To the Editor of the *Mathematical Gazette*

DEAR SIR,

I am indebted to Mr. A. Kodym of Prague for pointing out tha the expression for π as a continued fraction with unit numerator has been carried to 200 places by P. Pedersen (Nordisk Matematisk Tidskrift, Vol. 6, No. 2, 1958). In this paper it is shown tha π appears to satisfy two conditions which a random number satisfies. Both the geometrical mean of the first n denominators an the nth root of the denominator of the nth convergent appear t converge (somewhat erratically) to the limiting values of th corresponding quantities for a random number.

I should like to correct the assertion in my previous letter (Octobe 1959, p. 179) that the probability that, for a random number, an denominator is equal to r is $1/r(r+1)$. This is true only of th first denominator. Khintchine (Compositio Math. 3, pp. 276–285 has shown that the limit, as n tends to infinity, of the proportion c denominators which are equal to r is $\log[1 + 1/r(r+2)]/\log 2$ If we assume that 200 is a sufficiently large value of n for this to b approximately true, we obtain the following results.

Denominator	*Theoretical frequency*	*Actual frequency*
1	83	80
2	34	38
3	19	19
4	12	10
5	8	5
6	6	6
7	5	7
over 7	33	35

Finally, Khintchine (Compositio Math. 1, pp. 361–382) ha shown that the arithmetic mean of the first n denominators of a random number tends asymptotically to $\log n/\log 2$. For $n = 200$ this value is 7·64. The corresponding value for π is 8·45. Thi agreement is quite good, since the arithmetic mean is dominated to a much greater extent than the geometric mean, by the larg denominators.

Yours etc., E. J F PRIMROSE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Your issue of October 1959 contains two letters on the numerica properties of π. The first letter, by Misses Curphey, Kelley, and Moffat, gives a frequency count of the first 10,000 digits of π which I presume to be those computed by Mr. Felton and published in July 1957 in the *Proceedings of the Oxford Mathematical Conference*

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In April 1958, Mr. Felton informed me that this value of π was wrong after the 7480th decimal, due to a machine error.

The second letter, from Dr. Primrose, asks for an expression of π as a continued fraction with unit numerators and contains the sentence:—"For a random irrational number (assuming that "random" can be defined!) the probability that a denominator of the continued fraction is a given positive integer r is easily shown to be $1/r\,(r + 1)$". This sentence may be compared with the following assertion:—"The shortest path between two points (assuming that "shortest" can be defined!) is easily shown to be a spiral.' In each case, the adjective in question can be defined in many different ways, and the truth of the statement will depend upon which definition is used. The appropriate definitions, to ensure the truth of the two statements, are somewhat pathological and unlikely to be relevant in normal contexts. Without essential loss of generality, we may suppose that the random number lies between 0 and 1 and therefore has the form

$$\Delta = \frac{1}{\delta_1 +}\,\frac{1}{\delta_2 +} \tag{1}$$

where $\delta_1, \delta_2, \ldots$ are positive integers. Amongst possible ways of defining the random character of Δ, *either* we may postulate $F(x) = \text{Prob}\,(\Delta \leqq x)$ and deduce the joint distribution of $\delta_1 \delta_2, \ldots$, *or* we may postulate the distribution of $\delta_1, \delta_2, \ldots$ and deduce the function $F(x)$.

If, for example, we adopt the first alternative together with the additional postulate that Δ is to be uniformly distributed between 0 and 1, we have $F(x) = x$. Then the probability that δ_1 and δ_2 will respectively equal r_1 and r_2 is

$$p(r_1, r_2) = F\left(\frac{1}{r_1 +}\,\frac{1}{r_2 + 1}\right) - F\left(\frac{1}{r_1 +}\,\frac{1}{r_2}\right)$$

$$= \frac{1}{(r_1 r_2 + 1)(r_1 r_2 + r_1 + 1)} \tag{2}$$

Since the right-hand side of (2) does not factorize into the form $q_1(r_1) q_2(r_2)$, we see that δ_1 and δ_2 are not statistically independent. Further their marginal distributions are different. The probablity that δ_1 equals r_1 is

$$p_1(r_1) = \sum_{r_2=1}^{\infty} \frac{1}{(r_1 r_2 + 1)(r_1 r_2 + r_1 + 1)} = \frac{1}{r_1(r_1 + 1)}, \tag{3}$$

which is the form given by Dr. Primrose. But the probability that δ_2 equals r_2 is

$$p_2(r_2) = \sum_{r_1=1}^{\infty} \frac{1}{(r_1 r_2 + 1)(r_1 r_2 + r_1 + 1)} = \frac{\pi^2}{6 r_2 (r_2 + 1)} + 0\left(\frac{1}{r_2{}^3}\right) \tag{4}$$

If, on the other hand, we adopt the second alternative together with the additional postulate that the δ's are statistically independent and are all distributed with the same probability law (3), suggested by Dr. Primrose, then it can be proved $F(x)$ has the form

$$F\left(\frac{1}{r_1+}\frac{1}{r_2+}\cdots\right) = \frac{1}{r_1}\left\{1 - \frac{1}{(r_1+1)r_2}\left\{1 - \frac{1}{(r_2+1)r_3}\left\{1 - \right.\right.\right. \qquad (5)$$

The function given by (5) is a very pathological one, as the following properties show:—

(i) $F(x)$ is continuous and $F(0) = 0$ and $F(1) = 1$.

(ii) For almost all values of x, $F(x)$ is differentiable and its derivative equals 0.

(iii) $F(x)$ is rational if x is rational and also if x is a quadratic surd.

(iv) If x is rational, $F(x)$ has a left-hand and a right-hand derivative at x; and these two one-sided derivatives are equal if and only if the final denominator in the continued fraction form of x equals 3.

In (ii) the phrase (almost all x) means "all x which do not belong to a set of zero Lebesgue measure".

It can be shown, though I would not describe the proof as easy, that, for almost all values of Δ, the proportion of the first n denominators of Δ, which equal a prescribed integer r, tends to

$$\log_2\left\{\frac{(r+1)^2}{r(r+2)}\right\}$$

as $n \to \infty$. This and other results can be found in the following references:—

P. Lévy. *Théorie de l'Addition des Variables Aléatories* (pp. 311–313) Paris: Gauthier-Villars (1937).

P Lévy. "Fractions continues aléatoires" *Rend. Circ. Mat. Palermo* (2) 1 (1952) 1–39.

C. Ryll-Nardzewski. "Ergodic theory of continued fractions". *Studia Math.* 12 (1951) 74–79.

Yours etc., J. M. HAMMERSLEY

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Further to Dr. H. Martyn Cundy's letter and postscript about the "Stroud" system, not only was it given by Prof. Everett in 1879, but it was also "proposed" by M. Hospitalier at the International Congress of Electricians of 1891. Stroud was anticipated more than once!

Yours etc., T. W. HALL