = -\beta_{21} = \frac{1}{\rho}.$$

We proceed to evaluate $\beta_{23} = \zeta'_2 \cdot \zeta_3 = (\zeta'_2, \zeta_3)$. Since ζ_1 and ζ_2 are orthogonal vectors, (6.1), (6.2), and (6.3) yield

$$\zeta_3 = \zeta_1\zeta_2 = \rho x\bar{x}'\bar{x}(x'' + x).$$

Observing that $\bar{x}x'\bar{x}x' = -1$, we obtain

$$\zeta_3 = -\rho(\bar{x}'x'' + \bar{x}'x).$$

Moreover,

$$\zeta'_2 = \rho'\bar{x}(x'' + x) + \rho(\bar{x}'x'' + \bar{x}x''')$$

and, since $(\zeta_2, \zeta_3) = 0$,

$$\zeta'_2 \cdot \zeta_3 = (\zeta'_2, \zeta_3) = \rho(\bar{x}'x'' + \bar{x}x''', \zeta_3).$$

This can be reduced, by virtue of (2.4), (5.2), and (5.11), to

$$-1 - \rho^2[(\bar{x}x''', \bar{x}'x'') + (\bar{x}x''', \bar{x}'x)].$$

To evaluate the scalar products in the square brackets, write

$$p_{ij} = x_i x_j''' - x_j x_i''',\ q_{ij} = x'_i x''_j - x'_j x''_i,\ t_{ij} = x'_i x_j - x'_j x_i.$$

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Observing that, by reason of (5.5), $(x, x''') = 0$, we obtain

$$\bar{x}x''' = e_1(p_{01} - p_{23}) + e_2(p_{02} - p_{31}) + e_3(p_{03} - p_{12}),$$
$$\bar{x}'x'' = e_1(q_{01} - q_{23}) + e_2(q_{02} - q_{31}) + e_3(q_{03} - q_{12}).$$

Therefore

$$(\bar{x}x''', \bar{x}'x'') = \sum p_{ij}q_{ij} - \sum p_{ij}q_{kl},$$

where i,j and k,l are complementary pairs of subscripts. But

$$\sum p_{ij}q_{ij} = \begin{vmatrix} (x, x') & (x, x'') \\ (x''', x') & (x''', x'') \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ -(x'', x'') & (x''', x'') \end{vmatrix} = -(x'', x'')$$

and

$$\sum p_{ij}q_{kl} = \Delta.$$

Therefore

$$(\bar{x}x''', \bar{x}'x'') = -(x'', x'') - \Delta.$$

Similarly

$$(\bar{x}x''', \bar{x}'x) = \sum p_{ij}t_{ij} - \sum p_{ij}t_{kl}.$$

But

$$\sum p_{ij}t_{ij} = \begin{vmatrix} (x, x') & (x, x) \\ (x''', x') & (x''', x') \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -(x'', x'') & 0 \end{vmatrix} = (x'', x'')$$

and

$$\sum p_{ij}t_{kl} = 0.$$

Hence $(\bar{x}x''', \bar{x}'x) = (x'', x'')$. Consequently

$$\beta_{23} = -\beta_{32} = \rho^2\Delta - 1 = -1 - \frac{1}{\tau}.$$

Substituting the expressions we have found for the β_{ij} in (6.4), we obtain

$$(6.4^*) \qquad \begin{aligned} \zeta_1' &= \qquad \frac{1}{\rho}\zeta_2 \\ \zeta_2' &= -\frac{1}{\rho}\zeta_1 \qquad - \left(1 + \frac{1}{\tau}\right)\zeta_3 \\ \zeta_3' &= \qquad \left(1 + \frac{1}{\tau}\right)\zeta_2 \end{aligned}$$

If we change from the parameter σ to s (Euclidean arc-length), the system (6.4*) becomes

$$(6.4^{**}) \qquad \begin{aligned} \frac{d\zeta_1}{ds} &= \qquad \zeta_2 \\ \frac{d\zeta_2}{ds} &= -\zeta_1 \qquad - \rho\left(1 + \frac{1}{\tau}\right)\zeta_3 \\ \frac{d\zeta_3}{ds} &= \qquad \rho\left(1 + \frac{1}{\tau}\right)\zeta_2 \end{aligned}$$

The system (6.4**) is the set of Serret-Frenet formulae of $Z(s)$ regarded as a spherical curve.

Evidently $K_f = -\rho(1 + 1/\tau)$ is the geodesic curvature of the fixed centrode, and represents the rate of turning (bending) of the plane tangent to the fixed (space) cone as its point of tangency with the fixed centrode moves on it with unit speed. If R is the radius of curvature of $Z(s)$ regarded as a space curve, $R^{-2} = K_f^2 + 1$.

In a similar manner, the Serret-Frenet formulae for the mobile centrode $\pm W(\sigma)$ on S_m can be found. Letting η_1 be the unit vector $W(\sigma) = x'\bar{x}$ at the point σ, η_2 the unit vector tangent to $W(\sigma)$ at σ, and $\eta_3 = \eta_1 \times \eta_2 = \eta_1\eta_2$, we obtain

$$
(6.5)\qquad
\begin{aligned}
\frac{d\eta_1}{ds} &= \eta_2 \\
\frac{d\eta_2}{ds} &= -\eta_1 + \rho\left(1 - \frac{1}{\tau}\right)\eta_3 \\
\frac{d\eta_3}{ds} &= -\rho\left(1 - \frac{1}{\tau}\right)\eta_2
\end{aligned}
$$

where s is the Euclidean arc-length of the mobile centrode. The geodesic curvature of the mobile centrode is $K_m = \rho(1 - 1/\tau)$.

The geodesic curvatures $K_f(s)$ and $K_m(s)$ determine the fixed and mobile centrodes on S_f and S_m respectively within rotations around O. Hence the differential invariants $\rho(\sigma)$ and $\tau(\sigma)$ which determine a series $\mathfrak{S}$ within a whirl-rotation also determine, within rotations, the fixed and mobile centrodes of the continuous motion $\mathscr{M}$ defined by $\mathfrak{S}$. When K_f and K_m are constant, the two centrodes are circles, and the motion $\mathscr{M}$ becomes that of a circle of radius $(1 + K_m^2)^{-\frac{1}{2}}$ on S_m rolling without slipping on a circle of radius $(1 + K_f^2)^{-\frac{1}{2}}$ on S_f.

If we regard the four components x_1 of the quaternion x as the homogeneous coordinates of a point in projective three-space, we obtain a continuous one-to-one mapping of the lineal elements x on S upon the points x in three-space. By virtue of this mapping it is evident that to the ∞^6 whirl-rotations (2.3) on S correspond the ∞^6 displacements in elliptic space E_3; indeed, to the whirls correspond the left-translations in E_3 and to the rotations correspond the right-translations. A series $\mathfrak{S}$ as in (5.1) is mapped on a curve $\mathscr{C}$ in E_3, turbines being mapped on the straight lines. If $\mathfrak{S}$ is not a turbine, the moving frame of lineal elements ξ_i ($i = 1,2,3,4$) associated with $\mathfrak{S}$ is mapped on a frame of four points associated with $\mathscr{C}$. The invariants $1/\rho$ and $1/\tau$ can be interpreted as the elliptic curvature and torsion respectively of $\mathscr{C}$, and the equations (5.9*) become the Serret-Frenet formulae for a curve in E_3. A continuous motion of S_m over S_f which corresponds within a whirl-rotation to a series $\mathfrak{S}$ on S therefore also corresponds within an elliptic displacement to a curve $\mathscr{C}$ in E_3.

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